FINITE ELEMENT METHODS FOR COMPRESSIBLE STOKES EQUATIONS WITH INFLOW BOUNDARY CONDITION

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A finite element method for solving the compressible viscous Stokes equation with an inflow boundary condition is presented. The unique existence of the solution of the discrete problem is established, and an error analysis is given. It is shown that the error in pressure is dominated by the one in velocity and an error at the inflow portion of the boundary.

1. INTRODUCTION

We study and give an error analysis of numerical methods for the stationary linearised barotropic compressible viscous Navier-Stokes system with an inflow boundary condition, using finite element methods. Such a linearised system is often called a "compressible Stokes system" because it may also be obtained from the (incompressible) Stokes equations and certain modifications. It is well-known that the system is not elliptic in the ADN sense: the momentum equations are elliptic in velocity and the continuity one is hyperbolic in pressure, so that it is neither elliptic nor hyperbolic. Recently in [**4, 5**] we have studied the whole nonlinear steady-state barotropic compressible viscous Navier-Stokes equations with an inflow boundary condition. The reason that the inflow boundary condition is considered is: if one assigns specified nonzero values to the velocity components on the boundary of the region, then the velocity field gives the characteristic directions for the hyperbolic continuity equation, and so values of density must be specified on those portions of the boundary where the specified velocity vector points into the region. Naturally in the linearised problem a boundary condition for pressure is imposed, and zero boundary condition for velocity and zero initial condition for pressure are considered because it was convenient to use as dependant variables the deviation from the boundary values.

This paper consider the following problem (called a compressible Stokes system)

$$(1) \quad \begin{cases} -\mu\Delta\mathbf{u} - \nu\nabla\,\mathrm{div}\,\mathbf{u} + (\mathbf{U}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad \Omega, \\ \mathrm{div}\,\mathbf{u} + \mathbf{U}\cdot\nabla p = 0 \quad \text{in}\ \Omega, \\ \mathbf{u} = 0 \quad \text{on}\ \Gamma, \\ p = 0 \quad \text{on}\ \Gamma_{\text{in}}. \end{cases}$$

Here Ω is an open bounded domain in R^2 with smooth boundary Γ, and $\mathbf{u} = [u, v]$ is a velocity vector, p is the pressure, $\mathbf{U} = [U, V]$ is a given smooth vector function of (x, y), and $\mathbf{f}$ is a given function. The numbers μ and ν are viscous constants with $\mu > 0$ and $\mu > -\nu$. Throughout this paper it is assumed that $U \geqslant C_0 > 0$ for a constant C_0. The incoming and outgoing portions of the boundary, Γ_{in} and Γ_{out} are defined by

$$\Gamma_{\text{in}} = \big\{ (x, y) \in \Gamma : \mathbf{U} \cdot \mathbf{n} < 0 \big\},$$
$$\Gamma_{\text{out}} = \big\{ (x, y) \in \Gamma : \mathbf{U} \cdot \mathbf{n} \geqslant 0 \big\},$$

where $\mathbf{n} = (n_1, n_2)$ denotes the unit outward pointing normal vector to Γ.

In [5] we showed that the regularity of the solution of (1) depends on the geometry of the boundary of the domain, not on the regularity of the data because the singularities of the solution at the junction of inflow and outflow portions block further regularity in the solution, and we proved some regularity results for the linear and nonlinear problems. The regularity result for the linear case will be given in Section 2. In this paper we use this result and develop a finite element method for the system (1) by giving some modifications to the continuity equation. It is shown that the error in pressure is bounded by the one in velocity and an error at the inflow portion of the boundary. An error analysis for the finite element method is given.

To make our analysis simple, we assume throughout this paper that $U = 1$ and $V = 0$. In fact, if Γ has nonzero curvature at the points where $\mathbf{U} \cdot \mathbf{n}$ is zero, Theorem 1 in Section 2 can be obtained under suitable assumptions. Hence any bounded domain in R^2, with a C^2 boundary which has nonzero curvature at the points where $\mathbf{U} \cdot \mathbf{n}$ is zero, will be appropriate as a domain of the problem (1). Here a standard example of a domain is "a circle" in R^2 with centre the origin and radius 1.

In [3], an error analysis was given, and the error bound does not have optimal order in terms of approximation theory; the presence of the convective term in the continuity equation degrades the error bound, and also the problems in [3] have quite different character, namely, the flows just stay and play in a given domain, but the ones in our problem move in and out. In [3], a unique existence of the discrete solution was established by showing that the subspaces for velocity and pressure satisfy the "inf-sup" condition associated with the (incompressible) Stokes system.

Some advantages for the method to be presented here are that we have a nice error analysis for (1), and that in [2] and [3], the bubble function in the "MINI" element subspaces is necessary in proving stability, while in our problem, it is not. So the resulting finite element matrix order is much smaller than the one in [3]. A disadvantage is that the stiffness matrix resulting from our weak formulation is not sparse in some block of the matrix. Furthermore, it has been observed that an approximation of the

incoming portion boundary plays a role in resolving the accuracy of the discrete solution.

In Section 2 we propose a finite element method for (1), the "streamline" finite element method. The unique existence of the discrete solution for the finite element formulation is shown and its error analysis is given. It would be interesting to determine if the analysis given here for (1) can be extended to the system (1) with the nonconstant vector field $\mathbf{U} = [U, V]$.

In this paper, the following Sobolev spaces and norms shall be used: $H^{2,q}(\Omega)$ is the Sobolev space of real valued L^q functions such that their derivatives belong to $L^q(\Omega)$. We write $\|u\|_{0,q}$ and $\|u\|_{2,q}$ for the norms in $L^q(\Omega)$ and $H^{2,q}(\Omega)$, and (u,v), $\|u\|_0$ for the scalar product and norm in $L^2(\Omega)$. $H^1_0(\Omega) = \{u \in H^1(\Omega) : u|_\Gamma = 0\}$ and $H^{2,q}_0(\Omega) = H^{2,q}(\Omega) \cap H^1_0$ and $|u|_\infty \equiv \max\{|u(x)| : x \in \Omega\}$, $L^\infty(\Omega) = \{u : |u|_\infty < \infty\}$. We shall frequently use the following Sobolev imbedding result (see [1]): $H^{1,q}(\Omega) \subset L^\infty$ as well as the norm inequalities that correspond to this imbedding.

In our proofs, C denotes a generic constant, depending on certain quantities. We shall make this dependence explicit, for example, writing $C(\Omega)$ if C depends only on Ω (for example, in the Sobolev inequalities) or $C(\Omega, \mathbf{U}, C_0)$ if C depends on $\Omega, \mathbf{U}$, and C_0, and so on.

2. Finite element method

We start this section by citing the following result from [**5**, Theorem 2.1].

Theorem 1. *Let $2 < q < 3$. Suppose that $U \geqslant C_0 > 0$ for some constant C_0. Assume that μ is large enough. Then there is a unique solution $[\mathbf{u}, p] \in H^{2,q}_0 \times H^{1,q}$ of (1). Also the solution $[\mathbf{u}, p]$ satisfies*

$$\|\mathbf{u}\|_{2,q} + \|p\|_{1,q} \leqslant C\Big(\|\mathbf{f}\|_{0,q} + \|g\|_{1,q}\Big),$$

where $C = C\Big(\Omega, \|\mathbf{U}\|_{2,q}, C_0, (\mu + \nu)/\mu, \mathbf{U}\Big)$ and $g = 0$.

We let $\mathbf{V} = H^1_0(\Omega) \times H^1_0(\Omega)$ and $M = L^2(\Omega)$, and consider a real-valued function δ defined by $\delta(y) = -\sqrt{1 - y^2}$ $(-1 \leqslant y \leqslant 1)$. We define a linear map $T : \mathbf{V} \longrightarrow R$ by

$$(2) \qquad\qquad T\mathbf{u}(x,y) = \int_{\delta(y)}^x \nabla \cdot \mathbf{u}(s,y)\, ds.$$

An easy computation shows that $\|T\mathbf{u}\|_0 \leqslant \|\nabla\mathbf{u}\|_0$. With the observations described in the previous section, the domain Ω to be considered will be a unit circle with centre at the origin, which can be described in this way: $\Omega = \{(x,y) \in R^2 : x^2 + y^2 < 1\}$, $\Gamma = \{(x,y) \in R^2 : x^2 + y^2 = 1\}$, and $\Gamma_{\text{in}} = \{(\delta(y), y) : \delta(y) = -\sqrt{(1-y^2)}, -1 \leqslant y \leqslant 1\}$.

Hence using (2) with $U = 1$ and $V = 0$, the system (1) becomes

$$-\mu\Delta\mathbf{u} - \nu\nabla\,\mathrm{div}\,\mathbf{u} + (\mathbf{U}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega,$$

(3)
$$p + T\mathbf{u} = 0 \quad \text{in } \Omega,$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma.$$

Notice that (2) implies $p(\delta(y), y) = 0$. In order to formulate the problem (3) in a discrete version, several bilinear forms are considered. We define the bilinear form a by

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \left\{ \mu\nabla\mathbf{u}\cdot\nabla\mathbf{v} + \nu\,\mathrm{div}\,\mathbf{u}\,\mathrm{div}\,\mathbf{v} + [(\mathbf{U}\cdot\nabla)\mathbf{u}]\mathbf{v} \right\} dx, \quad \mathbf{u}, \mathbf{v} \in V.$$

We also define the bilinear forms b, c and d by

$$b(\mathbf{v}, p) = -\int_\Omega p\,\mathrm{div}\,\mathbf{v}\,dx, \quad \mathbf{v} \in V,\, p \in M,$$

$$c(p, \eta) = \int_\Omega p\eta\,dx, \quad p, \eta \in M,$$

$$d(\mathbf{u}, \eta) = \int_\Omega T\mathbf{u}\,\eta\,dx, \quad \mathbf{u} \in V,\, \eta \in M.$$

Using these, a weak formulation of (3) is given by: find $[\mathbf{u}, p]$ in $\mathbf{V} \times M$ such that

(4)
$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = \langle \mathbf{f}, \mathbf{v} \rangle, \quad \forall \in \mathbf{V},$$

$$c(p, \eta) + d(\mathbf{u}, \eta) = 0, \quad \forall\eta \in M.$$

This will be called the streamline weak formulation.

It is assumed that there is a triangulation $T_h = \{K\}$, which is a standard finite element subdivision of Ω into nonoverlapping triangles K of diameter h_K. Here, $h = \max_K h_K$. Let $P(K)$ be the space of linear functions on K. Let $\mathbf{V}_h$ and M_h be defined by

(5)
$$\mathbf{V}_h = \left\{ \mathbf{u}_h \in \mathbf{V} : \mathbf{u}_h \text{ is continuous on } \Omega,\ \mathbf{u}_h|_K \in (P(K))^2, \forall K \in T_h \right\},$$

$$M_h = \left\{ p_h \in M : p_h \text{ is continuous on } \Omega,\ p_h|_K \in P(K), \forall K \in T_h \right\}.$$

Let Ω_h be a given polygonal approximation of Ω. Let $(\delta_h(y), y)$, $-1 \leqslant y \leqslant 1$, describe the incoming portion of the boundary Γ_h of Ω_h. Notice that $\delta_h(y)$ is a piecewise linear approximation of $\delta(y) = -\sqrt{1 - y^2}$ for $-1 \leqslant y \leqslant 1$. For $\mathbf{u}_h \in \mathbf{V}_h$, define

$$T_h\mathbf{u}_h(x, y) = \int_{\delta_h(y)}^x \nabla\cdot\mathbf{u}_h(s, y)\,ds.$$

Hence $T_h\mathbf{u}_h \in M_h$. We define a difference between Γ_{in} and $\Gamma_{h,\mathrm{in}}$, the incoming portion of Γ_h:

$$E_{\Gamma_{\mathrm{in}}}(y) = |\delta_h(y) - \delta(y)| \left(1 - y^2\right)^{1/4}, \quad -1 \leqslant y \leqslant 1.$$

If f is a function of only y, we denote its L^2 norm by $\|f\|_{L^2[-1,1]} \equiv \left\{ \int_{-1}^1 |f(y)|^2\,dy \right\}^{1/2}$.

LEMMA 2.1. *Let $q > 2$. Then $\|(T - T_h)\mathbf{u}\|_0 \leqslant \sqrt{2}\,\|\nabla\mathbf{u}\|_{1,q}\,\|E_{\Gamma_{\text{in}}}\|_{L^2[-1,1]}$.*

PROOF: Using the Sobolev inequality $H^{1,q}(\Omega) \subset L^\infty$,

$$
\begin{aligned}
|(T - T_h)\mathbf{u}|^2 &= \left| \int_{\delta(y)}^{\delta_h(y)} \nabla \cdot \mathbf{u}(s,y)\,ds \right|^2 \\
&\leqslant \|\nabla\mathbf{u}\|_{1,q}^2 \, |\delta_h(y) - \delta(y)|^2 \, .
\end{aligned}
$$

Hence the result follows easily.   □

Let $\{(x_i, y_i) : x_i^2 + y_i^2 = 1,\ 0 \leqslant i \leqslant n+1\}$ be the set of the nodal points of the incoming portion $\Gamma_{h,\text{in}}$ for a given polygonal approximation Ω_h of Ω, where $x_0 = 0$, $y_0 = -1$, and $x_{n+1} = 0$, $y_{n+1} = 1$.

LEMMA 2.2. *Let $\overline{x} = \max\{|x_1|, |x_n|\}$. Then $\|E_{\Gamma_{\text{in}}}\|_{L^2[-1,1]} \leqslant 2\sqrt{2}\,\overline{x}^{2+1/2} + Ch$, where C is independent of h.*

PROOF: We have that

$$
\int_{-1}^{1} E_{\Gamma_{\text{in}}}(y)^2\,dy = \int_{y_0}^{y_1} E_{\Gamma_{\text{in}}}(y)^2\,dy + \int_{y_n}^{y_{n+1}} E_{\Gamma_{\text{in}}}(y)^2\,dy + \sum_{i=2}^{n} \int_{y_{i-1}}^{y_i} E_{\Gamma_{\text{in}}}(y)^2\,dy.
$$

The second term of the right hand side of the above equation is bounded by

$$
\begin{aligned}
\int_{y_n}^{y_{n+1}} E_{\Gamma_{\text{in}}}(y)^2\,dy &\leqslant 4 \int_{y_n}^{1} |\delta(y)|^3\,dy \\
&\leqslant 4\,|\delta(y_n)|^3\,(1 - y_n) \\
&\leqslant 4\,|\delta(y_n)|^5 = 4\,|x_n|^5\,,
\end{aligned}
$$

where we used $1 - y_n \leqslant 1 - y_n^2 = x_n^2$ in the last inequality. Similiarly, the first term of the right hand side of the above equation is bounded by $4\,|x_1|^5$. In order to compute the third term of the above equation, we have, using Lagrange's interpolation formula, $|\delta(y) - \delta_h(y)| = (1/2)\,(y_i - y)(y - y_{i-1})\big(1 - \xi_y^2\big)^{-3/2}$, $y_{i-1} \leqslant \xi_y \leqslant y_i$, $(i = 2, 3, \ldots, n)$. So

$$
\begin{aligned}
\sum_{i=2}^{n} \int_{y_{i-1}}^{y_i} E_{\Gamma_{\text{in}}}(y)^2\,dy &= \frac{1}{4} \sum_{i=2}^{n} \int_{y_{i-1}}^{y_i} (y - y_{i-1})^2 (y - y_i)^2 \frac{1}{\big(1 - \xi_y^2\big)^3} \sqrt{1 - y^2}\,dy \\
&\leqslant Ch^2 \sum_{i=2}^{n} \int_{y_{i-1}}^{y_i} \big(1 - y^2\big)^{-1/2}\,dy \\
&\leqslant Ch^2,
\end{aligned}
$$

where C is independent of h. This finishes our proof.   □

The discrete bilinear form of "d" is defined on $\mathbf{V}_h \times M_h$ in the following way: $d_h(\mathbf{u}_h, \eta) = \langle T_h \mathbf{u}_h, \eta \rangle$, $\eta \in M_h$. Then the finite element method corresponding to (4) is given by: find $[\mathbf{u}_h, p_h] \in \mathbf{V}_h \times M_h$ such that

$$
\begin{aligned}
a(\mathbf{u}_h, \mathbf{v}) + b(\mathbf{v}, p_h) &= \langle \mathbf{f}, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in \mathbf{V}_h, \\
c(p_h, \eta) + d_h(\mathbf{u}_h, \eta) &= 0, \qquad \forall \eta \in M_h.
\end{aligned}
$$

(6)

Since the definition of T_h includes an integration, the finite element matrix coming from (6) is not sparse. It also is easily proven that from $(6)_b$, $p_h(x,y) + T_h \mathbf{u}_h(x,y) = 0$ on Ω.

LEMMA 2.3. *If μ is sufficiently large, then there is a unique solution $[\mathbf{u}_h, p_h] \in \mathbf{V}_h \times M_h$ of (6). Also, there is a constant C, not depending on h, such that if $[\mathbf{u}_h, p_h]$ satisfies (6), then*

$$
\|\mathbf{u}_h\|_1 + \|p_h\|_0 \leqslant C \|\mathbf{f}\|_0 .
$$

(7)

PROOF: Let $\eta = p_h \in M_h$ in $(6)_b$. So we have $c(p_h, p_h) + d_h(\mathbf{u}_h, p_h) = 0$ and easily get $\|p_h\|_0 \leqslant \|\nabla \mathbf{u}_h\|_0$. Next, let $\mathbf{v} = \mathbf{u}_h \in \mathbf{V}_h$ in $(6)_a$. Then $a(\mathbf{u}_h, \mathbf{u}_h) + b(\mathbf{u}_h, p_h) = \langle \mathbf{f}, \mathbf{u}_h \rangle$ and we get easily $\mu_0 \|\nabla \mathbf{u}_h\|_0^2 \leqslant C \|\nabla \mathbf{u}_h\|_0 \|\mathbf{u}_h\|_0 + \|p_h\|_0 \|\nabla \mathbf{u}_h\|_0 + \|\mathbf{f}\|_0 \|\mathbf{u}_h\|_0$, where $C = C(|\mathbf{U}|_\infty)$, $\mu_0 = \min\{\mu, \mu + \nu\}$. Finally, choosing a Poincaré constant $\overline{C}$ such that $\|\mathbf{u}_h\|_0 \leqslant \overline{C} \|\nabla \mathbf{u}_h\|_0$, $\mathbf{u}_h \in \mathbf{V}_h$, and assuming that μ is large enough, we have $\|\nabla \mathbf{u}_h\|_0 \leqslant C \|\mathbf{f}\|_0$, $C = C(|\mathbf{U}|_\infty, \overline{C}, \mu_0)$. Thus, we obtain (7). A similiar argument shows the existence and uniqueness of the solution of (6). □

Let $\widehat{\mathbf{u}} \in V_h$ be an approximation to $\mathbf{u}$, and let

$$
\widehat{p}(x,y) = -T_h \widehat{\mathbf{u}}(x,y).
$$

(8)

REMARK. One need a requirement on the triangulation "no triangle has a horizontal side" in establishing that $\widehat{p} \in M_h$. However as will be seen this requirement is not necessary to obtain an error estimate for our finite element method. With this requirement, one easily see that the function $\widehat{p}$ belongs to M_h.

We now give an error analysis of the "streamline" finite element method. Let $[\mathbf{u}, p]$ be the true solution of (4) and let $[\mathbf{u}_h, p_h]$ be the finite element solution of (6).

THEOREM 2.4. *Let $q > 2$. If μ is sufficiently large, then there is a constant K depending on μ, ν, and $\mathbf{U}$ such that*

$$
\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \leqslant K \left\{ \inf \left[\|\nabla(\mathbf{u} - \widehat{\mathbf{u}})\|_0 \right] + \|\nabla \mathbf{u}\|_{1,q} \left\| E_{\Gamma_{\text{in}}} \right\|_{L^2[-1,1]} \right\},
$$

(9)

where the infimum is taken over all $\widehat{\mathbf{u}} \in \mathbf{V}_h$.

PROOF: Suppose first that the triangulation is such that no triangle has a horizontal side. Let $\widehat{\mathbf{u}} \in \mathbf{V}_h$, and let $\widehat{p} = -T_h \widehat{\mathbf{u}}$. By the assumption on the triangulation,

$\widehat{p} \in M_h$. Using (4) and (6), we have $a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) + b(\mathbf{v}, p - p_h) = 0$, $\quad \forall \mathbf{v} \in \mathbf{V}_h$ and $c(p - p_h, \eta) + d(\mathbf{u}, \eta) - d_h(\mathbf{u}_h, \eta) = 0$, $\quad \forall \eta \in M_h$. Hence one has

$$
\begin{aligned}
(10) \quad & a(\mathbf{u}_h - \widehat{\mathbf{u}}, \mathbf{v}) + b(\mathbf{v}, p_h - \widehat{p}) = a(\mathbf{u} - \widehat{\mathbf{u}}, \mathbf{v}) + b(\mathbf{v}, p - \widehat{p}), && \forall \mathbf{v} \in \mathbf{V}_h, \\
& c(p_h - \widehat{p}, \eta) + d_h(\mathbf{u}_h - \widehat{\mathbf{u}}, \eta) = c(p - \widehat{p}, \eta) + d(\mathbf{u}, \eta) - d_h(\widehat{\mathbf{u}}, \eta), && \forall \eta \in M_h.
\end{aligned}
$$

Picking $\eta = p_h - \widehat{p} \in M_h$ in $(10)_b$ and taking $d_h(\mathbf{u}_h - \widehat{\mathbf{u}}, p_h - \widehat{p})$ to the right hand side,

$$
\begin{aligned}
(11) \quad \|p_h - \widehat{p}\|_0 &\leqslant \|p - \widehat{p}\|_0 + \|T\mathbf{u} - T_h \widehat{\mathbf{u}}\|_0 + \|T_h \mathbf{u}_h - T_h \widehat{\mathbf{u}}\|_0 \\
&\leqslant C_1(\|\nabla(\mathbf{u}_h - \widehat{\mathbf{u}})\|_0 + \|\nabla(\mathbf{u} - \widehat{\mathbf{u}})\|_0 + \|p - \widehat{p}\|_0) \\
&\quad + C_1 \|(T - T_h)\mathbf{u}\|_0.
\end{aligned}
$$

The constant C_1 depends on $\|T_h\|$. Since the polygonal domain is contained in Ω, one see that $\|T_h\| \leqslant \|T\|$. Since $\|T\|$ depends only on Ω, one concludes that the constant C_1 in (11) depends only on Ω. Next pick $\mathbf{v} = \mathbf{u}_h - \widehat{\mathbf{u}} \in \mathbf{V}_h$ in $(10)_a$. So

$$
\begin{aligned}
(12) \quad a(\mathbf{u} - \widehat{\mathbf{u}}, \mathbf{u}_h - \widehat{\mathbf{u}}) + b(\mathbf{u}_h - \widehat{\mathbf{u}}, p - \widehat{p}) &\leqslant (\mu + |\nu|) \|\nabla(\mathbf{u} - \widehat{\mathbf{u}})\|_0 \|\nabla(\mathbf{u}_h - \widehat{\mathbf{u}})\|_0 \\
&\quad + C \|\nabla(\mathbf{u} - \widehat{\mathbf{u}})\|_0 \|\mathbf{u}_h - \widehat{\mathbf{u}}\|_0 \\
&\quad + \|p - \widehat{p}\|_0 \|\nabla(\mathbf{u}_h - \widehat{\mathbf{u}})\|_0,
\end{aligned}
$$

where $C = C(|\nabla \mathbf{U}|_\infty)$. Choose a Poincaré constant $\overline{C}$ such that $\overline{C} \|\mathbf{u}_h - \widehat{\mathbf{u}}\|_0 \leqslant \|\nabla(\mathbf{u}_h - \widehat{\mathbf{u}})\|_0$. Thus, using (12),

$$
(13) \quad a(\mathbf{u} - \widehat{\mathbf{u}}, \mathbf{u}_h - \widehat{\mathbf{u}}) + b(\mathbf{u}_h - \widehat{\mathbf{u}}, p - \widehat{p}) \leqslant C(\|\nabla(\mathbf{u} - \widehat{\mathbf{u}})\|_0 + \|p - \widehat{p}\|_0) \|\nabla(\mathbf{u}_h - \widehat{\mathbf{u}})\|_0,
$$

where $C = C\left(\mu, \nu, \overline{C}, \|\mathbf{U}\|_{1,\infty}\right)$. Furthermore we have

$$
\begin{aligned}
(14) \quad a(\mathbf{u}_h - \widehat{\mathbf{u}}, \mathbf{u}_h - \widehat{\mathbf{u}}) + b(\mathbf{u}_h - \widehat{\mathbf{u}}, p_h - \widehat{p}) &\geqslant \mu_0 \|\nabla(\mathbf{u}_h - \widehat{\mathbf{u}})\|_0^2 - \frac{1}{2} |\operatorname{div} \mathbf{U}|_\infty \|\mathbf{u}_h - \widehat{\mathbf{u}}\|_0^2 \\
&\quad - \|p_h - \widehat{p}\|_0 \|\nabla(\mathbf{u}_h - \widehat{\mathbf{u}})\|_0,
\end{aligned}
$$

where $\mu_0 = \min\{\mu, \mu + \nu\}$. Next, using (11), we have

$$
\begin{aligned}
(15) \quad a(\mathbf{u}_h - \widehat{\mathbf{u}}, \mathbf{u}_h - \widehat{\mathbf{u}}) + b(\mathbf{u}_h - \widehat{\mathbf{u}}, p_h - \widehat{p}) &\geqslant (M_0 - C_1) \|\nabla(\mathbf{u}_h - \widehat{\mathbf{u}})\|_0^2 \\
&\quad - C_1 \|\nabla(\mathbf{u}_h - \widehat{\mathbf{u}})\|_0 (\|\nabla(\mathbf{u} - \widehat{\mathbf{u}})\|_0 + \|p - \widehat{p}\|_0 + \|(T - T_h)\mathbf{u}\|_0),
\end{aligned}
$$

where $M_0 = \mu_0 - (1/2)\overline{C}^{-1} |\operatorname{div} \mathbf{U}|_\infty$ and C_1 is the constant in (11). Combining (11), (13) and (15), we get

$$
(16) \quad (M_0 - C_1) \|\nabla(\mathbf{u}_h - \widehat{\mathbf{u}})\|_0 \leqslant C(\|\nabla(\mathbf{u} - \widehat{\mathbf{u}})\|_0 + \|p - \widehat{p}\|_0 + \|(T - T_h)\mathbf{u}\|_0),
$$

where $C = C\Big(\Omega, \mu, \nu, \overline{C}, \|\mathbf{U}\|_{1,\infty}\Big)$. The constant $M_0 - C_1$ is positive for a sufficiently large μ. Finally using (11) and (16), we obtain

$$(17) \quad \|\nabla(\mathbf{u}_h - \widehat{\mathbf{u}})\|_0 + \|p_h - \widehat{p}\|_0 \leqslant C_2\Big(\|\nabla(\mathbf{u} - \widehat{\mathbf{u}})\|_0 + \|p - \widehat{p}\|_0 + \|(T - T_h)\mathbf{u}\|_0 \Big),$$

where $C_2 = C\Big(\Omega, \mu, \nu, \overline{C}, \|\mathbf{U}\|_{1,\infty}, M_0\Big)$. Thus, using the triangle inequality, we have

$$(18) \quad \|\nabla(\mathbf{u} - \mathbf{u}_h)\|_0 + \|p - p_h\|_0 \leqslant C_2\Big(\|\nabla(\mathbf{u} - \widehat{\mathbf{u}})\|_0 + \|p - \widehat{p}\|_0 + \|(T - T_h)\mathbf{u}\|_0 \Big).$$

Since $\|p - \widehat{p}\|_0 \leqslant C\Big(\|\nabla(\mathbf{u} - \widehat{\mathbf{u}})\|_0 + \|(T - T_h)\mathbf{u}\|_0 \Big)$, we have

$$(19) \quad \|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \leqslant C_2\Big(\|\nabla(\mathbf{u} - \widehat{\mathbf{u}})\|_0 + \|(T - T_h)\mathbf{u}\|_0 \Big).$$

Therefore we conclude that

$$(20) \quad \|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \leqslant C_2\Big\{ \inf\big[\|\nabla(\mathbf{u} - \widehat{\mathbf{u}})\|_0 \big] + \|(T - T_h)\mathbf{u}\|_0 \Big\},$$

where $C_2 = C\Big(\Omega, \mu, \nu, \overline{C}, \|\mathbf{U}\|_{1,\infty}, M_0\Big)$, and the infimum is taken over all $\widehat{\mathbf{u}} \in \mathbf{V}_h$. Thus, using Lemma 2.1, we can finish our proof in the case that no triangle in the triangulation has a horizontal side. Since the constant K in (9) has been shown to be independent of the triangulation, the result holds for any triangulation by a limiting argument. $\quad\square$

We next use the following approximation property to give a corollary to the theorem.

$$(21) \quad \inf_{\mathbf{v} \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}\|_1 \leqslant Ch^k \|\mathbf{u}\|_{k+1}, \quad \forall \mathbf{u} \in \mathbf{V} \cap \big(H^{k+1}(\Omega)\big)^2,$$

where C is a constant independent of h.

COROLLARY 2.5. *Let $q > 2$. Assume that the conditions in Theorem 2.4 hold. Then there exists a constant K, depending on μ, ν and $\mathbf{U}$ such that the finite element solution $[\mathbf{u}_h, p_h]$ of (6) satisfies*

$$(22) \quad \|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \leqslant Kh \|\mathbf{u}\|_{2,q}.$$

PROOF: Theorem 2.4 gives $\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \leqslant K\Big(h + \big\|E_{\Gamma_{\text{in}}}\big\|_{L^2[-1,1]} \Big) \|\mathbf{u}\|_{2,q}$. Hence the inequality (22) follows easily by Lemma 2.2. $\quad\square$

We note that with a large condition on μ, the inequality $\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \leqslant Kh$ for a positive constant K can be obtained easily.

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