J. Aust. Math. Soc. **79** (2005), 361–368

COHOMOLOGICAL CHARACTERIZATION OF THE HILBERT SYMBOL OVER $\mathbb{Q}_p^*$

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(Received 11 February 2004; revised 15 November 2004)

Communicated by K. F. Lai

Abstract

The aim of this work is to offer a new characterization of the Hilbert symbol over $\mathbb{Q}_p^*$ from the commutator of a certain central extension of groups. We obtain a characterization for $\mathbb{Q}_p^*$ ($p \neq 2$) and a different one for $\mathbb{Q}_2^*$.

2000 *Mathematics subject classification*: primary 19F15, 20F12.

1. Introduction

In recent years, characterizations of algebraic symbols have been obtained from the properties of infinite-dimensional vector spaces in order to provide new interpretations for these symbols and to deduce standard theorems from the new definitions in an easy way.

Thus, in 1968 Tate [7] gave a definition of the residues of differentials on curves in terms of traces of certain linear operators on infinite-dimensional vector spaces.

A few years later, in 1989, Arbarello, de Concini and Kac [1] obtained a definition of the tame symbol of an algebraic curve from the commutator of a certain central extension of group. More recently, the author has given an interpretation of this central extension in terms of determinants associated with infinite-dimensional vector subspaces [3], and has defined the Parshin symbol on a surface as iterated tame symbols [4].

This work is partially supported by the DGESYC research contract no. BFM2003-00078 and Castilla y León Regional Government contract SA071/04.

https://doi.org/10.1017/S1446788700010958 Published online by Cambridge University Press

In all articles referred to above, the respective reciprocity laws (in particular, the residue theorem) are deduced directly from the finiteness of the cohomology groups $H^0(C, \mathcal{O}_C)$ and $H^1(C, \mathcal{O}_C)$.

The purpose of the present work is to give a new characterization of the Hilbert symbol over $\mathbb{Q}_p^*$ by using the method described in [1] and [3]. This definition, which involves topics of Steinberg symbols, allows us to use the results of [1, 2, 3] to study the properties of this symbol. A remaining problem is to obtain a new proof of the Gauss Reciprocity Law from the statements of this characterization.

Similarly to the computation of the symbol ([6, page 20]), we obtain a characterization for $\mathbb{Q}_p^*$ with $p \neq 2$ and a different one for $\mathbb{Q}_2^*$.

For a detailed study of p-adic fields and the Hilbert symbol, the reader is referred to [6].

2. Preliminaries

This section is added for the sake of completeness.

2.1. Definition of the Hilbert symbol If k denotes either the field $\mathbb{R}$ of real numbers or the field $\mathbb{Q}_p$ of p-adic numbers (p being a prime number), Serre [6] defines the Hilbert symbol $(\cdot, \cdot)_k : k^* \times k^* \to \mu_2$ as:

$$(a, b)_k = \begin{cases} 1 & \text{if } z^2 - ax^2 - by^2 = 0 \text{ has a non-trivial solution in } k^3; \\ -1 & \text{otherwise,} \end{cases}$$

where $a, b \in k^*$ and $\mu_2 = \{1, -1\}$.

The Hilbert symbol is a Steinberg symbol ([2, page 94]) because it is bimultiplicative and satisfies $(a, 1 - a)_k = 1$. Moreover, $(a, -a)_k = 1$ and $(a, b)_k = (b, a)_k$.

If $k = \mathbb{Q}_p$, we shall write $(a, b)_p = (a, b)_k$.

It is known that $\mathbb{Q}_p^* \simeq \mathbb{Z} \times \mathcal{U}^p$, where $\mathcal{U}^p$ is the group of p-adic units. Hence, if v_p denotes the p-adic valuation, each element $a \in \mathbb{Q}_p^*$ can be written uniquely in the form $a = p^\alpha u$, with $\alpha = v_p(a)$ and $u \in \mathcal{U}^p$.

Moreover, if we denote by $\mathcal{U}^2$ the group of units of $\mathbb{Z}_2$, we can define the morphisms of groups $\epsilon, \omega : \mathcal{U}^2 \to \mathbb{Z}/2$ as follows:

$$\epsilon(u) = \frac{u-1}{2} \pmod 2 = \begin{cases} 0 & \text{if } u \equiv 1 \pmod 4; \\ 1 & \text{if } u \equiv -1 \pmod 4, \end{cases}$$

$$\omega(u) = \frac{u^2-1}{8} \pmod 2 = \begin{cases} 0 & \text{if } u \equiv \pm 1 \pmod 8; \\ 1 & \text{if } u \equiv \pm 5 \pmod 8. \end{cases}$$

Setting $\mathcal{U}_n^p = 1 + p^n\mathbb{Z}_p$, ϵ and ω determine isomorphisms of groups

$$\epsilon : \mathcal{U}^2/\mathcal{U}_2^2 \xrightarrow{\sim} \mathbb{Z}/2 \quad \text{and} \quad \omega : \mathcal{U}_2^2/\mathcal{U}_3^2 \xrightarrow{\sim} \mathbb{Z}/2.$$

Furthermore, $\mathbb{Q}_2^*/(\mathbb{Q}_2^*)^2 \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ and each element $a = 2^\alpha u \in \mathbb{Q}_2^*$ can be written in the form $(-1)^{\epsilon(u)}2^\alpha 5^{\omega(u)}\bar{u}$, where $\bar{u} \in (\mathbb{Q}_2^*)^2$.

There exist also isomorphisms of groups $\phi : \mathcal{U}^p/\mathcal{U}_1^p \xrightarrow{\sim} (\mathbb{Z}/p)^*$, and if $f \in \mathcal{U}^p$, we shall write $f(p) = \phi(\bar{f}) \in (\mathbb{Z}/p)^*$.

Thus, given $a, b \in \mathbb{Q}_p^*$, $a = p^\alpha u$ and $b = p^\beta v$, the value of the Hilbert symbol is:

$$(a, b)_p = (-1)^{\alpha\beta\epsilon(p)} \left(\frac{u^\beta}{v^\alpha}(p)\right)^{(p-1)/2} \quad \text{if } p \neq 2,$$
$$(a, b)_2 = (-1)^{\epsilon(u)\epsilon(v)+\alpha\omega(v)+\beta\omega(u)}.$$

2.2. The group $\mathrm{Gl}(V, A)$ and its canonical central extension Let V be a vector space over a field k (in general infinite-dimensional) and let A be a vector subspace of V. With the same notation as in [1] we set

$$\mathrm{Gl}(V, A) = \{f \in \mathrm{Aut}(V) \text{ such that } fA \sim A\},$$

where $f \sim fA$ when $\dim_k(A + fA/A \cap fA) < \infty$, which is the definition of commensurable subspaces of Tate [7].

If $f \in \mathrm{Gl}(V, A)$, we set $(A|fA) = \Lambda(A/A \cap fA)^* \bigotimes_k \Lambda(fA/A \cap fA)$, Λ being the maximal exterior power. Canonically, Arbarello, de Concini and Kac [1] defined a group $\widetilde{\mathrm{Gl}}(V, A) = \{(f, s) \text{ with } f \in \mathrm{Gl}(V, A) \text{ and } s \in (A|fA), s \neq 0\}$, which induces a central extension:

$$1 \to k^* \to \widetilde{\mathrm{Gl}}(V, A) \to \mathrm{Gl}(V, A) \to 1.$$

Let us set $A = k[[t]]$ and $V = k((t))$. Since $k((t))^* \subseteq \mathrm{Gl}(V, A)$, if we denote by $\{\cdot, \cdot\}_A$ the commutator of the above extension and we consider two elements, $f, g \in k((t))^*$ with $f = \lambda t^\alpha\left(1 + \sum_{i\geq 1} a_i t^i\right)$, and $g = \mu t^\beta\left(1 + \sum_{j\geq 1} b_j t^j\right)$, where $\lambda, \beta \in k^*$ and $\alpha, \beta \in \mathbb{Z}$, we have that

$$\{f, g\}_A = \frac{\lambda^\beta}{\mu^\alpha} \in k^*.$$

This computation of the commutator can also be found in [3].

2.3. Steinberg symbols For any field F, a bimultiplicative mapping $c : F^* \times F^* \to A$ to an abelian group, satisfying $c(x, 1 - x) = 1$ for $x \neq 1$, is called a 'Steinberg symbol' on the field F.

https://doi.org/10.1017/S1446788700010958 Published online by Cambridge University Press

If F_v is a discrete valuation field, $\mathcal{O}_v$ is the valuation ring, $\mathfrak{m}_v$ is the unique maximal ideal and $k(v) = \mathcal{O}_v/\mathfrak{m}_v$ is the residue class field, the tame symbol

$$\begin{aligned} d_v : F_v^* \times F_v^* &\to k(v)^* \\ (x, y) &\mapsto (-1)^{v(x)v(y)} \frac{x^{v(y)}}{y^{v(x)}} \pmod{\mathfrak{m}_v} \end{aligned}$$

is an easy example of a Steinberg symbol on F_v^*.

For a detailed study of Steinberg symbols, we refer the reader to [2].

3. Characterization of the Hilbert symbol

Let us now consider $k = \mathbb{Q}_p$, $A_p = \mathbb{Q}_p[[t]]$ and $V_p = \mathbb{Q}_p((t))$.

Since each element $f \in \mathbb{Q}_p^*$ can be written uniquely in the form $a = p^{v_p(f)}u$ with $u \in \mathcal{U}^p$, we can consider the injective group morphism

$$\begin{aligned} \varphi : \mathbb{Q}_p^* &\hookrightarrow \mathbb{Q}_p((t))^*, \\ p^\alpha u &\mapsto ut^\alpha, \end{aligned}$$

and we deduce that $\mathbb{Q}_p^*$ is a commutative subgroup of $\mathrm{Gl}(V_p, A_p)$ by considering the homotheties $h_{\varphi(f)}$. Thus the commutator of the following central extension of groups

$$1 \to \mathbb{Q}_p^* \to \widetilde{\mathrm{Gl}}(V_p, A_p) \to \mathrm{Gl}(V_p, A_p) \to 1$$

determines a 2-cocycle $\{\cdot, \cdot\}_p : \mathbb{Q}_p^* \times \mathbb{Q}_p^* \to \mathcal{U}^p$ whose value is $\{f, g\}_p = u^\beta/v^\alpha$, where $f = p^\alpha u$ and $g = p^\beta v$.

3.1. Hilbert symbol over $\mathbb{Q}_p^*$ ($p \neq 2$) From the morphism of groups $\psi_p : \mathcal{U}^p \to \mu_2$ defined as $\psi_p(u) = (u(p))^{(p-1)/2}$ we have a 2-cocycle $\widetilde{\{\cdot, \cdot\}}_p : \mathbb{Q}_p^* \times \mathbb{Q}_p^* \to \mu_2$, whose value is

$$\widetilde{\{f, g\}}_p = \psi_p(\{f, g\}_p) = \left(\frac{u^\beta}{v^\alpha}(p)\right)^{(p-1)/2}.$$

In general , one has that $\widetilde{\{\cdot, \cdot\}}_p$ is not a Steinberg symbol because

$$\{p^{-1}, \widetilde{1 - p^{-1}}\}_p = -1$$

when $p \equiv 3 \pmod 4$.

We shall now give a cohomological definition of the Hilbert symbol as a distinguished element in the cohomology class $[\widetilde{\{\cdot, \cdot\}}_p] \in H^2(\mathbb{Q}_p^*, \mu_2)$, where $H^2(A, B)$ is the group of classes of 2-cocycles $f : A \times A \to B$ (mod 2-coboundaries) [5].

LEMMA 3.1. *For each $a \in \mathbb{Z}$, there exists a unique 2-coboundary $c_a : \mathbb{Z} \times \mathbb{Z} \to \mu_2$ satisfying the conditions*:

- $c_a(\alpha, \beta + \gamma) = c_a(\alpha, \beta)c_a(\alpha, \gamma)$;
- $c_a(\alpha, \alpha) = (-1)^{\alpha a}$

for $\alpha, \beta, \gamma \in \mathbb{Z}$.

PROOF. Recall that a 2-cocycle $c_a : \mathbb{Z}\times\mathbb{Z} \to \mu_2$ is a 2-coboundary when there exists a map $\phi: \mathbb{Z} \to \mu_2$ such that $c_a(\alpha, \beta) = \phi(\alpha+\beta)\phi(\alpha)^{-1}\phi(\beta)^{-1}$. Let $\phi(\alpha) = \lambda_\alpha \in \mu_2$. It follows from the conditions of the lemma that

$$\lambda_\alpha = (-1)^{\alpha(\alpha-1)a/2}\lambda_1^\alpha \quad \text{for each } \alpha \in \mathbb{Z}.$$

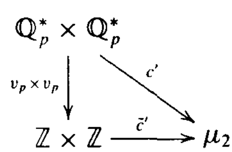
Hence $c_a(\alpha, \beta) = (-1)^{\alpha\beta a}$ is the unique 2-coboundary that satisfies the statement of the lemma. □

THEOREM 3.2. *There exists a unique Steinberg symbol $(\cdot, \cdot)_p$ in the cohomology class $[\widetilde{\{\cdot, \cdot\}}_p] \in H^2(\mathbb{Q}_p^*, \mu_2)$ satisfying the condition*

$$(f, g)_p = \widetilde{\{f, g\}}_p \quad \textit{if } v_p(f) = 0.$$

This Steinberg symbol is the Hilbert symbol over $\mathbb{Q}_p^*$.

PROOF. Let $\nu(f, g) = c'(f, g)\widetilde{\{f, g\}}_p$ be a Steinberg symbol in the cohomology class $[\widetilde{\{\cdot, \cdot\}}_p] \in H^2(\mathbb{Q}_p^*, \mu_2)$ such that $c'(f, g) = 1$ for $v_p(f) = 0$. Since c' is a 2-coboundary, one has that $c'(f, g) = 1$ when $v_p(g) = 0$ and, therefore, there exists a commutative diagram



where $\tilde{c}'$ is a 2-coboundary satisfying $\tilde{c}'(\alpha, \beta + \gamma) = \tilde{c}'(\alpha, \beta)\tilde{c}'(\alpha, \gamma)$.

Furthermore, since $\nu(f, -f) = 1$ for all $f \in \mathbb{Q}_p^*$, one has that $\tilde{c}'(\alpha, \alpha) = (-1)^{\alpha(p-1)/2} = (-1)^{\alpha\epsilon(p)}$. It then follows from Lemma 3.1 that $\tilde{c}'(\alpha, \beta) = (-1)^{\alpha\beta\epsilon(p)}$ and $c'(f, g) = (-1)^{v_p(f)v_p(g)\epsilon(p)}$.

Thus, the unique Steinberg symbol in $[\widetilde{\{\cdot, \cdot\}}_p] \in H^2(\mathbb{Q}_p^*, \mu_2)$ is

$$\nu(f, g) = (-1)^{v_p(f)v_p(g)\epsilon(p)}\widetilde{\{f, g\}}_p,$$

which is the Hilbert symbol. □

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REMARK 3.3. The above property, which characterizes the Hilbert symbol in $\mathbb{Q}_p^*$ is equivalent to one of the conditions that Serre gave to define local symbols on algebraic curves ([5]) which are also Steinberg symbols.

Let us now consider in $\mathbb{Q}_p^*$ the structure of the topological group induced by the p-adic valuation and let us consider μ_2 as a topological group with the discrete topology.

CONJECTURE 3.4. $(\cdot,\cdot)_p$ *is the unique continuous Steinberg symbol in the cohomology class* $[\widetilde{\{\cdot,\cdot\}}_p] \in H^2(\mathbb{Q}_p^*, \mu_2)$.

3.2. Hilbert symbol over $\mathbb{Q}_2^*$ Let us consider the morphism of groups $\psi_2 : \mathcal{U}^2 \to \mu_2$ defined as $\psi_2(u) = (-1)^{\omega(u)}$. This map induces a 2-cocycle $\widetilde{\{\cdot,\cdot\}}_2 : \mathbb{Q}_2^* \times \mathbb{Q}_2^* \to \mu_2$ whose value is

$$\widetilde{\{f,g\}}_2 = \psi_2(\{f,g\}_2) = (-1)^{\beta\omega(u)+\alpha\omega(v)},$$

where $f = 2^\alpha u$ and $g = 2^\beta v$.

Again, $\widetilde{\{\cdot,\cdot\}}_2$ is not a Steinberg symbol because $\widetilde{\{6,-5\}}_2 = -1$. We shall now determine the relation between the commutator $\widetilde{\{\cdot,\cdot\}}_2$ and the Steinberg symbol $(\cdot,\cdot)_2$ in the group of 2-cocycles $Z^2(\mathbb{Q}_2^*, \mu_2)$.

THEOREM 3.5. *There exists a unique 2-cocycle* $c_2 \in Z^2(\mathbb{Q}_2^*, \mu_2)$ *such that* $c_2\widetilde{\{\cdot,\cdot\}}_2$ *is a non-trivial Steinberg symbol. This symbol is the Hilbert symbol* $(\cdot,\cdot)_2$. *Moreover,* c_2 *is not a 2-coboundary and hence* $(\cdot,\cdot)_2 \notin [\widetilde{\{\cdot,\cdot\}}_2] \in H^2(\mathbb{Q}_2^*, \mu_2)$.

PROOF. Let $c_2 \in Z^2(\mathbb{Q}_2^*, \mu_2)$ such that $\nu = c_2\widetilde{\{\cdot,\cdot\}}_2$ is a Steinberg symbol. Since $\widetilde{\{\cdot,\cdot\}}_2$ is a bimultiplicative map, c_2 must be bimultiplicative and one has that

$$c_2(f, 1-f) = \widetilde{\{f, 1-f\}}_2 \quad \text{for } f \neq 1.$$

Moreover, since $\widetilde{\{f,-f\}}_2 = 1$, the condition $c_2(f,-f) = 1$ must be satisfied and it follows from the equality

$$\widetilde{\{f,g\}}_2 = \widetilde{\{g,f\}}_2$$

that c_2 must be a symmetric 2-cocycle.

Furthermore, since $c_2(f^2, g) = c_2(f, g^2) = 1$, we have that c_2 is characterized by its values in -1, 2 and 5, which are the generators of the group $\mathbb{Q}_2^*/(\mathbb{Q}_2^*)^2$.

Bearing in mind the previous considerations, we have that

$$c_2(2,-1) = \widetilde{\{2,-1\}}_2 = 1, \quad c_2(2,2) = c_2(2,-2)c_2(2,-1) = 1.$$

From the equalities

$$c_2(5,-4) = \widetilde{\{5,-4\}}_2 = 1 \quad \text{and} \quad c_2(5,2^2) = 1,$$

we deduce that $c_2(5,-1) = 1$. It is now clear that $c_2(5,5) = 1$.

To conclude, let us first assume that $c_2(-1,-1) = 1$. Then, since $\epsilon(3) = 1$ and $\omega(3) = 1$ we have that $3 = -5u^2$. Moreover, $c_2(6,-5) = \widetilde{\{6,-5\}}_2 = -1$, and thus

$$\begin{aligned} c_2(2,5) &= c_2(2,-5) = -c_2(12,-5) = -c_2(3,-5) \\ &= -c_2(-5,-5) = -c_2(-1,-1) = -1. \end{aligned}$$

Hence if we write $f = (-1)^{\epsilon(u)} 2^\alpha 5^{\omega(u)} \bar{u}^2$ and $g = (-1)^{\epsilon(v)} 2^\beta 5^{\omega(v)} \bar{v}^2$, we have in this case that

$$c_2(f,g) = c_2(2^\alpha, 5^{\omega(v)}) c_2(5^{\omega(u)}, 2^\beta) = (-1)^{\alpha\omega(v)+\beta\omega(u)} = \widetilde{\{f,g\}}_2$$

and $\nu = c_2\widetilde{\{\cdot,\cdot\}}_2 = 1$ is the trivial Steinberg symbol.

Finally, when $c_2(-1,-1) = -1$ we deduce, using a similar argument, that $c_2(2,5) = 1$. Hence, $c_2(f,g) = (-1)^{\epsilon(u)\epsilon(v)}$, and

$$\nu(f,g) = (-1)^{\epsilon(u)\epsilon(v)} \widetilde{\{f,g\}}_2 = (f,g)_2.$$

Furthermore, since $c_2(-1,-1) \neq 1$ we have that c_2 is not a 2-coboundary and we conclude the proof. □

Acknowledgements

The author wishes to thank Professors José Mourao, Joao Nunes and Carlos Florentino at the Instituto Superior Técnico of Lisbon (Portugal) for their hospitality during a three-month period in 2001 when much of the work of this paper was done.

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