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Characterizations and Representations of Core and Dual Core Inverses

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Abstract. In this paper, double commutativity and the reverse order law for the core inverse are considered. Then new characterizations of the Moore–Penrose inverse of a regular element are given by one-sided invertibilities in a ring. Furthermore, the characterizations and representations of the core and dual core inverses of a regular element are considered.

1 Introduction

Let R be an associative ring with unity 1. We say that $a \in R$ is (von Neumann) regular if there exists $x \in R$ such that $axa = a$. Such x is called an *inner inverse* of a, and is denoted by a^-. Let $a\{1\}$ be the set of all inner inverses of a. Recall that an element $a \in R$ is said to be *group invertible* if there exists $x \in R$ such that $axa = a$, $xax = x$ and $ax = xa$. The element x satisfying the conditions above is called a *group inverse* of a. The group inverse of a is unique if it exists, and is denoted by $a^\#$.

Throughout this paper, assume that R is a unital $*$-ring, that is a ring with unity 1 and an involution $a \mapsto a^*$ such that $(a^*)^* = a$, $(a+b)^* = a^* + b^*$ and $(ab)^* = b^*a^*$ for all $a, b \in R$. An element $a \in R$ is called *Moore–Penrose invertible* [7] if there exists $x \in R$ satisfying the following equations

$$\text{(i) } axa = a, \text{ (ii) } xax = x, \text{ (iii) } (ax)^* = ax, \text{ (iv) } (xa)^* = xa.$$

Any element x satisfying the equations (i)-(iv) is called a Moore-Penrose inverse of a. If such x exists, it is unique and is denoted by $a^\dagger$. If x satisfies conditions (i) and (iii), then x is called a $\{1,3\}$-inverse of a, and is denoted by $a^{(1,3)}$. If x satisfies the conditions (i) and (iv), then x is called a $\{1,4\}$-inverse of a, and is denoted by $a^{(1,4)}$. The symbols $a\{1,3\}$ and $a\{1,4\}$ denote the sets of all $\{1,3\}$-inverses and $\{1,4\}$-inverses of a, respectively.

The concept of core inverse of a complex matrix was first introduced by Baksalary and Trenkler [2]. Recently, Rakić et al. [9] generalized the definition of core inverse to the ring case. An element $a \in R$ is core invertible (see [9, Definition 2.3]) if there exists

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$x \in R$ such that $axa = a$, $xR = aR$, and $Rx = Ra^*$. It is known that the core inverse x of a is unique if it exists, and is denoted by $a^{\textcircled{\#}}$. The dual core inverse of a, when it exists, is defined as the unique $a_{\textcircled{\#}}$ such that $aa_{\textcircled{\#}}a = a$, $a_{\textcircled{\#}}R = a^*R$, and $Ra_{\textcircled{\#}} = Ra$. By R^{-1}, $R^{\#}$, $R^{\dagger}$, $R^{(1,3)}$, $R^{(1,4)}$, $R^{\textcircled{\#}}$ and $R_{\textcircled{\#}}$ we denote the sets of all invertible, group invertible, Moore–Penrose invertible, $\{1,3\}$-invertible, $\{1,4\}$-invertible, core invertible, and dual core invertible elements in R, respectively.

In this paper, double commutativity and the reverse order law for the core inverse proposed in [1] are considered. Also, we characterize the Moore–Penrose inverse of a regular element by one-sided invertibilities in a ring R. Furthermore, new existence criteria and representations of core inverse and dual core inverse of a regular element are given by units.

2 Some Lemmas

The following lemmas will be useful in the sequel.

Lemma 2.1 *Let $a, b \in R$.*

(i) *If there exists $x \in R$ such that $(1+ab)x = 1$, then $(1+ba)(1-bxa) = 1$.*
(ii) *If there exists $y \in R$ such that $y(1+ab) = 1$, then $(1-bya)(1+ba) = 1$.*

According to Lemma 2.1, we know that $1+ab \in R^{-1}$ if and only if $1+ba \in R^{-1}$. In this case, $(1+ba)^{-1} = 1 - b(1+ab)^{-1}a$, which is known as Jacobson's Lemma.

Lemma 2.2 ([5, p. 201]) *Let $a, x \in R$.*

(i) *x is a $\{1,3\}$-inverse of a if and only if $a^* = a^*ax$.*
(ii) *x is a $\{1,4\}$-inverse of a if and only if $a = aa^*x^*$.*

It is known that $a \in R^{\dagger}$ if and only if $a \in aa^*R \cap Ra^*a$ if and only if $a \in R^{(1,3)} \cap R^{(1,4)}$. In this case, $a^{\dagger} = a^{(1,4)}aa^{(1,3)}$. By Lemma 2.2, we know that $a = xa^*a = aa^*y$ implies $a \in R^{\dagger}$ and $a^{\dagger} = y^*ax^*$.

Lemma 2.3 ([11, Theorems 2.16, 2.19, and 2.20]) *Let S be a $*$-semigroup and let $a \in S$. Then a is Moore–Penrose invertible if and only if $a \in aa^*aS$ if and only if $a \in Saa^*a$. Moreover, if $a = aa^*ax = yaa^*a$ for some $x, y \in S$, then $a^{\dagger} = a^*ax^2a^* = a^*y^2aa^*$.*

Lemma 2.4 ([5, Proposition 7]) *Let $a \in R$. Then $a \in R^{\#}$ if and only if $a = a^2x = ya^2$ for some $x, y \in R$. In this case, $a^{\#} = yax = y^2a = ax^2$.*

Lemma 2.5 ([10, Theorems 2.6 and 2.8]) *Let $a \in R$. Then*

(i) *$a \in R^{\textcircled{\#}}$ if and only if $a \in R^{\#} \cap R^{(1,3)}$. In this case, $a^{\textcircled{\#}} = a^{\#}aa^{(1,3)}$.*
(ii) *$a \in R_{\textcircled{\#}}$ if and only if $a \in R^{\#} \cap R^{(1,4)}$. In this case, $a_{\textcircled{\#}} = a^{(1,4)}aa^{\#}$.*

Lemma 2.6 ([9, Theorem 2.14] and [10, Theorem 3.1]) *Let $a \in R$. Then $a \in R^{\textcircled{\#}}$ with core inverse x if and only if $axa = a$, $xax = x$, $(ax)^* = ax$, $xa^2 = a$ and $ax^2 = x$ if and only if $(ax)^* = ax$, $xa^2 = a$, and $ax^2 = x$.*

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3 Double Commutativity and Reverse Order Law for Core Inverses

First, we give the following lemma to prove the double commutativity of core inverse.

Lemma 3.1 *Let $a, b, x \in R$ with $xa = bx$ and $xa^* = b^*x$. If $a, b \in R^{(1,3)}$, then*

$$xaa^{(1,3)} = bb^{(1,3)}x.$$

Proof From $xa = bx$, it follows that

$$xaa^{(1,3)} = bxa^{(1,3)} = bb^{(1,3)}bxa^{(1,3)} = bb^{(1,3)}xaa^{(1,3)}.$$

The condition $xa^* = b^*x$ implies that

$$\begin{aligned} bb^{(1,3)}x &= (b^{(1,3)})^*b^*x = (b^{(1,3)})^*xa^* = (b^{(1,3)})^*x(aa^{(1,3)}a)^* \\ &= (b^{(1,3)})^*xa^*aa^{(1,3)} = (b^{(1,3)})^*b^*xaa^{(1,3)} = bb^{(1,3)}xaa^{(1,3)}. \end{aligned}$$

Hence, $xaa^{(1,3)} = bb^{(1,3)}x$. ∎

Theorem 3.2 *Let $a, b, x \in R$ with $xa = bx$ and $xa^* = b^*x$. If $a, b \in R^{(\#)}$, then $xa^{(\#)} = b^{(\#)}x$.*

Proof As $a, b \in R^{(\#)}$, then $a, b \in R^{\#}$ from Lemma 2.5. Applying [4, Theorem 2.2], we get $b^{\#}x = xa^{\#}$, since $xa = bx$.

So, $xa^{(\#)} = b^{(\#)}x$. Indeed, $xa^{(\#)} = xa^{\#}aa^{(1,3)} = b^{\#}xaa^{(1,3)} = b^{\#}bb^{(1,3)}x = b^{(\#)}x$. ∎

Remark 3.3 Theorem 3.2 can also been obtained from [4, Theorem 2.3]. Indeed, note in [9, Theorem 4.4] that a has (a, a^*)-inverse if and only if $a \in R^{(\#)}$.

Corollary 3.4 *Let $a, x \in R$ with $xa = ax$ and $xa^* = a^*x$. If $a \in R^{(\#)}$, then $xa^{(\#)} = a^{(\#)}x$.*

In 2012, Baksalary and Trenkler [1] asked the following question. Given complex matrices A and B, if $A^{(\#)}$, $B^{(\#)}$ and $(AB)^{(\#)}$ exist, does it follow that $(AB)^{(\#)} = B^{(\#)}A^{(\#)}$? Later, Cohen, Herman, and Jayaraman [3] presented several counterexamples for this problem.

Next, we show that the reverse order law for the core inverse holds under certain conditions in a general ring case.

Theorem 3.5 *Let $a, b \in R^{(\#)}$ with $ab = ba$ and $ab^* = b^*a$. Then $ab \in R^{(\#)}$ and $(ab)^{(\#)} = b^{(\#)}a^{(\#)} = a^{(\#)}b^{(\#)}$.*

Proof It follows from Theorem 3.2 that $b^{(\#)}a = ab^{(\#)}$ and $a^{(\#)}b = ba^{(\#)}$.

Also, the conditions $b^*a = ab^*$ and $a^*b^* = b^*a^*$ guarantee that $b^*a^{(\#)} = a^{(\#)}b^*$, which together with $a^{(\#)}b = ba^{(\#)}$ imply $a^{(\#)}b^{(\#)} = b^{(\#)}a^{(\#)}$ according to Theorem 3.2.

Given the above conditions, it is straightforward to check

(a) By Lemma 3.1, we have $abb^{(1,3)} = bb^{(1,3)}a$. Hence,

$$abb^{(\#)}a^{(\#)}ab = abb^{(1,3)}aa^{\#}b = bb^{(1,3)}aaa^{\#}b = bb^{(1,3)}ba = ab.$$

(b) Since $abb^{(1,3)} = bb^{(1,3)}a$, it follows that

$$\begin{aligned} b^{(\#)}a^{(\#)} &= b^{\#}bb^{(1,3)}a^{\#}aa^{(1,3)} = b^{\#}bb^{(1,3)}aa^{\#}a^{(1,3)} \\ &= b^{\#}abb^{(1,3)}a^{\#}a^{(1,3)} = ab^{\#}bb^{(1,3)}a^{\#}a^{(1,3)} = abb^{\#}b^{(1,3)}a^{\#}a^{(1,3)} \end{aligned}$$

and

$$\begin{aligned} ab &= b^{\#}b^2a = b^{\#}bb^{(1,3)}b^2a \\ &= b^{(\#)}ab^2 = b^{(\#)}a^{\#}aa^{(1,3)}a^2b^2 = b^{(\#)}a^{(\#)}a^2b^2. \end{aligned}$$

Hence, $abR = b^{(\#)}a^{(\#)}R$.

(c) If x in Lemma 3.1 is group invertible, then $aa^{(1,3)}x^{\#} = x^{\#}aa^{(1,3)}$. We have

$$\begin{aligned} b^{(\#)}a^{(\#)} &= b^{\#}bb^{(1,3)}a^{\#}aa^{(1,3)} = b^{\#}a^{\#}bb^{(1,3)}aa^{(1,3)} \\ &= b^{\#}a^{\#}(aa^{(1,3)}bb^{(1,3)})^* = b^{\#}a^{\#}(baa^{(1,3)}b^{(1,3)})^* = b^{\#}a^{\#}(a^{(1,3)}b^{(1,3)})^*(ab)^* \end{aligned}$$

and

$$\begin{aligned} (ab)^* &= b^*a^*aa^{(1,3)} = a^*b^*aa^{(1,3)} = a^*b^*bb^{(1,3)}aa^{(1,3)} \\ &= b^*a^*aa^{\#}abb^{(1,3)}a^{(1,3)} = b^*a^*abb^{(1,3)}a^{\#}aa^{(1,3)} \\ &= b^*a^*abb^{\#}bb^{(1,3)}a^{\#}aa^{(1,3)} = b^*a^*abb^{(\#)}a^{(\#)}. \end{aligned}$$

Thus, $Rb^{(\#)}a^{(\#)} = R(ab)^*$.

So, $ab \in R^{(\#)}$ and $(ab)^{(\#)} = b^{(\#)}a^{(\#)} = a^{(\#)}b^{(\#)}$. ∎

4 Characterizations of Core Inverses by Units

In this section, we give existence criteria for the core inverse of ring elements in terms of units. Representations based on classical inverses are also given. By duality, all the results apply to the dual core inverse.

We now present an existence criterion of group inverse of a regular element.

Proposition 4.1 *Let $k \geq 1$ be an integer and suppose that $a \in R$ is regular with an inner inverse a^-. Then the following conditions are equivalent:*

(i) $a \in R^{\#}$.

(ii) $u = a^k + 1 - aa^- \in R^{-1}$.

(iii) $v = a^k + 1 - a^-a \in R^{-1}$.

In this case, $a^{\#} = u^{-1}a^{2k-1}v^{-1}$.

Proof (i) ⇒ (ii). Since

$$\begin{aligned} u(a(a^{\#})^ka^- + 1 - aa^{\#}) &= (a^k + 1 - aa^-)(a(a^{\#})^ka^- + 1 - aa^{\#}) \\ &= a^{k+1}(a^{\#})^ka^- + 1 - aa^- = aa^- + 1 - aa^- = 1, \end{aligned}$$

it follows that u is right invertible.

Similarly, we can prove $(a(a^{\#})^ka^- + 1 - aa^{\#})u = 1$, *i.e.*, u is left invertible.

Hence, $u = a^k + 1 - aa^- \in R^{-1}$.

(ii) $\Leftrightarrow$ (iii). Note that $u = 1 + a(a^{k-1} - a^-) \in R^{-1}$ if and only if $1 + (a^{k-1} - a^-)a = v \in R^{-1}$.

(iii) $\Rightarrow$ (i). As $v \in R^{-1}$, then $u \in R^{-1}$. Since $ua = a^{k+1} = av$, it follows that $a = a^{k+1}v^{-1} = u^{-1}a^{k+1} \in a^2R \cap Ra^2$, *i.e.*, $a \in R^{\#}$.

Note that $a = u^{-1}a^{k-1}a^2 = a^2a^{k-1}v^{-1} \in a^2R \cap Ra^2$. It follows from Lemma 2.4 that $a^{\#} = u^{-1}a^{k-1}aa^{k-1}v^{-1} = u^{-1}a^{2k-1}v^{-1}$. ■

Theorem 4.2 *Let $a \in R$ be regular. Then the following conditions are equivalent:*

(i) $a \in R^{(\#)}$.
(ii) *$a + 1 - aa^-$ and $a^* + 1 - aa^-$ are invertible for some $a^- \in a\{1\}$.*
(iii) *$a + 1 - aa^-$ is invertible and $a^* + 1 - aa^-$ is left invertible for some $a^- \in a\{1\}$.*
(iv) *$a^*a + 1 - aa^-$ and $(a^*)^2 + 1 - aa^-$ are invertible for some $a^- \in a\{1\}$.*
(v) *$a^*a + 1 - aa^-$ and $(a^*)^2 + 1 - aa^-$ are left invertible for some $a^- \in a\{1\}$.*
(vi) *$a + 1 - aa^-$ and $(a^*)^2 + 1 - aa^-$ are left invertible for some $a^- \in a\{1\}$.*

In this case,

$$a^{(\#)} = (a^*a + 1 - aa^-)^{-1}a^* = a\big[((a^*)^2 + 1 - aa^-)^{-1}\big]^* = (a + 1 - aa^-)^{-1}a((a^* + 1 - aa^-)^{-1})^*.$$

Proof (i) $\Rightarrow$ (ii). Since $a \in R^{(\#)}$, $a \in R^{\#} \cap R^{(1,3)}$ by Lemma 2.5. Let $a^- \in a\{1,3\}$. Then $a + 1 - aa^-$ is invertible by Proposition 4.1, and hence $a^* + 1 - aa^- = (a + 1 - aa^-)^*$ is invertible.

(ii) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (i). As $a^* + 1 - aa^-$ is left invertible, there exists $s \in R$ such that $s(a^* + 1 - aa^-) = 1$. Hence, $a = s(a^* + 1 - aa^-)a = sa^*a \in Ra^*a$, *i.e.*, $a^{(1,3)}$ exists by Lemma 2.2(i). Also, $a + 1 - aa^- \in R^{-1}$ implies that $a^{\#}$ exists by Proposition 4.1. So, $a \in R^{(\#)}$ by Lemma 2.5.

(i) $\Rightarrow$ (iv). Let $a^- \in a\{1,3\}$. Then $a + 1 - aa^-$ and $a^* + 1 - aa^-$ are invertible. Hence, $a^*a + 1 - aa^- = (a^* + 1 - aa^-)(a + 1 - aa^-)$ is invertible.

Also, it follows from Proposition 4.1 that $a^2 + 1 - aa^- \in R^{-1}$ since $a \in R^{\#}$. So, $(a^*)^2 + 1 - aa^- = (a^2 + 1 - aa^-)^* \in R^{-1}$.

(iv) $\Rightarrow$ (v) is clear.

(v) $\Rightarrow$ (i). Since $a^*a + 1 - aa^-$ and $(a^*)^2 + 1 - aa^-$ are both left invertible, there exist $m, n \in R$ such that $m(a^*a + 1 - aa^-) = 1 = n((a^*)^2 + 1 - aa^-)$. As

$$a = m(a^*a + 1 - aa^-)a = ma^*a^2 \quad \text{and} \quad a = n((a^*)^2 + 1 - aa^-)a = n(a^*)^2a,$$

$ma^* = m(n(a^*)^2a)^* = (ma^*a^2)n^* = an^*$.

Let $x = ma^* = an^*$. Then $a = (na^*)a^*a = x^*a^*a$, and hence x is a {1,3}-inverse of a by Lemma 2.2. So, we have $axa = a$ and $(ax)^* = ax$. Also, $xa^2 = ma^*a^2 = a$ and $ax^2 = ax(an^*) = (axa)n^* = an^* = x$. It follows from Lemma 2.6 that $a \in R^{(\#)}$ and $a^{(\#)} = ma^* = an^*$.

(i) $\Rightarrow$ (vi) by (i) $\Rightarrow$ (iv) and Proposition 4.1.

(vi) $\Rightarrow$ (i). Let $u = a + 1 - aa^-$ and $v = (a^*)^2 + 1 - aa^-$. As u and v are left invertible, there exist $s, t \in R$ such that $su = tv = 1$. Hence, $a = tva = t(a^*)^2a \in$

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Ra^*a, which implies that $a \in R^{(1,3)}$ according to Lemma 2.2(i). Also, $a = t(a^*)^2a = ta^*(t(a^*)^2a)^*a = (t(a^*)^2a)at^*a = a^2t^*a \in a^2R$, which combined with $a = sua = sa^2 \in Ra^2$ implies $a \in a^2R \cap Ra^2$, *i.e.*, $a \in R^{\#}$. So, $a \in R^{(\#)}$ by Lemma 2.5.

We next give another formula of $a^{(\#)}$.

Note that (iv) ⇔ (v). In the proof of (v) ⇒ (i), taking $m = (a^*a + 1 - aa^-)^{-1}$ and $n = ((a^*)^2 + 1 - aa^-)^{-1}$.

We obtain

$$a^{(\#)} = ma^* = (a^*a + 1 - aa^-)^{-1}a^* = an^* = a\Big[\big((a^*)^2 + 1 - aa^-\big)^{-1}\Big]^*.$$

As $(a+1-aa^-)a = a^2$, then $a = (a+1-aa^-)^{-1}a^2$ and hence $a^{\#} = (a+1-aa^-)^{-2}a$ by Lemma 2.4.

From $(a^* + 1 - aa^-)a = a^*a$, it follows that $a = (a^* + 1 - aa^-)^{-1}a^*a$. Using Lemma 2.2(i), we know that $((a^* + 1 - aa^-)^{-1})^*$ is a {1,3}-inverse of a.

So,

$$\begin{aligned} a^{(\#)} = a^{\#}aa^{(1,3)} &= (a + 1 - aa^-)^{-2}a^2((a^* + 1 - aa^-)^{-1})^* \\ &= (a + 1 - aa^-)^{-1}a((a^* + 1 - aa^-)^{-1})^*. \end{aligned}$$

The proof is completed. ∎

Remark 4.3 If $a \in R$ satisfies $a^*a = 1$ and $aa^* \neq 1$, then $a^* + 1 - aa^-$ is not left invertible for any $a^- \in a\{1\}$. In fact, if $a^* + 1 - aa^-$ is left invertible for some $a^- \in a\{1\}$, then there exists s such that $s(a^* + 1 - aa^-) = 1$. As $a^*a = 1$, $a = s(a^* + 1 - aa^-)a = sa^*a = s$. Hence, $a(a^* + 1 - aa^-) = 1$ and $a \in R^{-1}$. So, $aa^* = 1$, which is a contradiction.

Proposition 4.4 *Let $k \geq 1$ be an integer and suppose that $a \in R$ is regular. If $(a^*)^k + 1 - aa^- \in R^{-1}$ for any $a^- \in a\{1\}$, then $a \in R^{(\#)}$.*

Proof Let $u = (a^*)^k + 1 - aa^-$. As u is invertible, then $a = u^{-1}(a^*)^k a \in Ra^*a$, hence a is $\{1,3\}$-invertible by Lemma 2.2(i).

As $((a^*)^k + 1 - aa^{(1,3)})^* = a^k + 1 - aa^{(1,3)}$ is invertible for $a^{(1,3)} \in a\{1\}$, then $a \in R^{\#}$ by Proposition 4.1. So, $a \in R^{(\#)}$ from Lemma 2.5. ∎

Remark 4.5 If $a^* + 1 - aa^- \in R^{-1}$ for some $a^- \in a\{1\}$, then $a \notin R^{(\#)}$ in general. Such as let $R = M_2(\mathbb{C})$ be the ring of all 2×2 complex matrices and suppose that involution $*$ is the conjugate transpose. Let $A = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \in R$. Then $A^- = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right] \in A\{1\}$. Hence, $A^* + I - AA^- = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right] \in R^{-1}$, but $A \notin R^{\#}$. So, $A \notin R^{(\#)}$.

The converse of Proposition 4.4 may not be true. In the following Example 4.6, we find a core invertible, but there exists some $a^- \in a\{1\}$ such that none of $a^* + 1 - aa^-$, $(a^*)^2 + 1 - aa^-$ and $a^*a + 1 - aa^-$ are invertible.

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Example 4.6 Let R be the ring as Remark 4.5. Given $A = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \in R$, then $A^2 = -A$ and hence $A^{\#}$ exists. So, $A^{(\#)}$ exists. Taking $A^- = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ 0 & 0 \end{bmatrix}$,

$$A^* + I - AA^- = \frac{1}{3}\begin{bmatrix} 4 & 2 \\ -8 & -4 \end{bmatrix}, \quad (A^*)^2 + I - AA^- = \frac{1}{3}\begin{bmatrix} -2 & -4 \\ 4 & 8 \end{bmatrix},$$
$$A^*A + I - AA^- = \frac{1}{3}\begin{bmatrix} 7 & -13 \\ -14 & 26 \end{bmatrix}$$

are not invertible.

Theorem 4.7 *Let $k \geq 1$ be an integer and suppose $a \in R^{(\#)}$. Then the following conditions are equivalent, for any $a^- \in a\{1\}$:*

(i) $(a^*)^k + 1 - aa^- \in R^{-1}$.
(ii) $(a^*)^{k+1} + 1 - aa^- \in R^{-1}$.
(iii) $a^*a + 1 - aa^- \in R^{-1}$.

In this case, $a^{(\#)} = (a^*a + 1 - aa^-)^{-1}a^* = a^k[((a^*)^{k+1} + 1 - aa^-)^{-1}]^*$.

Proof As $a \in R^{(\#)}$, $a \in R^{\#}$ by Lemma 2.5. Hence, $a+1-aa^{(\#)} \in R^{-1}$ from Proposition 4.1. Note that $aa^{(\#)} = aa^{(1,3)}$ and $a^*aa^{(\#)} = a^*$. Hence, $1 + (a^* - 1)aa^{(\#)} = a^* + 1 - aa^{(\#)} = (a + 1 - aa^{(\#)})^* \in R^{-1}$. From Jacobson's Lemma, it follows that $aa^{(\#)}a^* + 1 - aa^{(\#)} = 1 + aa^{(\#)}(a^* - 1) \in R^{-1}$.

As $(a^*+1-aa^-)(a+1-aa^{(\#)}) = a^*a+1-aa^-$ and $a+1-aa^{(\#)} \in R^{-1}$, $a^*+1-aa^- \in R^{-1}$ if and only if $a^*a + 1 - aa^- \in R^{-1}$.

Also, $((a^*)^n+1-aa^-)(aa^{(\#)}a^*+1-aa^{(\#)}) = (a^*)^n aa^{(\#)}a^*+1-aa^- = (a^*)^{n+1} + 1 - aa^-$; then $(a^*)^n + 1 - aa^- \in R^{-1}$ if and only if $(a^*)^{n+1} + 1 - aa^- \in R^{-1}$, since $aa^{(\#)}a^* + 1 - aa^{(\#)} \in R^{-1}$.

Thus, $a^*a + 1 - aa^- \in R^{-1}$ if and only if $(a^*)^k + 1 - aa^- \in R^{-1}$ if and only if $(a^*)^{k+1} + 1 - aa^- \in R^{-1}$.

Set $m_1 = (a^*a + 1 - aa^-)^{-1}$ and $n_1 = ((a^*)^{k+1} + 1 - aa^-)^{-1}(a^*)^{k-1}$. Then

$$a = (a^*a + 1 - aa^-)^{-1}(a^*a + 1 - aa^-)a = m_1a^*a^2$$

and

$$\begin{aligned} a &= \big((a^*)^{k+1} + 1 - aa^-\big)^{-1}\big((a^*)^{k+1} + 1 - aa^-\big)a \\ &= \big((a^*)^{k+1} + 1 - aa^-\big)^{-1}(a^*)^{k-1}(a^*)^2a = n_1(a^*)^2a. \end{aligned}$$

From Theorem 4.2 (v) ⇒ (i), it follows that

$$a^{(\#)} = m_1a^* = an_1^* = (a^*a + 1 - aa^-)^{-1}a^* = a^k\big[((a^*)^{k+1} + 1 - aa^-)^{-1}\big]^*. \qquad \blacksquare$$

Remark 4.8 Even though $a^*a + 1 - aa^- \in R^{-1}$ for any $a^- \in a\{1\}$, it does not imply the core invertibility of a. Let R be the infinite matrix ring as in Remark 5.4 and let $a = \Sigma_{i=1}^{\infty} e_{i+1,i}$. Then $a^*a = 1$, $aa^* \neq 1$. For any $a^- \in a\{1\}$, as $(2 - aa^-)^{-1} = \frac{1}{2}(1 + aa^-)$; then $2 - aa^- = a^*a + 1 - aa^- \in R^{-1}$. But $a \notin R^{\#}$ and hence $a \notin R^{(\#)}$.

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Proposition 4.9 *Let $k \geq 1$ be an integer and suppose $a \in R$. Then the following conditions are equivalent:*

(i) $a \in R^{(\#)}$.
(ii) $a \in R^{(1,3)}$ *and* $(a^*)^k + 1 - aa^{(1,3)} \in R^{-1}$ *for any* $a^{(1,3)} \in a\{1,3\}$.
(iii) $a \in R^{(1,3)}$ *and* $(a^*)^k + 1 - aa^{(1,3)} \in R^{-1}$ *for some* $a^{(1,3)} \in a\{1,3\}$.

In this case, $a^{(\#)} = (u^{-1})^* a^{2k-1}(u^{-1})^* = (u^{-1})^* a^{k-1} u^{-1}(a^k)^*$, *where* $u = (a^*)^k + 1 - aa^{(1,3)}$.

Proof (i) ⇒ (ii). It follows from Lemma 2.5 that $a \in R^{(\#)}$ implies $a \in R^{\#} \cap R^{(1,3)}$. Hence, $a^k + 1 - aa^{(1,3)} \in R^{-1}$ from Proposition 4.1.

So, $(a^*)^k + 1 - aa^{(1,3)} = (a^k + 1 - aa^{(1,3)})^* \in R^{-1}$ for any $a^{(1,3)} \in a\{1,3\}$.

(ii) ⇒ (iii) is clear.

(iii) ⇒ (i). Let $u = (a^*)^k + 1 - aa^{(1,3)}$. Then $a^k + 1 - aa^{(1,3)} = u^* \in R^{-1}$, and hence $a \in R^{\#}$ by Proposition 4.1.

As $u^* a = a^{k+1}$, $a = (u^{-1})^* a^{k+1} = (u^{-1})^* a^{k-1} a^2$. Lemma 2.4 guarantees that $a^{\#} = ((u^{-1})^* a^{k-1})^2 a$.

Also, $ua = (a^*)^k a$ implies $a = (u^{-1}(a^*)^{k-1}) a^* a$. So, applying Lemma 2.2(i), we know that $a \in R^{(1,3)}$ and $(u^{-1}(a^*)^{k-1})^* = a^{k-1}(u^{-1})^*$ is a {1,3}-inverse of a.

Hence, we have

$$\begin{aligned} a^{(\#)} &= a^{\#} aa^{(1,3)} = ((u^{-1})^* a^{k-1})^2 a^2 a^{k-1} (u^{-1})^* \\ &= (u^{-1})^* a^{k-1} (u^{-1})^* a^{k+1} a^{k-1} (u^{-1})^* \\ &= (u^{-1})^* a^{k-1} a a^{k-1} (u^{-1})^* = (u^{-1})^* a^{2k-1} (u^{-1})^*. \end{aligned}$$

From $ua^k = (a^*)^k u^*$, it follows that $u^{-1}(a^*)^k = a^k (u^{-1})^*$. Thus,

$$a^{(\#)} = (u^{-1})^* a^{k-1} a^k (u^{-1})^* = (u^{-1})^* a^{k-1} u^{-1} (a^*)^k. \qquad \blacksquare$$

Remark 4.10 In Proposition 4.9, if $k \geq 2$, then the expression of the core inverse of a can be given as $a^{(\#)} = a^{k-1}(u^{-1})^*$, where $u = (a^*)^k + 1 - aa^{(1,3)}$. Indeed, as $u^* a^{k-1} = a^{2k-1}$, $(u^*)^{-1} a^{2k-1} = a^{k-1}$. Hence, $a^{(\#)} = (u^{-1})^* a^{2k-1} (u^{-1})^* = a^{k-1}(u^{-1})^*$.

Taking $k = 1$ in Proposition 4.9, we have the following corollary.

Corollary 4.11 *Let $a \in R$. Then the following conditions are equivalent:*

(i) $a \in R^{(\#)}$.
(ii) $a \in R^{(1,3)}$ *and* $a^* + 1 - aa^{(1,3)} \in R^{-1}$ *for any* $a^{(1,3)} \in a\{1,3\}$.
(iii) $a \in R^{(1,3)}$ *and* $a^* + 1 - aa^{(1,3)} \in R^{-1}$ *for some* $a^{(1,3)} \in a\{1,3\}$.

In this case, $a^{(\#)} = (u^{-1})^* a (u^{-1})^* = (u^{-1})^* u^{-1} a^*$, *where* $u = a^* + 1 - aa^{(1,3)}$.

Remark 4.12 Let $a \in R$ be regular with an inner inverse a^-. If $u = a^* + 1 - aa^- \in R^{-1}$, then $(u^{-1})^* a (u^{-1})^* = (u^{-1})^* u^{-1} a^*$. In fact, as $ua = a^* a$, $a = u^{-1} a^* a$, and hence $(u^{-1})^* \in a\{1,3\}$ by Lemma 2.2. Thus, $a(u^{-1})^* = (a(u^{-1})^*)^* = u^{-1} a^*$ and $(u^{-1})^* a (u^{-1})^* = (u^{-1})^* u^{-1} a^*$. Moreover, if $a \in R^{(\#)}$, then $a^{(\#)} \neq (u^{-1})^* u^{-1} a^*$ in general. Indeed, take $A = \left[\begin{smallmatrix} 1 & -2 \\ 1 & -2 \end{smallmatrix}\right]$ in Remark 4.5. Then $A^{\#} = A$ and $A^{\dagger} = \frac{1}{10}\left[\begin{smallmatrix} 1 & 1 \\ -2 & -2 \end{smallmatrix}\right]$.

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Hence, $A^{(\#)} = A^{\#}AA^{\dagger} = -\frac{1}{2}\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$. If $A^{-} = \left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right] \in A\{1\}$, then $U = A^* + I - AA^- = \left[\begin{smallmatrix}2&0\\-2&-2\end{smallmatrix}\right]$ is invertible. But $(U^{-1})^*U^{-1}A^* = (U^{-1})^*A(U^{-1})^* = \frac{1}{4}\left[\begin{smallmatrix}0&0\\-1&-1\end{smallmatrix}\right] \neq A^{(\#)}$.

Proposition 4.13 *Let $a \in R^{(\#)}$ and suppose $u = a^* + 1 - aa^- \in R^{-1}$ for some $a^- \in a\{1\}$. Then $a^{(\#)} = (u^{-1})^*a(u^{-1})^*$ if and only if $a^- \in a\{1,3\}$.*

Proof "⇒" As $ua = a^*a$, $a = u^{-1}a^*a$. It follows from Lemma 2.2 that $(u^{-1})^* \in a\{1,3\}$ and $a = a(u^{-1})^*a$. Also, as $a = a^{(\#)}a^2 = (u^{-1})^*a(u^{-1})^*a^2 = (u^{-1})^*a^2 = (u^*)^{-1}a^2$, $a^2 = u^*a = (a + 1 - (aa^-)^*)a = a^2 + a - (aa^-)^*a$, which implies $a = (aa^-)^*a = (a^-)^*a^*a$. Again, Lemma 2.2 guarantees that $a^- \in a\{1,3\}$.

"⇐" See Corollary 4.11. ∎

Recall that a ring R is called *Dedekind-finite* if $ab = 1$ implies $ba = 1$, for all $a, b \in R$. We next give characterizations of core inverse in such a ring.

Corollary 4.14 *Let R be a Dedekind-finite ring. Then the following conditions are equivalent:*

(i) *$a \in R^{(\#)}$.*
(ii) *$a \in R^{(1,3)}$ and $a^*a + 1 - aa^{(1,3)}$ is invertible for any $a^{(1,3)}$.*
(iii) *$a \in R^{(1,3)}$ and $a^*a + 1 - aa^{(1,3)}$ is invertible for some $a^{(1,3)}$.*

In this case, $a^{(\#)} = v^{-1}a^*$, where $v = a^*a + 1 - aa^{(1,3)}$.

Proof Let $u = a^* + 1 - aa^{(1,3)}$ and $v = a^*a + 1 - aa^{(1,3)}$. Then $v = uu^*$. As R is a Dedekind-finite ring, $v \in R^{-1}$ if and only if $u \in R^{-1}$. By Corollary 4.11, $a^{(\#)} = (u^{-1})^*u^{-1}a^* = (uu^*)^{-1}a^* = v^{-1}a^*$. ∎

5 Core, Dual Core, and Moore–Penrose Invertibility

In this section, we mainly characterize the core inverse and dual core inverse of ring elements. First, new characterizations of the Moore–Penrose inverse of a regular element are given by one-sided invertibilities. One can find that some parts of the following Theorem 5.1 were given in [12, Theorem 3.3]. Herein, a new proof is given.

Theorem 5.1 *Let $a \in R$ be regular with an inner inverse a^-. Then the following conditions are equivalent:*

(i) *$a \in R^{\dagger}$.*
(ii) *$aa^* + 1 - aa^-$ is right invertible.*
(iii) *$a^*a + 1 - a^-a$ is right invertible.*
(iv) *$aa^*aa^- + 1 - aa^-$ is right invertible.*
(v) *$a^-aa^*a + 1 - a^-a$ is right invertible.*
(vi) *$aa^* + 1 - aa^-$ is left invertible.*
(vii) *$a^*a + 1 - a^-a$ is left invertible.*
(viii) *$aa^*aa^- + 1 - aa^-$ is left invertible.*
(ix) *$a^-aa^*a + 1 - a^-a$ is left invertible.*

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Proof (ii) ⇔ (iii), (ii) ⇔ (iv), (iii) ⇔ (v), (vi) ⇔ (vii), (vi) ⇔ (viii), and (vii) ⇔ (ix) follow from Lemma 2.1.

(i) ⇒ (ii). If $a \in R^\dagger$, then there exists $x \in R$ such that $a = aa^*ax$ from Lemma 2.3. As $(aa^*aa^- + 1 - aa^-)(axa^- + 1 - aa^-) = 1$, $aa^*aa^- + 1 - aa^-$ is right invertible. Hence, $aa^* + 1 - aa^-$ is right invertible by Lemma 2.1.

(ii) ⇒ (i). As $aa^* + 1 - aa^-$ is right invertible, $a^*a + 1 - a^-a$ is also right invertible by Lemma 2.1. Hence, there is $s \in R$ such that $(a^*a + 1 - a^-a)s = 1$. We have $a = a(a^*a + 1 - a^-a)s = aa^*as \in aa^*aR$. So, $a \in R^\dagger$ by Lemma 2.3.

(i) ⇒ (vi). This is similar to the proof of (i) ⇒ (ii).

(vi) ⇒ (i). As $aa^* + 1 - aa^-$ is left invertible, $t(aa^* + 1 - aa^-) = 1$ for some $t \in R$. Also, $a = 1 \cdot a = t(aa^* + 1 - aa^-)a = taa^*a \in Raa^*a$, which ensures $a \in R^\dagger$ according to Lemma 2.3. ∎

As a special result of Theorem 5.1, we have the following corollary.

Corollary 5.2 ([6, Theorem 1.2]) *Let $a \in R$ be regular with an inner inverse a^-. Then the following conditions are equivalent:*

(i) $a \in R^\dagger$.
(ii) $aa^* + 1 - aa^-$ *is invertible.*
(iii) $a^*a + 1 - a^-a$ *is invertible.*
(iv) $aa^*aa^- + 1 - aa^-$ *is invertible.*
(v) $a^-aa^*a + 1 - a^-a$ *is invertible.*

Theorems 5.3 and 5.5 were given in [12] by the authors. Next, we give different purely ring theoretical proofs.

Theorem 5.3 *Let $a \in R$ be regular with an inner inverse a^-. Then the following conditions are equivalent:*

(i) $a \in R^\dagger$ *and* $aR = a^2R$.
(ii) $u = aa^*a + 1 - aa^-$ *is right invertible.*
(iii) $v = a^*a^2 + 1 - a^-a$ *is right invertible.*

Proof (i) ⇒ (ii). As $aR = a^2R$, $a + 1 - aa^-$ is right invertible by [8, Theorem 1]. Also, from $a \in R^\dagger$ we can conclude $aa^*aa^- + 1 - aa^-$ is invertible by Corollary 5.2. Hence, $u = aa^*a + 1 - aa^- = (aa^*aa^- + 1 - aa^-)(a + 1 - aa^-)$ is right invertible.

(ii) ⇔ (iii) follows from Lemma 2.1.

(iii) ⇒ (i). Since v is right invertible, there exists $v_1 \in R$ such that $vv_1 = 1$. Then $a = avv_1 = a(a^*a^2 + 1 - a^-a)v_1 = aa^*a^2v_1 \in aa^*aR$, and hence $a \in R^\dagger$ by Lemma 2.3. It follows from Corollary 5.2 that $a \in R^\dagger$ implies that $w = a^*a + 1 - a^-a \in R^{-1}$. As $v = (a^*a + 1 - a^-a)(a^-a^2 + 1 - a^-a)$ is right invertible, $a^-a^2 + 1 - a^-a = w^{-1}v$ is right invertible, and hence $a + 1 - a^-a$ is also right invertible. So, $aR = a^2R$ by [8, Theorem 1]. ∎

Remark 5.4 In general, $a \in R^\dagger$ and $aR = a^2R$ may not imply $a \in R^\#$. For example, let R be the ring of all infinite complex matrices with finite nonzero elements in each column with transposition as involution. Let $a = \Sigma_{i=1}^{\infty} e_{i,i+1} \in R$, where $e_{i,j}$ denotes

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the infinite matrix whose (i, j)-entry is 1 and other entries are zero. Then $aa^* = 1$ and $a^*a = \Sigma_{i=2}^{\infty} e_{i,i}$. So, $a^\dagger = a^*$ and $aR = a^2R$. But $a \notin R^\#$. In fact, if $a \in R^\#$, then $a^\# a = aa^\# = aa^\# aa^* = aa^* = 1$, which would imply that a is invertible. This is a contradiction.

Dually, we have the following result.

Theorem 5.5 *Let $a \in R$ be regular with an inner inverse a^-. Then the following conditions are equivalent:*

(i) *$a \in R^\dagger$ and $Ra = Ra^2$.*
(ii) *$u = aa^*a + 1 - a^-a$ is left invertible.*
(iii) *$v = a^2a^* + 1 - aa^-$ is left invertible.*

We next give existence criteria and representations of the core inverse and of the dual core inverse of a regular element in a ring.

Theorem 5.6 *Let $a \in R$ be regular with an inner inverse a^-. Then the following conditions are equivalent:*

(i) $a \in R^\# \cap R^\dagger$.
(ii) $a \in R^{(\#)} \cap R_{(\#)}$.
(iii) $u = aa^*a + 1 - aa^- \in R^{-1}$.
(iv) $v = aa^*a + 1 - a^-a \in R^{-1}$.
(v) $s = a^*a^2 + 1 - a^-a \in R^{-1}$.
(vi) $t = a^2a^* + 1 - aa^- \in R^{-1}$.

In this case,

$$\begin{aligned} a^{(\#)} &= u^{-1}aa^*,\ a_{(\#)} = a^*av^{-1}, \\ a^\dagger &= (t^{-1}a^2)^* = (a^2s^{-1})^*, \\ a^\# &= (aa^*t^{-1})^2a = a(s^{-1}a^*a)^2. \end{aligned}$$

Proof (i) $\Leftrightarrow$ (ii) by Lemma 2.5.

(iii) $\Leftrightarrow$ (v) and (iv) $\Leftrightarrow$ (vi) are obtained by Jacobson's Lemma.

(i) $\Rightarrow$ (iii). From Proposition 4.1 and Corollary 5.2, $a \in R^\# \cap R^\dagger$ implies that $a + 1 - aa^-$ and $aa^*aa^- + 1 - aa^-$ are both invertible. Hence, $u = aa^*a + 1 - aa^- = (aa^*aa^- + 1 - aa^-)(a + 1 - aa^-)$ is invertible.

(iii) $\Rightarrow$ (i). Suppose that $u = aa^*a + 1 - aa^-$ is invertible. Then $a \in R^\dagger$ from Theorem 5.3, and hence $aa^*aa^- + 1 - aa^-$ is invertible by Corollary 5.2. Since

$$u = (aa^*aa^- + 1 - aa^-)(a + 1 - aa^-)$$

is invertible,

$$a + 1 - aa^- = (aa^*aa^- + 1 - aa^-)^{-1}u$$

is invertible, *i.e.*, $a \in R^\#$ by Proposition 4.1.

(i) $\Leftrightarrow$ (iv) can be obtained by a proof similar to that of (i) $\Leftrightarrow$ (iii).

Next, we give representations of $a^{(\#)}$, $a_{(\#)}$, $a^\dagger$ and $a^\#$, respectively.

Since $ua = aa^*a^2$, $a = (u^{-1}aa^*)a^2$. As $a^\#$ exists, then $a^\# = (u^{-1}aa^*)^2a$ by Lemma 2.4. From Lemma 2.5, we have

$$a^{(\#)} = a^\# aa^{(1,3)} = u^{-1}aa^*u^{-1}aa^*a^2a^{(1,3)}$$
$$= u^{-1}aa^*aa^{(1,3)} = u^{-1}aa^*(aa^{(1,3)})^* = u^{-1}aa^*.$$

Similarly, it follows that $a^\# = a(a^*av^{-1})^2$ and $a_{(\#)} = a^*av^{-1}$.

As $as = aa^*a^2$ and $ta = a^2a^*a$, we then have $a = aa^*(a^2s^{-1}) = (t^{-1}a^2)a^*a$. It follows from Lemma 2.2 that $a \in R^\dagger$ and

$$a^\dagger = (a^2s^{-1})^*a(t^{-1}a^2)^* = (s^{-1})^*(a^2)^*a(a^2)^*(t^{-1})^*$$
$$= (s^{-1})^*(aa^*a^2)^*a^*(t^{-1})^* = (s^{-1})^*(as)^*a^*(t^{-1})^*$$
$$= (a^*)^2(t^{-1})^* = (t^{-1}a^2)^*.$$

Similarly, $a^\dagger = (a^2s^{-1})^*$.

Noting $sa^-a = a^*a^2$, we have $a^-a = s^{-1}a^*a^2$ and $a = aa^-a = (as^{-1}a^*)a^2$. Hence, it follows that $a^\# = (as^{-1}a^*)^2a = a(s^{-1}a^*a)^2$ since $a \in R^\#$. We can also get $a^\# = (aa^*t^{-1})^2a$ by a similar way. ∎

Corollary 5.7 *Let $a \in R^\dagger$. Then the following conditions are equivalent:*

(i) $a \in R^{(\#)}$.
(ii) $a \in R_{(\#)}$.
(iii) $u = aa^*a + 1 - aa^\dagger \in R^{-1}$.
(iv) $v = aa^*a + 1 - a^\dagger a \in R^{-1}$.
(v) $s = a^*a^2 + 1 - a^\dagger a \in R^{-1}$.
(vi) $t = a^2a^* + 1 - aa^\dagger \in R^{-1}$.

In this case,

$$a^{(\#)} = u^{-1}aa^* = aa^*t^{-1},$$
$$a_{(\#)} = a^*av^{-1} = s^{-1}a^*a.$$

Proof As $a \in R^\dagger$, $a \in R^{(\#)}$ if and only if $a \in R^\#$ if and only if $a \in R_{(\#)}$ by Lemma 2.5. So (i)–(vi) are equivalent by Theorem 5.6. Moreover, $a^{(\#)} = u^{-1}aa^*$ and $a_{(\#)} = a^*av^{-1}$. Note that $uaa^* = aa^*t$ and $a^*av = sa^*a$. Then $u^{-1}aa^* = aa^*t^{-1}$ and $a^*av^{-1} = s^{-1}a^*a$, as required. ∎

Proposition 5.8 *Let $a \in R^\dagger$. Then the following conditions are equivalent:*

(i) $a \in R^{(\#)}$.
(ii) $a \in R^\#$.
(iii) $a^* + 1 - aa^\dagger \in R^{-1}$.

In this case, $a^\# = (u^{-2})^*a$ and $a^{(\#)} = (u^{-1})^*u^{-1}a^*$, where $u = a^* + 1 - aa^\dagger$.

Proof (i) ⇔ (ii) by Theorem 5.6 (i) ⇔ (ii).

(ii) ⇔ (iii). Note that $a^* + 1 - aa^\dagger = (a + 1 - aa^\dagger)^*$. It follows from Proposition 4.1 that $a \in R^\#$ if and only if $a + 1 - aa^\dagger \in R^{-1}$ if and only if $a^* + 1 - aa^\dagger \in R^{-1}$.

Let $u = a^* + 1 - aa^\dagger$. Then $u^* a = a^2$ and $a = (u^*)^{-1}a^2$. As $a \in R^\#$, $a^\# = (u^*)^{-2}a = (u^{-2})^* a$ by Lemma 2.4.

Since $a \in R^\dagger$, it follows that

$$\begin{aligned} a^{(\#)} &= a^\# aa^{(1,3)} = a^\# aa^\dagger = (u^*)^{-2}a^2a^\dagger \\ &= (u^*)^{-1}(u^*)^{-1}a^2a^\dagger = (u^*)^{-1}aa^\dagger \\ &= (u^*)^{-1}u^{-1}uaa^\dagger = (u^{-1})^* u^{-1}a^*. \end{aligned}$$

The proof is completed. ∎

Proposition 5.9 *Let $a \in R^\#$. Then the following conditions are equivalent:*

(i) $a \in R^{(\#)} \cap R_{(\#)}$.
(ii) $a \in R^\dagger$.
(iii) $a^* + 1 - aa^\# \in R^{-1}$.

In this case, $a^\dagger = (u^{-1})^* a(u^{-1})^*$, $a^{(\#)} = a^\# a(u^{-1})^*$ and $a_{(\#)} = (u^{-1})^* aa^\#$, where $u = a^* + 1 - aa^\#$.

Proof (i) ⇔ (ii) by Theorem 5.6 (i) ⇔ (ii).

(ii) ⇒ (iii). Note that $a \in R^\dagger$ implies $a^*a + 1 - a^\# a \in R^{-1}$ by Corollary 5.2. As $a \in R^\#$, then $a + 1 - aa^\dagger \in R^{-1}$ from Proposition 4.1. Since $a^*a + 1 - a^\# a = (a^* + 1 - aa^\#)(a + 1 - aa^\dagger)$, it follows that $a^* + 1 - aa^\# = (a^*a + 1 - a^\# a)(a + 1 - aa^\dagger)^{-1} \in R^{-1}$.

(iii) ⇒ (ii). Let $u = a^* + 1 - aa^\#$. Then $ua = a^*a$ and $au = aa^*$. As $u \in R^{-1}$, then $a = aa^*u^{-1} = u^{-1}a^*a \in aa^*R \cap Ra^*a$. So, $a \in R^\dagger$ and $(u^{-1})^*$ is both a {1,3}-inverse and a {1,4}-inverse of a. Moreover, $a^\dagger = a^{(1,4)}aa^{(1,3)} = (u^{-1})^* a(u^{-1})^*$.

Hence, $a^{(\#)} = a^\# aa^{(1,3)} = a^\# a(u^{-1})^*$ and $a_{(\#)} = a^{(1,4)}aa^\# = (u^{-1})^* aa^\#$. ∎

It is known that if $a \in R^\dagger$, then $aa^{(1,3)} = aa^\dagger$. Applying Corollary 4.14, we have the following corollary.

Corollary 5.10 *Let R be a Dedekind-finite ring. If $a \in R^\dagger$, then $a \in R^{(\#)}$ if and only if $a^*a + 1 - aa^\dagger \in R^{-1}$. In this case, $a^{(\#)} = (a^*a + 1 - aa^\dagger)^{-1}a^*$.*

Remark 5.11 Suppose $2 \in R^{-1}$. If $a^*a + 1 - aa^\dagger \in R^{-1}$ implies $a \in R^{(\#)}$ for any $a \in R^\dagger$, then $a^*a = 1$ can conclude $aa^* = 1$. Indeed, if $a^*a = 1$, then $a \in R^\dagger$ and $a^\dagger = a^*$. Hence, $a^*a + 1 - aa^\dagger = 2 - aa^\dagger \in R^{-1}$ with inverse $\frac{1}{2}(1 + aa^\dagger)$. Thus, $a \in R^{(\#)}$ and $a \in R^\#$. As $aa^\# = a^\# a = (a^*a)a^\# a = a^*a = 1$, $a \in R^{-1}$, and hence $aa^* = 1$.

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