Compositio Mathematica **130:** 49–73, 2002. 49

Computations and Stability of the Fukui Invariant

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(Received: 2 November 1999; accepted in final form: 29 September 2000)

Abstract. T. Fukui introduced an invariant for the blow-analytic equivalence of real singularities. For a nondegenerate analytic function (germ) f, he discovered a formula for computing the one-dimensional invariant, denoted by $A(f) := A_1(f)$. We find a formula for $A(f)$ for any f (real or complex, degenerate or not). We then define, and characterise, various notions of stability of $A(f)$, using the formula. For real analytic f, the Fukui invariant with sign is defined, and computed by a similar formula. In the case where f is an analytic function of two complex variables, $A(f)$ can also be computed using the tree-model of f.

Mathematics Subject Classifications (2000). 57R45, 58Kxx.

Key words. Fukui invariant, blow-analyticity, simplification, tree-model.

1. Introduction

Let $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be a given germ of analytic function, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, not identically zero. Take any analytic arc

$$\lambda(t) = (\lambda_1(t), \ldots, \lambda_n(t))\colon (\mathbb{K}, 0) \to (\mathbb{K}^n, 0),$$

where $\lambda_i(t)$ are convergent power series in t, $\lambda_i(0) = 0$. As $f(\lambda(t))$ is a power series in t, its order in t, $\mathrm{O}(f(\lambda(t)))$, is a positive integer, or ∞. We call the set of orders $A(f) = \{\mathrm{O}(f(\lambda(t)))\}$ for all choices of λ, *the Fukui invariant* of f. This was introduced by Fukui [3] as an invariant for the *blow-analytic equivalence* of singularities defined in [8]. In his paper, Fukui actually introduced an invariant, $A_m(f)$, for each positive integer m; but he only gave a formula for computing $A(f) := A_1(f)$, for nondegenerate functions f.

We shall give, in Section 3, a formula for computing $A(f)$, for *any* f, using a simplification (desingularisation) of $f^{-1}(0)$.

The reader is referred to the survey article [4] for more on blow-analyticity.

Let us exclude a trivial case in the outset. Suppose that f is already a normal crossing:

$$f(x_1, \ldots, x_n) = (\text{unit})\ x_1^{e_1} \ldots x_n^{e_n}\ ,\ e_i \geqslant 0.$$

It is easy to see that

$$A(f) = \left\{ \sum m_i e_i \mid m_i \geqslant 1 \right\} \cup \{\infty\}.$$

From now on, we assume that f is not a normal crossing, whence at least one exceptional divisor must appear when f is desingularised.

Let $\mathbb{N}$ denote the set of positive integers.

PROPERTY 1.1. *If $c \in A(f)$ and $k \in \mathbb{N}$, then $kc \in A(f)$.*

This is clear because $\mathrm{O}(f(\lambda(t^k))) = k\mathrm{O}(f(\lambda(t)))$.

It follows from this property that $A(f)$ is an infinite set (unless f is identically zero). Let us write

$$A(f) = \{a_1, a_2, \ldots, a_l, \ldots\} \cup \{\infty\},\ \ a_1 < a_2 < \cdots < a_l < \cdots\ \ (l \in \mathbb{N}).$$

We say that $A(f)$ is *stably periodic* if there exist $c, q, d \in \mathbb{N}$ such that $a_{j+kq} = a_j + kd$ for $c \leqslant j < c + q$ and $k \in \mathbb{N}$. The smallest value of q for which this holds is called *the stable period.*

We say $A(f)$ is *stably interval-like* if there exist $c, d, m \in \mathbb{N}$ such that $a_{m+i} = c + id,\ \ i = 0, 1, 2, \ldots$.

PROPERTY 1.2. *$A(f)$ is stably interval-like if and only if there exist $s, d, m \in \mathbb{N}$ such that $a_{m+i} = (s + i)d,\ \ i = 0, 1, 2, \ldots$.*

This is also clear. By Property 1.1, there exists $s \in \mathbb{N}$ such that $c + sd = 2c$. Hence, $c = sd,\ a_{m+i} = (s + i)d$.

In case $d = 1$, we say $A(f)$ is *stably unit-interval-like.*

EXAMPLE 1.3. Let $f\colon (\mathbb{K}^2, 0) \to (\mathbb{K}, 0)$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, be a polynomial function defined by $f(x, y) = x^4 + y^6$. Then

$$\begin{aligned} A(f) &= \{4, 6, 8, 12, 13, 14, \cdots\} \cup \{\infty\} \text{ in the case } \mathbb{K} = \mathbb{C}, \\ A(f) &= \{4, 6, 8, 12, 16, 18, 20, 24, \ldots\} \cup \{\infty\} \\ &= 4\mathbb{N} \cup 6\mathbb{N} \cup \{\infty\} \text{ in the case } \mathbb{K} = \mathbb{R}. \end{aligned}$$

In both cases, $A(f)$ is stably periodic. In the complex case $A(f)$ is stably unit-interval-like, but in the real case $A(f)$ is not even stably interval-like.

This example gives rise to the following natural questions:

QUESTION 1.4. In either case, is $A(f)$ always stably periodic?

QUESTION 1.5. In the complex case, is $A(f)$ always stably interval-like?

In Section 3, we shall give a formula to compute the Fukui invariant $A(f)$, using a simplification of $f^{-1}(0)$.

Using the formula for $A(f)$ in Section 3, we can answer Question (1.4) in the positive.

However, the answer to Question 1.5 is 'No'. An example is given in the next section. This example leads to the discovery of the characterisation of the stable interval-likeness in Section 5. We can then see why it is easier for the Fukui invariant to be stably interval-like in the complex case than in the real case.

In the real case, we define, in Section 7, some new invariants, which are slightly better than the Fukui invariant. We call them the Fukui invariants with sign. They can be computed by a similar formula (Theorem VII).

In the complex two variables case, we shall give another formula (Section 8) to compute the Fukui invariant, using the tree-model. The notion of tree-model was introduced in [7]. In Sections 9 and 10, we prove this formula.

The Fukui invariant is a kind of dual to the valuation in algebraic geometry. To be more precise, let us consider the 'inner product'

$$\langle g, \lambda \rangle := \mathrm{O}(g(\lambda(t))), \quad \lambda \text{ any arc.}$$

If we take a fixed λ on $f^{-1}(0)$, and vary g in the function field of $f^{-1}(0)$, we get a valuation. But if we take g to be f, and vary λ, we find $A(f)$.

Some interesting observations, in the case of two complex variables, are as follows.

Suppose $f(x, y)$ is irreducible. Let C denote the germ $f^{-1}(0)$, and $I(C, C')$ denote its intersection multiplicity with any other germ C'.

Each analytic arc, λ, can be identified with an irreducible curve germ. Hence

$$A(f) = \{I(C, C') \mid C' \text{ irreducible}\}.$$

On the other hand, it is well known in Algebraic Curves that the semigroup

$$\Gamma(C) := \{I(C, C') \mid C' \text{ any germ}\},$$

which is obviously generated by $A(f)$, has a *conductor*, c, so that, in particular, $\Gamma(C)$ contains all integers $c + i$ for all $i \geqslant 0$. Thus, $\Gamma(C)$ is stably unit-interval-like in our sense. Moreover, $A(f)$ is also unit-interval-like. This follows from our Theorem (III.C).

However, if $f(x, y)$ is not irreducible, then Example (2.1) shows that, in general, $A(f)$ need not be stably interval-like.

2. A Negative Example for Question 1.5

For a positive integer $e \in \mathbb{N}$, let $\mathbb{N}_{\geqslant e} = \{m \in \mathbb{N} \mid m \geqslant e\}$.

EXAMPLE 2.1. Let $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a homogeneous polynomial function defined by

$$f(x, y) = (x - y)^2(x - 2y)^3(x - 3y)^3(x - 4y)^4.$$

Then $A(f) = ((2\mathbb{N} \cup 3\mathbb{N}) \cap \mathbb{N}_{\geqslant 12}) \cup \{\infty\}$. Therefore $A(f)$ is not stably interval-like.

Proof. Let $\lambda\colon (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ be an analytic arc. Then $\lambda(t) = (X(t), Y(t))$ can be expressed in the following way:

$$X(t) = a_0 t^u + a_1 t^{u+1} + \cdots, \quad Y(t) = b_0 t^v + b_1 t^{v+1} + \cdots,$$

where $a_0, b_0 \neq 0$ and $u, v \geqslant 1$. If $X \equiv 0$ (resp. $Y \equiv 0$), let $u = \infty$ (resp. $v = \infty$).

In the case $\lambda \equiv 0$ or $X(t) = kY(t)$, $k = 1, 2, 3, 4$, we have $\mathrm{O}(f \circ \lambda) = \infty$.

We next consider the case where λ is not identically zero and $X(t) \neq kY(t)$, $k = 1, 2, 3, 4$. If $u < v$, then $\mathrm{O}(f \circ \lambda) = 12u$. Therefore $\{\mathrm{O}(f \circ \lambda) \mid \lambda$ with $u < v\} = 12\mathbb{N}$. Similarly we have $\{\mathrm{O}(f \circ \lambda) \mid \lambda$ with $u > v\} = 12\mathbb{N}$. Thus it remains to consider the case $u = v$.

Case I: $a_0 = b_0$. In this case, we have

$$X(t) - Y(t) = c_1 t^{w+1} + c_2 t^{w+2} + \cdots (c_1 \neq 0, w \geqslant u),$$
$$X(t) - kY(t) = d_k t^u + \cdots (d_k \neq 0, k = 2, 3, 4).$$

Therefore $\{\mathrm{O}(f \circ \lambda) \mid \lambda\} = \{12 + 2p \mid p \in \mathbb{N}\}$.

Case II: $a_0 = 2b_0$. Similarly we have $\{\mathrm{O}(f \circ \lambda) \mid \lambda\} = \{12 + 3p \mid p \in \mathbb{N}\}$.
Case III: $a_0 = 3b_0$. We have $\{\mathrm{O}(f \circ \lambda) \mid \lambda\} = \{12 + 3p \mid p \in \mathbb{N}\}$.
Case IV: $a_0 = 4b_0$. We have $\{\mathrm{O}(f \circ \lambda) \mid \lambda\} = \{12 + 4p \mid p \in \mathbb{N}\}$.
Case V: Otherwise. $\{\mathrm{O}(f \circ \lambda) \mid \lambda\} = 12\mathbb{N}$.

It follows that

$$\begin{aligned} A(f) &= 12\mathbb{N} \cup \{12 + 2p \mid p \in \mathbb{N}\} \cup \{12 + 3p \mid p \in \mathbb{N}\} \cup \{\infty\} \\ &= ((2\mathbb{N} \cup 3\mathbb{N}) \cap \mathbb{N}_{\geqslant 12}) \cup \{\infty\}. \end{aligned}$$

For a finite number of positive integers $s_1, \ldots, s_k$, let $\mathrm{GCD}(s_1, \ldots, s_k)$ denote the greatest common divisor of them. In case $k = 2$, we write $(s_1, s_2) = \mathrm{GCD}(s_1, s_2)$ as usual.

OBSERVATION 2.2. In Example 2.1, set $s_1 = 2$, $s_2 = 3$, $s_3 = 3$ and $s_4 = 4$. Then $s = s_1 + s_2 + s_3 + s_4 = 12$ and $\mathrm{GCD}(s_1, s_2, s_3, s_4) = 1$. Note that there is no j, $j = 1, 2, 3, 4$, such that $(s, s_j) = 1$.

Let $s_1, \ldots, s_k$ be positive integers. Set $s = s_1 + \cdots + s_k$, $r = \mathrm{GCD}(s_1, \ldots, s_k)$ and $d_0 = \mathrm{GCD}((s, s_1), \ldots, (s, s_k))$. Then it is easy to see that $r = d_0$.

OBSERVATION 2.3. The following conditions are equivalent.

(1) There is j, $1 \leqslant j \leqslant k$, such that $(s, s_j) = r$.
(2) There is j, $1 \leqslant j \leqslant k$, such that $(s, s_j) = d_0$.

3. A Formula for $A(f)$ Using Simplification

Let $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be an analytic function germ, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. In [3], Fukui gave a formula to compute $A(f)$ using a toric resolution, in case f is a nondegenerate function. In this section, we give a formula for $A(f)$ in the general case, using the Hironaka–Bierstone–Milman desingularisation ([1, 2, 5]).

Let X be a complex manifold. By *an arc through* $x \in X$, we mean a complex analytic mapping λ into X of a neighbourhood of $0 \in \mathbb{C}$ such that $\lambda(0) = x$.

Let $\Pi\colon (X, E) \to (\mathbb{C}^n, 0)$, $E = \Pi^{-1}(0)_{\mathrm{red}}$, be a *simplification* of $f^{-1}(0)$, namely, Π is a composition of a finite number of blowings up, X is smooth and $f \circ \Pi$ is a normal crossing. Here, we call a function *normal crossing* if it can be locally expressed as a product of powers of a number of local coordinates. Let $D = (f \circ \Pi)^{-1}(0)_{\mathrm{red}}$ be its reduction and $D = D_1 \cup \cdots \cup D_s$ the decomposition into irreducible components. Since we are concerned with divisors around E, we may assume that $D_i \cap E \neq \emptyset$ ($i = 1, \ldots, s$). For a subset

$$I = \{i_1, \ldots, i_p\} \subset S = \{1, \ldots, s\}$$

of subscripts, let $\{j_1, \ldots, j_q\}$, $p + q = s$, denote the complement I in S and put

$$D_I = (D_{i_1} \cap \cdots \cap D_{i_p}) \setminus (D_{j_1} \cup \cdots \cup D_{j_q}).$$

This is a manifold of codimension p (if it is not empty). The family $\{D_I\}$ gives a stratification of D. We put $\mathcal{C} = \{I \ :\ D_I \cap E \neq \emptyset\}$ for a simplification Π. Since $D_i \cap E \neq \emptyset$, $i \in S$, we have $\bigcup_{I \in \mathcal{C}} I = S$.

Remark 3.1. By choosing a suitable Π, we can assume that E is also a normal crossing divisor. Then E is a union of some D_i and if $I \in \mathcal{C}$ and $I \subset J \subset S$, then $J \in \mathcal{C}$.

Each divisor D_i has natural *multiplicity* m_i, which is the multiplicity of $f \circ \Pi$ at a generic point of D_i. Let us put

$$\Omega_I(f) = (m_{i_1}\mathbb{N} + \cdots + m_{i_p}\mathbb{N}) \cup \{\infty\},$$

for $I = (i_1, \dots, i_p) \in \mathcal{C}$. This is nothing but the set of values of the orders of $f \circ \lambda$ for various arcs on X through a point of $D_I \cap E$.

Next let us consider the real case. Exactly as in the complex case, a simplification Π: $(X^{\mathbb{R}}, E^{\mathbb{R}}) \to (\mathbb{R}^n, 0)$, $E^{\mathbb{R}} = \Pi^{-1}(0)$, exists by the real desingularisation theorem [5, 1, 2]. Then we can similarly define $D^{\mathbb{R}} = D_1^{\mathbb{R}} \cup \dots \cup D_s^{\mathbb{R}}$, $\mathcal{C}^{\mathbb{R}} \subset 2^S$, $D_I^{\mathbb{R}}$ $(I \in \mathcal{C}^{\mathbb{R}})$, the multiplicities $m_i^{\mathbb{R}}$ and $\Omega_I(f)$.

Remark 3.2. In the real case, there is a simplification of the complexification which is the composition of a finite number of blowings up with real centres. Then its real part is a real simplification and the real hypersurfaces $D^{\mathbb{R}}$ and $D_i^{\mathbb{R}}$ are, respectively, the real parts of the complex divisors D and D_j which are invariant with respect to the auto-conjugation. The multiplicities $m_i^{\mathbb{R}}$ of $D_i^{\mathbb{R}}$ are equal to m_j of the corresponding complex divisors D_j.

We can write down the Fukui invariant $A(f)$, using $\Omega_I(f)$.

THEOREM I. *Let* f: $(\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ *be an analytic function germ,* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$*, and* Π *be a simplification of* $f^{-1}(0)$. *Then we have* $A(f) = \bigcup_{I \in \mathcal{C}} \Omega_I(f)$.

Proof. Obviously we only have to prove that

$$A(f) \setminus \{\infty\} = \bigcup_{I \in \mathcal{C}} \Omega_I(f) \setminus \{\infty\}.$$

Suppose that $k \in A(f) \setminus \{\infty\}$. Then there exists an arc λ through $0 \in \mathbb{K}^n$ such that $\mathrm{O}(f \circ \lambda) = k$. By the universality of blowing up ([6], Definition 1), there exists a lifting μ: $U \to X$. Let ξ be the unique intersection point of the image of λ and E, which belongs to an unique D_I $(I = \{i_1, \dots, i_p\})$. Obviously $\xi \in E$ implies $I \in \mathcal{C}$. There exists a local coordinate system $(x_1, \dots, x_n)$ centered at ξ such that $f \circ \Pi = x_1^{l_1} \cdots x_n^{l_n}$ and μ is given by

$$\begin{aligned} x_1 &= \alpha_{11} t^{a_{11}} + \alpha_{12} t^{a_{12}} + \cdots, \\ &\cdots \\ x_n &= \alpha_{n1} t^{a_{n1}} + \alpha_{n2} t^{a_{n2}} + \cdots, \end{aligned}$$

at ξ, where $\alpha_{i1} \neq 0$, $0 < a_{i1} < a_{i2} < \cdots$, $i = 1, \dots, n$. Since the set of nonzero elements of $\{l_1, \dots, l_n\}$ coincides with $\{m_{i_1}, \dots, m_{i_p}\}$, we have

$$k = O(f \circ \lambda) = O(f \circ \Pi \circ \mu) = a_{11} l_{i_1} + \cdots + a_{n1} l_{i_n} \in \Omega_I(f).$$

The converse is now obvious.

COROLLARY II. *The Fukui invariant* $A(f)$ *is stably periodic.*

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4. A Formula for $A(f)$ in the Two Variables Case

Although the formula for the Fukui invariant given in the previous section looks simple, it is not so easy to compute it in general, for the actual task of finding a desingularisation of $f^{-1}(0)$ can be horrendous. In the two variables case, however, we can give a more explicit formula for the Fukui invariant, because in this case at most two divisors can intersect at a given point.

Let $f\colon (\mathbb{K}^2, 0) \to (\mathbb{K}, 0)$ be an analytic function germ, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. As stated in the introduction, we exclude the case where f is already a normal crossing.

Let us factor $f(x, y)$ into irreducible components

$$f(x, y) = \prod_{j=1}^{k} f_j(x, y)^{s_j},$$

where f_j are irreducible over $\mathbb{K}\{x, y\}$. Let $\Pi\colon (X, E) \to (\mathbb{K}^2, 0)$, $E = \Pi^{-1}(0)$, be a simplification of $f^{-1}(0)$ and $E = E_1 \cup \cdots \cup E_N$, where E_i are the irreducible exceptional divisors. Then the E_i's together with the strict transforms Z_j of $f_j^{-1}(0)$, $1 \leqslant j \leqslant k$, form a normal crossing family of smooth curves in a neighbourhood of E.

Let m_i denote the multiplicity of E_i, $1 \leqslant i \leqslant N$. The multiplicity of Z_j is s_j, $1 \leqslant j \leqslant k$ (if Z_j is not empty).

There are three kinds of points on E which interest us.

Take any E_i. Then take a generic point P_i on E_i. In terms of our stratification in Section 3, P_i is a point of the unique one-dimensional stratum contained in E_i. Let

$$\Delta(P_i) = m_i\mathbb{N}, \qquad \Delta_G = \bigcup_{i=1}^{N} \Delta(P_i).$$

Next, take any pair (E_i, E_j) such that $E_i \cap E_j \neq \emptyset$. Let H_{ij} denote the point of intersection. The point H_{ij} is also a stratum of our stratification. Define $\Delta_H = \bigcup_{i,j} (m_i\mathbb{N} + m_j\mathbb{N})$,where the union is taken over all pairs (i, j), $i \neq j$, with $E_i \cap E_j \neq \emptyset$.

The third kind are those points where the strict transforms Z_j meet the exceptional divisors. Take any Z_j. Let $E_{u(j)}$ denote the exceptional divisor which meets Z_j. Define $\Delta_S = \bigcup_{j=1}^{k} (s_j\mathbb{N} + m_{u(j)}\mathbb{N})$.

THEOREM IIIC. *For* $\mathbb{K} = \mathbb{C}$, $A(f) = \Delta_G \cup \Delta_H \cup \Delta_S \cup \{\infty\}$.

EXAMPLE 4.1. Consider $f(x, y) = x^2 - y^4$. In Figure 1, the second component of each bracket indicates the multiplicity of the divisor. The resolution tree gives:

$$\Delta(P_1) = \Delta(P_2) = 2\mathbb{N}, \quad \Delta_H = 2\mathbb{N} + 4\mathbb{N}, \quad \Delta_S = 4\mathbb{N} + \mathbb{N}.$$

Hence, $A(f) = \{2, 4, 5, 6, \ldots\} \cup \{\infty\}$.

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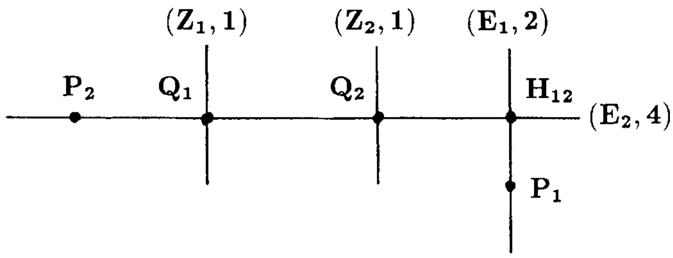

Figure 1.

When $\mathbb{K} = \mathbb{R}$, the formula in Theorem IIIC is still valid. However, Δ_S has to be interpreted properly, because the (real) strict transform Z_j is empty if $f_j^{-1}(0) = \{0\}$. For instance, $x^2 + y^2$ is desingularised by one blow-up, to $X^2(1 + Y^2)$. The real strict transform, defined by $1 + Y^2$, is empty. For each f_j with $f_j^{-1}(0) \neq \{0\}$, the real strict transform Z_j meets some $E_{u(j)}$ at a real point, say Q_j. Define

$$\Delta_S^{\mathbb{R}} = \bigcup_j (s_j \mathbb{N} + m_{u(j)} \mathbb{N}),$$

where the union is taken over all j with $f_j^{-1}(0) \neq \{0\}$. Thus we have

THEOREM IIIR. *For* $\mathbb{K} = \mathbb{R}$, $A(f) = \Delta_G \cup \Delta_H \cup \Delta_S^{\mathbb{R}} \cup \{\infty\}$.

EXAMPLE 4.2. Consider the function of Example 1.3, $f(x, y) = x^4 + y^6$, in $\mathbb{R}\{x, y\}$. The resolution tree (Figure 2) gives

$$\Delta_G = 4\mathbb{N} \cup 6\mathbb{N} \cup 12\mathbb{N} = 4\mathbb{N} \cup 6\mathbb{N},$$
$$\Delta_H = \{4\mathbb{N} + 12\mathbb{N}\} \cup \{6\mathbb{N} + 12\mathbb{N}\}, \quad \Delta_S^{\mathbb{R}} = \emptyset.$$

Hence $A(f) = 4\mathbb{N} \cup 6\mathbb{N} \cup \{\infty\}$.

We next compute the Fukui invariant of a *degenerate* function, using our formula.

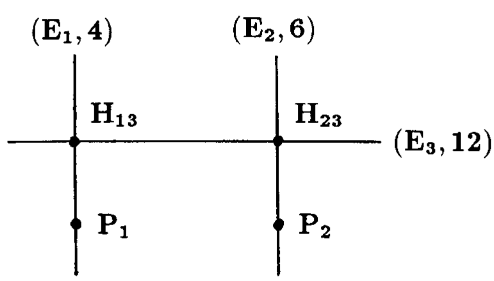

Figure 2.

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EXAMPLE 4.3. Let $f\colon (\mathbb{K}^2, 0) \to (\mathbb{K}, 0)$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, be a polynomial function defined by

$$f(x, y) = x^{15} + 3x^{12}y^2 + 3x^9y^4 + x^6y^6 + 2x^3y^{10} + y^{14}.$$

Note that

$$x^{15} + 3x^{12}y^2 + 3x^9y^4 + x^6y^6 = x^6(x^3 + y^2)^3,$$
$$x^6y^6 + 2x^3y^{10} + y^{14} = (x^3 + y^4)^2y^6.$$

We consider the resolution tree (Figure 3). Here, E_i' means E_{i+6}, $i = 2, 3, 4, 5$. We have used the notation E_i' to clarify our resolution process. In the real case, the strict transforms Z_4, Z_5 do not appear. The resolution tree gives:

$$\begin{aligned}
\Delta_G &= 12\mathbb{N} \cup 14\mathbb{N} \cup 28\mathbb{N} \cup 42\mathbb{N} \cup 44\mathbb{N} \cup 46\mathbb{N} \cup 48\mathbb{N} \cup 15\mathbb{N} \cup 30\mathbb{N} \\
&\quad \cup 33\mathbb{N} \cup 36\mathbb{N}, \\
\Delta_H &= (12\mathbb{N} + 42\mathbb{N}) \cup (28\mathbb{N} + 42\mathbb{N}) \cup (14\mathbb{N} + 28\mathbb{N}) \cup (42\mathbb{N} + 44\mathbb{N}) \\
&\quad \cup (44\mathbb{N} + 46\mathbb{N}) \cup (46\mathbb{N} + 48\mathbb{N}) \cup (12\mathbb{N} + 30\mathbb{N}) \cup (15\mathbb{N} + 30\mathbb{N}) \\
&\quad \cup (30\mathbb{N} + 33\mathbb{N}) \cup (33\mathbb{N} + 36\mathbb{N}), \\
\Delta_S &= (\mathbb{N} + 48\mathbb{N}) \cup (\mathbb{N} + 36\mathbb{N}).
\end{aligned}$$

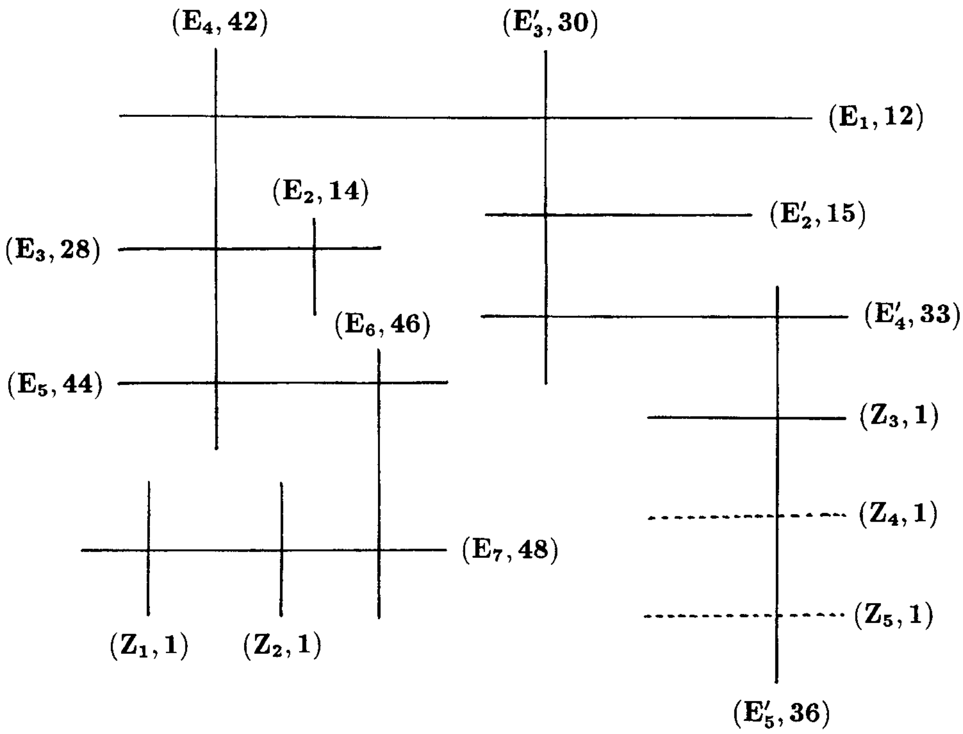

Figure 3.

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Hence,

$$\begin{aligned} A(f) &= 12\mathbb{N} \cup 14\mathbb{N} \cup 15\mathbb{N} \cup 33\mathbb{N} \cup (\mathbb{N} + 36\mathbb{N}) \cup \{\infty\} \\ &= \{12, 14, 15, 24, 28, 30, 33, 36, 37, 38, \ldots\} \cup \{\infty\}. \end{aligned}$$

In this case, $A(f)$ is stably unit-interval-like.

We can also compute this $A(f)$ by the orthodox method as in Example 2.1. But it is complicated. In fact, it is not so easy to find an arc λ by which $37 \in A(f)$ is attained.

Let $\lambda(t) = (X(t), Y(t))$ be an analytic arc defined by

$$X(t) = \alpha t^2, \quad Y(t) = t^3 + b_2 t^5 + b_3 t^6,$$

where $\alpha^3 = -1$, $4b_2^3 = 1$ and $b_3 \neq 0$. Then $\mathrm{O}(f \circ \lambda) = 37$.

5. Stable Interval-Likeness

We first consider the case $n = 2$. We keep the notations in Section 4. Namely, $f(x, y)$ is factored into irreducible components $f(x, y) = \prod_{j=1}^{k} f_j(x, y)^{s_j}$, and Π: $(X, E) \to (\mathbb{K}^2, 0)$ is a simplification of $f^{-1}(0)$, where $E = \Pi^{-1}(0)$ is the union of the exceptional divisors $E_1, \ldots, E_N$. Then m_i is the multiplicity of E_i, $1 \leqslant i \leqslant N$, and s_j is the multiplicity of the strict transform Z_j of $f_j^{-1}(0)$, $1 \leqslant j \leqslant k$.

If $E_i \cap E_j \neq \emptyset$, $i \neq j$, we define

$$\gamma_{ij} = (m_i, m_j), \qquad \Gamma_E = \bigcup \{\gamma_{ij}\},$$

where the union is taken over all pairs (i, j), $i \neq j$, with $E_i \cap E_j \neq \emptyset$. For each strict transform Z_j meeting some $E_{u(j)}$, define

$$\gamma_j = (s_j, m_{u(j)}), \qquad \Gamma_S = \bigcup_{j=1}^{k} \{\gamma_j\}.$$

When $\mathbb{K} = \mathbb{R}$, we need the following interpretation: Γ_S consists only of those γ_j for which $f_j^{-1}(0) \neq \{0\}$.

In the case where $f^{-1}(0) = \{0\}$ and the simplification is given by one blow-up, the resolution tree consists of only one exceptional divisor. Let m be the multiplicity. Then $A(f) = m\mathbb{N} \cup \{\infty\}$, thus $A(f)$ is stably interval-like. In this case, set $\Gamma = \{m\}$.

In the case where $f^{-1}(0) \neq \{0\}$ or the simplification cannot be given by only one blow-up, each exceptional divisor intersects another exceptional divisor or some strict transform. In other words, Γ_E or Γ_S is not empty. Set $\Gamma = \Gamma_E \cup \Gamma_S$.

We can characterise the stable interval-likeness as follows:

THEOREM IV. *Let d denote the greatest common divisor of the numbers in Γ. Then $A(f)$ is stably interval-like if and only if $d \in \Gamma$. In particular, $A(f)$ is stably unit-interval-like if and only if $1 \in \Gamma$.*

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Let $M = \bigcup_{i=1}^{N} \{m_i\}$ and $M_0 = \bigcup_{j=1}^{k} \{m_{u(j)}\}$. The next corollary follows immediately from Theorem IV.

COROLLARY V. *Let d_0 denote the greatest common divisor of the numbers in Γ_S. Suppose that every $m_i \in M \setminus M_0$ is divisible by some $\gamma_j \in \Gamma_S$. Then $A(f)$ is stably interval-like if and only if $d_0 \in \Gamma_S$.*

This corollary may not be so useful in the real case, because Γ_S may be empty.

EXAMPLE 5.1. Take a homogeneous form

$$f(x, y) = (a_1 x + b_1 y)^{s_1} \ldots (a_k x + b_k y)^{s_k}, \qquad a_i b_j - a_j b_i \neq 0 \ (i \neq j).$$

There is only one exceptional divisor in the resolution tree, whose multiplicity is $s = s_1 + \cdots + s_k$. In this case, $M = M_0 = \{s\}$. Thus $M \setminus M_0$ is empty and $\Gamma_S = \bigcup_{j=1}^{k} \{(s, s_j)\}$. By Corollary V, $A(f)$ is stably interval-like if and only if $d_0 = \mathrm{GCD}((s, s_1), \ldots, (s, s_k)) \in \Gamma_S$. Let $r = \mathrm{GCD}(s_1, \ldots, s_k)$. It follows from Observation (2.3) that $A(f)$ is stably interval-like if and only if $r \in \Gamma_S$.

Using a criterion for stable interval-likeness in Example (5.1), we can easily construct negative examples to Question (1.5). We give an example different from Example 2.1:

EXAMPLE 5.2. Let $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a homogeneous polynomial function defined by

$$f(x, y) = (x - y)^2 (x - 2y)^3 (x - 3y)^{25}.$$

Then $s = 2 + 3 + 25 = 30$ and $r = \mathrm{GCD}(2, 3, 25) = 1$. It follows that $r \notin \Gamma_S = \{2, 3, 5\}$. Therefore $A(f)$ is not stably interval-like.

As seen in Example 5.1, stable interval-likeness is determined by $\{s_1, \ldots, s_k\}$ in the homogeneous case. It is natural to ask if this is valid in general. The answer is no. Namely, stable interval-likeness cannot be determined merely by $\{s_1, \ldots, s_k\}$.

EXAMPLE 5.3. Let $f\colon (\mathbb{K}^2, 0) \to (\mathbb{K}, 0)$ be a polynomial function defined by

$$f(x, y) = (x^2 + y^3)^2 (x^2 + y^5)^3.$$

The resolution tree (Figure 4) gives

$$M \setminus M_0 = \{10, 18, 21\}, \qquad \Gamma_S = \{(2, 30), (3, 42)\} = \{2, 3\}.$$

Thus $m \in M \setminus M_0$ is divisible by some $\gamma \in \Gamma_S$. On the other hand, $d_0 = (2, 3) = 1 \notin \Gamma_S$. It follows from Corollary V that $A(f)$ is not stably interval-like.

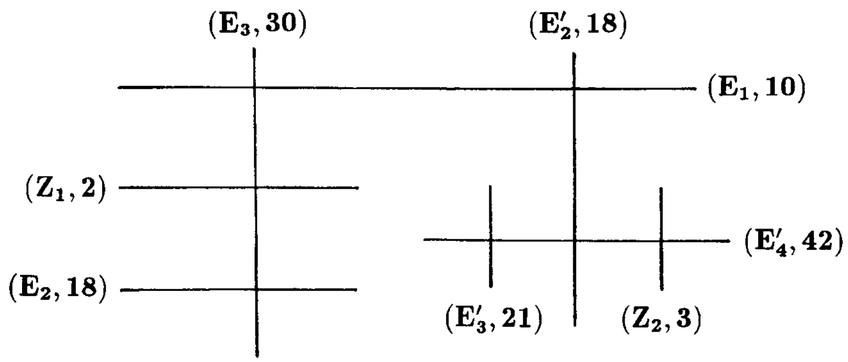

Figure 4.

EXAMPLE 5.4. Let f: $(\mathbb{K}^2, 0) \to (\mathbb{K}, 0)$ be a polynomial function defined by

$$f(x, y) = (x^2 + y^3)^2(x + y)^3.$$

The resolution tree (Figure 5) gives $1 = (3, 7) \in \Gamma$. By Theorem IV, $A(f)$ is stably unit-interval-like.

We next consider the general case. Let f: $(\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be an analytic function germ, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and let Π: $(X, E) \to (\mathbb{K}^n, 0)$ be a simplification of $f^{-1}(0)$. For the simplification Π, we define $\mathcal{C}$ in the same way as in Section 3. For $I = (i_1, \ldots, i_p) \in \mathcal{C}$, set

$$M_I = \mathrm{GCD}(m_{i_1}, \ldots, m_{i_p}).$$

Then we have

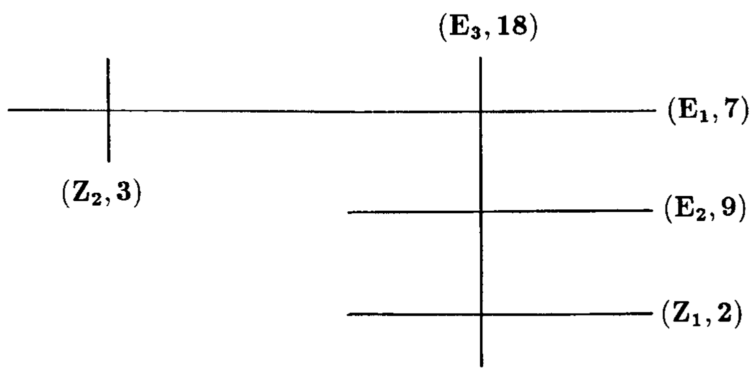

Figure 5.

THEOREM VI. *Let d denote the greatest common divisor of the numbers in $\{M_I \mid I \in \mathcal{C}\}$. Then $A(f)$ is stably interval-like if and only if $d \in \{M_I \mid I \in \mathcal{C}\}$.*

Theorem IV, Corollary V, Example 5.1 and Theorem VI are criteria for stable interval-likeness. We call them GCD tests.

Attention should be paid to the fact that although the statements of the tests are the same for the real and the complex cases, the actual meaning in each case is quite different.

Thus, let us take an analytic function with real coefficients. When considered as a complex analytic function, it has a (complex) resolution tree which, in general, can contain many more exceptional divisors than the real resolution tree. As a result, the complex Fukui set can be much larger than the real Fukui set; the former can be easily stably interval-like while the latter is not.

This explains why stably interval-like examples are much more numerous in the complex case than in the real case.

6. Proofs of Theorems IV and VI

Before starting the proofs of Theorems IV and VI, we prepare some lemmas and recall the notion of the conductor of two positive integers.

LEMMA 6.1. *Let $\tau_1, \dots, \tau_q$ be positive integers such that*

$$\tau_j \geqslant 2,\ 1 \leqslant j \leqslant q \quad \text{and} \quad \mathrm{GCD}(\tau_1, \dots, \tau_q) = 1.$$

Then $\tau_1\mathbb{N} \cup \cdots \cup \tau_q\mathbb{N}$ is not stably interval-like in our sense.

Proof. Assume that $\tau_1\mathbb{N} \cup \cdots \cup \tau_q\mathbb{N}$ is stably interval-like. Then there are positive integers k, d such that

$$(\tau_1\mathbb{N} \cup \cdots \cup \tau_q\mathbb{N}) \cap \mathbb{N}_{\geqslant k\tau_1\cdots\tau_q} = \{k\tau_1 \dots \tau_q,\ k\tau_1 \dots \tau_q + d,\ k\tau_1 \dots \tau_q + 2d,\ \dots\}. \tag{6.2}$$

Set $A = (\tau_1\mathbb{N} \cup \cdots \cup \tau_q\mathbb{N}) \cap \mathbb{N}_{\geqslant k\tau_1\cdots\tau_q}$. For simplicity, let $\tau_1 = \min(\tau_1, \dots, \tau_q)$. Then $d = \tau_1$. Since $\mathrm{GCD}(\tau_1, \dots, \tau_q) = 1$, there is j, $2 \leqslant j \leqslant q$, such that τ_j is not divisible by τ_1. Then there is a positive integer m such that $m\tau_1 < \tau_j < (m+1)\tau_1$. By (6.2),

$$\begin{aligned}
&k\tau_1 \dots \tau_q + m\tau_1 = k\tau_1 \dots \tau_q + md \in A,\\
&k\tau_1 \dots \tau_q + (m+1)\tau_1 = k\tau_1 \dots \tau_q + (m+1)d \in A,\\
&k\tau_1 \dots \tau_q + \tau_j \notin A.
\end{aligned}$$

On the other hand, $k\tau_1 \cdots \tau_q + \tau_j = (k\tau_1 \cdots \tau_{j-1}\tau_{j+1} \cdots \tau_q + 1)\tau_j \in A$. This is a contradiction.

The next lemma follows from this lemma.

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LEMMA 6.3. *Let $d, \tau_1, \ldots, \tau_q$ be positive integers such that*

$$\tau_j \geqslant 2,\ 1 \leqslant j \leqslant q, \quad \textit{and} \quad \mathrm{GCD}(\tau_1, \ldots, \tau_q) = 1.$$

Then $\tau_1 d\mathbb{N} \cup \cdots \cup \tau_q d\mathbb{N}$ is not stably interval-like.

Let $a,\ b$ be positive integers such that $(a, b) = d$. It is well-known that there is a positive integer c such that

$$(a\mathbb{N} + b\mathbb{N}) \cap \mathbb{N}_{\geqslant c} = \{c,\ c+d,\ c+2d, \ldots\}.$$

The smallest integer c for which this holds is called the *conductor* of a and b.

Proof of Theorem IV. We prove the complex case. The real case follows similarly, because Theorem IV is obvious when $f^{-1}(0) = \{0\}$ and the simplification is given by one blow-up. Let us consider the complex case. Each exceptional divisor intersects another exceptional divisor or some strict transform. By Theorem IIIC,

$$A(f) = \Delta_G \cup \Delta_H \cup \Delta_S \cup \{\infty\}.$$

Let e be an arbitrary element of $A(f)$. By the definition of Γ, e is divisible by some $\alpha_i \in \Gamma$. Therefore e is divisible by d, where d is the greatest common divisor of the numbers in Γ.

We first show that $d \in \Gamma$ implies the stable interval-likeness of $A(f)$. By assumption, there is $\alpha_i \in \Gamma$ such that $\alpha_i = d$. Then there is $\gamma_{ij} \in \Gamma_E$ with $\gamma_{ij} = d$ or $\gamma_j \in \Gamma_S$ with $\gamma_j = d$. Assume that $\gamma_{ij} = d$. In case $\gamma_j = d$, the argument is similar. Let c be the conductor of m_i and m_j. Then

$$(m_i\mathbb{N} + m_j\mathbb{N}) \cap \mathbb{N}_{\geqslant c} = \{c,\ c+d,\ c+2d, \ldots\}.$$

It follows from the divisibility of any element of $A(f)$ by d that

$$A(f) \cap \mathbb{N}_{\geqslant c} = \{c,\ c+d,\ c+2d, \ldots\}.$$

Therefore $A(f)$ is stably interval-like.

We next show the converse. Let $\Gamma = \Gamma_E \cup \Gamma_S = \{\alpha_1, \ldots, \alpha_q\}$. It is obvious in case $q = 1$. Therefore we assume that $q \geqslant 2$. Set $I = \{(i, j) \mid \gamma_{ij} \in \Gamma_E\}$.

Let c_{ij} be the conductor of m_i and m_j for $(i, j) \in I$, and let c_j be the conductor of s_j and $m_{u(j)}$ for $1 \leqslant j \leqslant k$. Then

$$(m_i\mathbb{N} + m_j\mathbb{N}) \cap \mathbb{N}_{\geqslant c_{ij}} = c_{ij} + \gamma_{ij}(\{0\} \cup \mathbb{N}),$$

$$(s_j\mathbb{N} + m_{u(j)}\mathbb{N}) \cap \mathbb{N}_{\geqslant c_j} = c_j + \gamma_j(\{0\} \cup \mathbb{N}).$$

Note that each γ_{ij} or γ_j is some $\alpha_v \in \Gamma$. Set

$$B = \prod_{(i,j)\in I} c_{ij} \prod_{j=1}^{k} c_j.$$

Then

$$A(f)\cap\mathbb{N}_{\geqslant B}=\left(\left(\bigcup_{(i,j)\in I}\gamma_{ij}\mathbb{N}\right)\cup\left(\bigcup_{j=1}^{k}\gamma_j\mathbb{N}\right)\right)\cap\mathbb{N}_{\geqslant B}=\left(\bigcup_{i=1}^{q}\alpha_i\mathbb{N}\right)\cap\mathbb{N}_{\geqslant B}.$$

Set $\alpha_i=\tau_i d$ for $1\leqslant i\leqslant q$. Then

$$A(f)\cap\mathbb{N}_{\geqslant B}=d\left(\bigcup_{i=1}^{q}\tau_i\mathbb{N}\right)\cap\mathbb{N}_{\geqslant B}$$

such that $\mathrm{GCD}(\tau_1,\dots,\tau_q)=1$. If $A(f)$ is stably interval-like, then $d(\bigcup_{i=1}^{q}\tau_i\mathbb{N})$ is also stably interval-like. Therefore it follows from Lemma 6.3 that there is i_0, $1\leqslant i_0\leqslant q$, such that $\tau_{i_0}=1$. Then $\alpha_{i_0}=\tau_{i_0}d=d$, namely, $d\in\Gamma$.

Proof of Theorem VI. Using the following lemma, we can show this theorem in the same way as above.

LEMMA 6.4. *Let $\alpha_1,\dots,\alpha_r$ be positive integers, and let $d=\mathrm{GCD}(\alpha_1,\dots,\alpha_r)$. There is a positive integer c such that*

$$(\alpha_1\mathbb{N}+\cdots+\alpha_r\mathbb{N})\cap\mathbb{N}_{\geqslant c}=\{c,\ c+d,\ c+2d,\dots\}.$$

7. The Fukui Invariants with Sign

Let $f\colon(\mathbb{R}^n,0)\to(\mathbb{R},0)$ be an analytic function germ, and let $\Pi\colon(X^{\mathbb{R}},E^{\mathbb{R}})\to(\mathbb{R}^n,0)$, $E^{\mathbb{R}}=\Pi^{-1}(0)$, be a simplification of $f^{-1}(0)$ in the sense of Section 3. As in Section 3, we can likewise define $D^{\mathbb{R}}=D_1^{\mathbb{R}}\cup\cdots\cup D_s^{\mathbb{R}}$, $C^{\mathbb{R}}\subset 2^s$, $D_I^{\mathbb{R}}$ ($I\in C^{\mathbb{R}}$), the multiplicities $m_i^{\mathbb{R}}$ and $\Omega_I(f)$.

Let us put

$$P(f)=\{x\in X\mid f\circ\Pi(x)>0\},\qquad N(f)=\{x\in X\mid f\circ\Pi(x)<0\},$$
$$\mathcal{C}^+=\{I\in\mathcal{C}\mid D_I^{\mathbb{R}}\cap E^{\mathbb{R}}\cap\overline{P(f)}\neq\emptyset\},\qquad \mathcal{C}^-=\{I\in\mathcal{C}\mid D_I^{\mathbb{R}}\cap E^{\mathbb{R}}\cap\overline{N(f)}\neq\emptyset\},$$

where the overlines indicate the closures in X.

Recall that an *arc* through $0\in\mathbb{R}^n$ is the germ of a real analytic map $\lambda\colon U\to\mathbb{R}^n$ with $\lambda(0)=0$, where U denotes a neighbourhood of $0\in\mathbb{R}$. An arc λ through 0 is *nonnegative* (resp. *nonpositive*) *for f* if $f\circ\lambda(t)\geqslant 0$ (resp. $\leqslant 0$) in a positive half neighbourhood $[0,\delta)\subset U$. Then we define *the Fukui invariants with sign* by

$$A^+(f)=\{\mathrm{O}(f\circ\lambda)\mid \lambda \text{ is a nonnegative arc through 0 for } f\},$$
$$A^-(f)=\{\mathrm{O}(f\circ\lambda)\mid \lambda \text{ is a nonpositive arc through 0 for } f\},$$

respectively. It is obvious that $A(f)=A^+(f)\cup A^-(f)$.

Remark 7.1. Fukui [3] introduced a set $A_n(f)$ of blow-analytic equivalence classes of real analytic function germs $\varphi\colon(X,D)\to(\mathbb{R},0)$, where X are n-dimensional manifolds and D are compact subspaces such that $f \circ \Pi$ is not a zero divisor. Our $A(f)$ is obtained from Fukui's $A_1(f)$ forgetting the sign and $A^+(f)$ (resp. $A^-(f)$) can be interpreted as the set of Fukui's $[(k)^+] \in A_1(f)$ (resp. $[(k)^-] \in A_1(f)$). Fukui gave a formula to compute $A_1(f)$ for a *nondegenerate* function.

Using an argument similar to the proof of Theorem I, we can show the following.

THEOREM VII. *Let* $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ *be an analytic function. Then we have*

$$A^+(f) = \bigcup_{I \in \mathcal{C}^+} \Omega_I(f), \quad A^-(f) = \bigcup_{I \in \mathcal{C}^-} \Omega_I(f).$$

EXAMPLE 7.2. Let f, g: $(\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be polynomial functions defined by

$$f(x, y) = x^3 + y^8 \quad \text{and} \quad g(x, y) = x^3 - y^8.$$

Then $A(f) = A(g) = \{3, 6, 8, 9, 12, 15, 16, 18, 21, 24, 25, 26, \dots\} \cup \{\infty\}$. Therefore, by merely using $A(f)$, we cannot distinguish the blow-analytic types of f and g.

We consider the resolution trees of f and g with sign (Figures 6 and 7). By Theorem VII, we have $8 \in A_+(f)$, $8 \notin A_-(f)$, $8 \notin A_+(g)$, $8 \in A_-(g)$. Therefore f and g are not blow-analytically equivalent.

The functions in Example 7.2 are nondegenerate. Thus, we can distinguish f from g, using the Fukui's result on $A_1(f)$. But Theorem VII is also applicable in the degenerate case.

8. A Formula for $A(f)$ Using the Tree-Model

Take a germ of holomorphic function

$$f(x, y) = H_m(x, y) + H_{m+1}(x, y) + \cdots$$

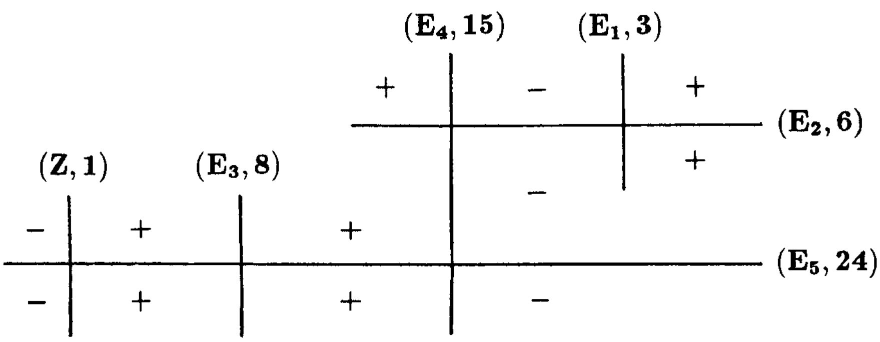

Figure 6.

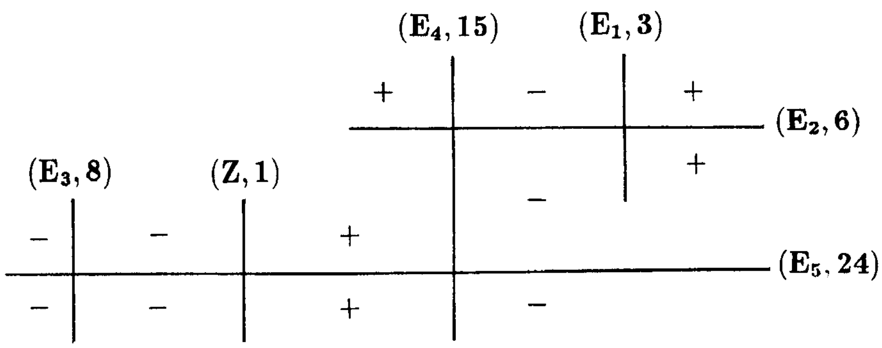

Figure 7.

which is *mini-regular in x of order m*, namely, $H_m(1,0) \neq 0$. The Newton–Puiseux factorisation has the form

$$f(x,y) = (\text{unit}) \prod_{i=1}^{m} (x - \beta_i(y)),$$

where the roots β_i are fractional power series with $\mathrm{O}_y(\beta(y)) \geqslant 1$.

Given a fractional power series $\gamma(y)$, $\mathrm{O}_y(\gamma) \geqslant 1$, we can write, as in Zariski [10],

$$\gamma(y) = a_1 y + \cdots + b_1 y^{\mu_1/d_1} + \cdots + b_2 y^{\mu_2/d_1 d_2} + \cdots + b_g y^{\mu_g/d_1 \cdots d_g} + \cdots,$$

where $b_i \neq 0$, $d_i > 1$, $(\mu_i, d_i) = 1$, $1 \leqslant i \leqslant g$, to expose the Puiseux characteristic sequence $\{(\mu_1, d_1), \ldots, (\mu_g, d_g)\}$. We call $\mu_i/d_1 \cdots d_i$ the ith characteristic exponent of γ. For convenience, we set $\mu_0 = d_0 = 1$ and also call $1 = \mu_0/d_0$ the 0th characteristic exponent. We write, as abreviation, $D_k = d_0 \cdots d_k$.

Take $\gamma_1(y)$ and $\gamma_2(y)$. Their *order of contact* is defined by

$$\mathrm{O}(\gamma_1, \gamma_2) = \mathrm{O}_y(\gamma_1(y) - \gamma_2(y)).$$

Take a positive rational number $q \in \mathbb{Q}^+$. We say γ_1 and γ_2 are *congruent modulo q*, written as $\gamma_1 \equiv \gamma_2 \bmod q$, if $\mathrm{O}(\gamma_1, \gamma_2) \geqslant q$. This equivalence relation gives rise to a lattice of subsets of $\{\beta_1, \ldots, \beta_m\}$, as follows:

Take any β_i. Then take any β_j and let

$$B = \{\beta_k \mid \mathrm{O}(\beta_k, \beta_i) \geqslant \mathrm{O}(\beta_j, \beta_i)\}.$$

The lattice, by definition, consists of all B obtained in this manner. It is partially ordered by the inclusion of sets.

In [7], this lattice is called the *tree-model* of $f(x,y)$ and B is called a *bar*. The number of β_i in B is the *multiplicity* of B, denoted by $m(B)$. We also call

$$h(B) = \min\{\mathrm{O}(\beta_j, \beta_k) \mid \beta_j, \beta_k \in B\}$$

the *height* of B.

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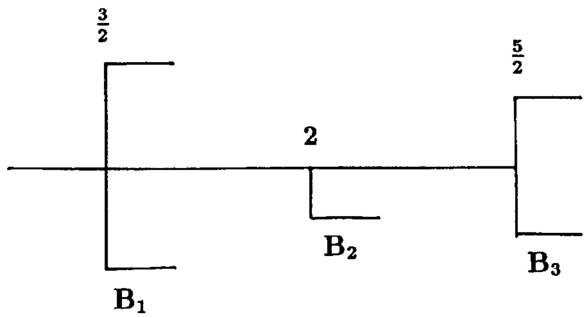

Figure 8.

EXAMPLE 8.1. Consider a function defined by $f(x, y) = (x - y^2)(x^2 - y^3)(x^2 - y^5)$. The tree model is shown in Figure 8 with $h(B_1) = 3/2$, $m(B_1) = 5$, $h(B_2) = 2$, $m(B_2) = 3$, $h(B_3) = 5/2$, $m(B_3) = 2$.

Note that by taking $\beta_i = \beta_j$, we obtain a bar of height ∞, whose multiplicity is that of the root β_i. In Example 8.1, there are 5 bars of height ∞, with multiplicity 1.

In the following, we shall define two sets of integers, $\hat{I}(B)$ and $\hat{J}(B)$, for each bar B, then prove the next result.

THEOREM VIII. *Let $f(x, y)$ be a holomorphic function mini-regular in x of order m. Then we have*

$$A(f) = m\mathbb{N} \cup \left(\bigcup_B (\hat{I}(B) \cup \hat{J}(B)) \right) \cup \{\infty\}.$$

Take a bar B. Take any $\beta_i \in B$. Let β_B denote β_i with all terms y^e, $e \geqslant h(B)$, omitted. We call β_B the *truncation* of B.

The largest characteristic exponent of β_B is called the *characteristic* of B, denoted by $\mathrm{char}(B)$. Note that $\mathrm{char}(B) < h(B)$. In Example 8.1, $\mathrm{char}(B_i) = 1$, $i = 1, 2, 3$.

If $B \supset B'$ with $B \neq B'$ and there is no other bar in between, we call B' a *postbar* of B.

ASSERTION 8.2 (proved in Section 9). *Every bar B must be one of the following three kinds*:

First kind. *For all (postbar)* B', $\mathrm{char}(B') = h(B)$.
Second kind. *For all* B', $\mathrm{char}(B') = \mathrm{char}(B)$.
Third kind. *There is a unique (postbar)* B^*, $\mathrm{char}(B^*) = \mathrm{char}(B)$ *and* $\mathrm{char}(B') = h(B)$ *for all* $B' \neq B^*$.

In Example 8.1, B_3 is of the first kind, B_2 the second kind and B_1 the third kind.

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Take any $\gamma(y)$. We define the following function:

$$\mathcal{M}_\gamma(r) = \text{number of } \beta_i \text{ such that } \mathrm{O}(\gamma, \beta_i) \geqslant r, 0 < r < \infty.$$

This is an integer-valued, decreasing, step function.

Let us take β_B as γ. The resulting function will be written simply as $\mathcal{M}_B(r)$, $0 < r < \infty$.

Take any bar B. Let us write

$$\mathrm{char}(B) = \frac{\mu_{k-1}}{d_1 \ldots d_{k-1}}, \quad d_i > 1, \ (d_i, \mu_i) = 1,$$

and then write

$$h(B) = \frac{\mu_k}{d_1 \ldots d_{k-1} d_k}, \quad d_k \geqslant 1.$$

Note that if $d_k = 1$, then B is of the second kind. The converse is also true as we shall see in Section 9. Note also that $\beta_B(y)$ is a finite series and $\beta_B(t^{D_{k-1}})$ is a polynomial in t.

Take a postbar B' of B. We can write

$$h(B') = h(B) + \frac{\mu'}{d_1 \ldots d_k d'}, \quad d' \geqslant 1, \ (\mu', d') = 1.$$

Suppose that B is of the first kind. Then, clearly, $\mathrm{O}(\beta_B, \beta_j) = h(B)$ for all $\beta_j \in B$. Using Abel's identity, we find

$$\int_0^{h(B)} \mathcal{M}_B(r)\, \mathrm{d}r = \mathrm{O}_y(f(\beta_B(y), y)) = \frac{1}{D_{k-1}} \mathrm{O}_t(f(\beta_B(t^{D_{k-1}}), t^{D_{k-1}})).$$

Hence the following numbers

$$J(B) = D_{k-1} \int_0^{h(B)} \mathcal{M}_B(r)\, \mathrm{d}r, \qquad I(B) = D_k \int_0^{h(B)} \mathcal{M}_B(r)\, \mathrm{d}r$$

are integers. We define

$$\hat{J}(B) = J(B)\mathbb{N} \text{ (integral multiples).}$$

We also define

$$\hat{I}(B, B') = \left\{ \mathrm{d}I(B) + \mu m(B') \mid d \geqslant 1, \ \mu \geqslant 1, \ \frac{\mu}{d} < \frac{\mu'}{d'} \right\},$$

and, taking union over all postbars B', $\hat{I}(B) = \bigcup_{B'} \hat{I}(B, B')$.

Next, suppose that B is of the second kind. We still define $I(B)$, $J(B)$ as above. Since $d_k = 1$, they are integers. We also define $\hat{I}(B)$, $\hat{J}(B)$ as above.

Finally, suppose that B is of the third kind. In this case,

$$\mathrm{char}(B) = \mathrm{char}(B^*) = \frac{\mu_{k-1}}{d_1 \dots d_{k-1}},$$

$$\mathrm{char}(B') = h(B) = \frac{\mu_k}{d_1 \dots d_k} \qquad (B' \neq B^*).$$

We still define $I(B)$ as above. This is clearly an integer. We also define $\hat{I}(B, B')$ and $\hat{I}(B)$ as above. As for $\hat{J}(B)$, the definition is more subtle, the reason being that $D_{k-1} \int_0^{h(B)} \mathcal{M}_B(r)\, \mathrm{d}r$ may not be an integer. (In Example 8.1, $\int_0^{h(B_1)} \mathcal{M}_{B_1}(r)\, \mathrm{d}r = 5 \times 3/2$.) Let us write

$$h(B^*) = \mathrm{char}(B) + \frac{\mu^*}{d_1 \dots d_{k-1} d^*}, \ d^* \geqslant 1,$$

and rewrite

$$h(B) = \mathrm{char}(B) + \frac{\bar{\mu}}{d_1 \dots d_{k-1} \bar{d}},$$

where, of course, $\bar{d} = d_k$, $\bar{\mu} = \mu_k - \mu_{k-1} d_k$.

ASSERTION 8.3. *The number*

$$J(B) = D_{k-1} \left\{ \int_0^{h(B)} \mathcal{M}_B(r)\, \mathrm{d}r - (h(B) - \mathrm{char}(B)) m(B^*) \right\}$$

is an integer.

We then define

$$\hat{J}(B) = \left\{ \mathrm{d}J(B) + \mu m(B^*) \mid d \geqslant 1,\ \mu \geqslant 1,\ \frac{\bar{\mu}}{\bar{d}} \leqslant \frac{\mu}{d} < \frac{\mu^*}{d^*} \right\}.$$

This completes the definitions of $\hat{I}(B)$ and $\hat{J}(B)$.

EXAMPLE 8.4. Consider $x^p - y^q$, $p < q$. There is only one bar of finite height, q/p, which is of the first kind,

$$A(x^p - y^q) = p\mathbb{N} \cup q\mathbb{N} \cup (pq\mathbb{N} + \mathbb{N}) \cup \{\infty\}.$$

Next, consider $x(x^2 - y^3)$. This time the bar is of the third kind,

$$m(B) = 3, \qquad I(B) = 9, \qquad J(B) = 4,$$
$$\hat{I}(B) = 9\mathbb{N} + \mathbb{N}, \qquad \hat{J}(B) = \{4d + \mu \mid 1/2 \leqslant \mu/d < \infty\} = \{5, 6, 7, \dots\}.$$

Hence

$$A(x^3 - xy^3) = \{3, 5, 6, 7, \cdots\} \cup \{\infty\}.$$

9. Proofs of Assertions 8.2 and 8.3

Take a weighted homogeneous form, $W(x, y)$, say with weights $w(x) = q$, $w(y) = p$, $(p, q) = 1$. Suppose that $1 < p < q$. Then

$$W(x, y) = ax^e y^{e'} \prod_i (x^p - c_i y^q)^{e_i}, \quad c_i \neq 0, \quad a \neq 0.$$

Ignoring the factor $y^{e'}$, all roots have the form $x = cy^{q/p}$ and characteristic exponent q/p, with one exception: if $e > 0$, then $x = 0$ is a root with characteristic exponent 1. If $p = 1$, the above is no longer true, all roots have characteristic exponent 1.

Assertion 8.2 is basically a consequence of the above phenomenon.

Let us take a bar B with truncation β_B and

$$\mathrm{char}(B) = \frac{\mu_{k-1}}{d_0 \cdots d_{k-1}}, \quad h(B) = \frac{\mu_k}{d_0 \cdots d_k} \quad (d_k \geqslant 1).$$

There is a polynomial $\phi(z)$ of degree $m(B)$, having the following property. Take any root b of $\phi(z) = 0$, ($b = 0$ allowed,) say of multiplicity m'. Then there are exactly m' elements β_j in B of the form

$$\beta_B(y) + by^{h(B)} + \text{term of order} > h(B).$$

More precisely, $\phi(z)$ can be obtained as follows:

Consider $X = x - \beta(y)$, $Y = y$, and

$$F(X, Y) = f(X + \beta_B(Y), Y) = \sum c_{ij} X^i Y^{\frac{j}{D}}, \quad D = D_{k-1}.$$

Let us plot a dot at $(i, j/D)$ for every $c_{ij} \neq 0$, then construct the Newton polygon of F. (In [9], this is called *the Newton polygon of f relative to the arc $x = \beta_B(y)$*.) There is a Newton edge E_B with angle Θ_B, associated to B, such that $\tan \Theta_B = h(B)$. Let us collect terms of F on E_B:

$$\Phi_B(X, Y) = \sum c_{ij} X^i Y^{\frac{j}{D}}, \qquad \left(i, \frac{j}{D}\right) \in E_B.$$

This is a weighted homogeneous form with weights $w(X) = \mu_k$ and $w(Y) = D_k$. Putting $Y = 1$, we obtain $\phi(z) = \Phi_B(z, 1)$.

In Example 8.1, for B_2, $\Phi = -X^3 Y^3 + X^2 Y^5$, and $\phi(z) = z^2(1 - z)$. The root $z = 0$ leads to $x = \pm y^{5/2}$, and $z = 1$ leads to $x = y^2$.

Now, in case $d_k = 1$, $h(B)$ is not a new characteristic exponent, B is a bar of the second kind. In case $d_k > 1$ and X is *not* a factor of Φ, B is of the first kind, while if X *is* a factor, B is of the third kind.

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Turning to Assertion 8.3, let us consider $f^*(x, y) = \prod_j (x - \beta_j(y))$, where the product is taken over all β_j which are conjugate to some root in B^*. We know f^* is a holomorphic function of x, y, and so is the quotient $\tilde{f}(x, y) = f(x, y)/f^*(x, y)$.

Take any γ, let $\mathcal{M}^*_\gamma$ and $\tilde{\mathcal{M}}_\gamma$ denote respectively the step functions for γ defined for f^* and $\tilde{f}$. Of course,

$$\mathcal{M}_\gamma(r) = \tilde{\mathcal{M}}_\gamma(r) + \mathcal{M}^*_\gamma(r), \ \ 0 < r < \infty.$$

In the tree-model of $\tilde{f}$, $\tilde{B} = B - B^*$ is a bar of the first kind. Hence, $D_{k-1} \int_0^{h(B)} \tilde{\mathcal{M}}_{\tilde{B}}(r)\, \mathrm{d}r$ is an integer. Note that B^* is a bar of the tree-model of f^*, hence $D_{k-1} \int_0^{char(B^*)} \mathcal{M}^*_{B^*}(r)\, dr$ is also an integer. The following identity is obvious:

$$\begin{aligned}\int_0^{h(B)} & \mathcal{M}_B(r)\, \mathrm{d}r - (h(B) - \mathrm{char}(B))m(B^*) \\ &= \int_0^{h(B)} \tilde{\mathcal{M}}_{\tilde{B}}(r)\, \mathrm{d}r + \int_0^{\mathrm{char}(B)} \mathcal{M}^*_{B^*}(r)\, \mathrm{d}r,\end{aligned}$$

whence $J(B)$ is an integer.

10. Proof of Theorem VIII

Take $\gamma(y)$. We write $\mathrm{O}(\gamma, f) = \max\{\mathrm{O}(\gamma, \beta_i) \mid 1 \leqslant i \leqslant m\}$. Let B_L denote the largest bar in the lattice, $m(B_L) = m$.

Take any γ with $\mathrm{O}(\gamma, f) = \mu/d \leqslant h(B_L)$. Then, clearly, $\mathrm{dO}_y(f(\gamma(y), y)) = \mu m$. This kind of γ gives rise to numbers in $m\mathbb{N}$. Taking $\mu = d = 1$ gives $m \in A(f)$, hence $m\mathbb{N} \subset A(f)$.

Now, take γ with $\mathrm{O}(\gamma, f) > h(B_L)$. Choose β_i such that $\mathrm{O}(\gamma, f) = \mathrm{O}(\gamma, \beta_i)$. Then choose a bar B whose height $h(B)$ is largest such that $\beta_i \in B$ and $\mathrm{O}(\gamma, f) \geqslant h(B)$.

Case 1. B is a bar of the first kind, $h(B) = \mu_k/d_0 \cdots d_k$.

Let us first examine a very special case, namely, $\mathrm{O}(\gamma, f) = h(B)$ and $h(B)$ is not a characteristic exponent of γ. (For instance, take β_B and $\gamma \equiv 0$ in Example 8.1.) The situation is modelled in Figure 9.

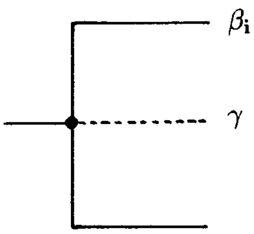

Figure 9.

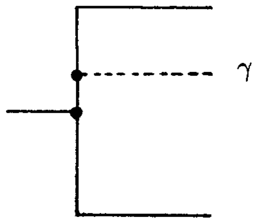

Figure 10.

When we omit all terms y^e, $e > h(B)$, in $\gamma(y)$, we obtain β_B. We can take β_B to be γ. Then we find

$$J(B) = D_{k-1}\mathrm{O}_y(f(\beta_B(y), y)) \in A(f).$$

Let $\mu_s/d_0 \cdots d_s$ be the largest characteristic exponent of γ. Then

$$D_s\mathrm{O}_y(f(\gamma(y), y)) = D_s\mathrm{O}_y(f(\beta_B(y), y))$$

is just a multiple of $J(B)$. Thus, this kind of γ gives rise to multiples of $J(B)$.

Next, suppose that $\mathrm{O}(\gamma, \beta_j) = h(B)$ for all $\beta_j \in B$, and $h(B)$ is a characteristic exponent of γ, as illustrated in Figure 10. (In Example 8.1, take B_3 and $\gamma(y) = 2y^{5/2}$.) In this case, D_k is the smallest integer for which $\tilde{\gamma}(t^{D_k})$ is integral in t. Here $\tilde{\gamma}$ denotes γ with terms y^e, $e > h(B)$, omitted. We have

$$D_k\mathrm{O}_y(f(\tilde{\gamma}(y), y)) = d_kJ(B).$$

Therefore, this kind of γ also gives rise to multiples of $I(B) = d_kJ(B)$.

Finally, consider the case $\mathrm{O}(\gamma, \beta_i) > h(B)$. There is a postbar B', containing β_i and $\mathrm{O}(\gamma, \beta_i) < h(B')$, as illustrated in Figure 11. We can write

$$O(\gamma, \beta_i) = h(B) + \frac{\mu}{d_0 \ldots d_kd}, \quad d \geqslant 1,$$

$$h(B') = h(B) + \frac{\mu'}{d_0 \ldots d_kd'}, \quad d' \geqslant 1,$$

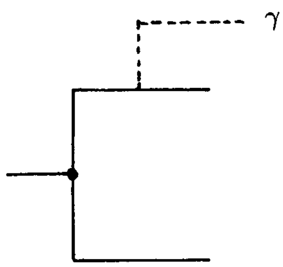

Figure 11.

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where $0 < \mu/d < \mu'/d'$. Let $\tilde{\gamma}$ be γ with terms y^e, $e > \mathrm{O}(\gamma, \beta_i)$, omitted. Then dD_k is the smallest integer for which $\tilde{\gamma}(t^{dD_k})$ is integral. Therefore

$$\mathrm{O}_y(f(\tilde{\gamma}(y), y)) = \int_0^{h(B)} \mathcal{M}_B(r)\, \mathrm{d}r + \frac{\mu}{D_k d} m(B'),$$

and

$$\mathrm{d}D_k \mathrm{O}_y(f(\tilde{\gamma}(y), y)) = \mathrm{d}I(B) + \mu m(B').$$

This number and its multiples are in $\hat{I}(B, B')$. All numbers in $\hat{I}(B)$ can be realised in this way.

Case 2. B is of the second kind.

This case is similar and simpler. First suppose that $\mathrm{O}(\gamma, \beta_j) = h(B)$ for all $\beta_j \in B$. Then γ gives rise to $J(B)$ and its multiples. Now suppose that $\mathrm{O}(\gamma, \beta_j) > h(B)$. Let us take B' as before. Then γ gives rise to numbers in $\hat{I}(B, B')$. All numbers in $\hat{I}(B)$ can be attained in this way.

Case 3. B is of the third kind.

The subtle case is where $\beta_i \in B^*$, $\mathrm{O}(\gamma, \beta_i) \geqslant h(B)$. Let us write

$$\mathrm{O}(\gamma, \beta_i) = \mathrm{char}(B^*) + \frac{\mu}{d_0 \cdots d_{k-1} d}, \quad d \geqslant 1.$$

Then

$$\begin{aligned}\mathrm{O}_y(f(\gamma(y), y)) &= \int_0^{h(B)} \mathcal{M}_B(r)\, \mathrm{d}r + (\mathrm{O}(\gamma, \beta_i) - h(B)) m(B^*)\\ &= \left\{ \int_0^{h(B)} \mathcal{M}_B(r)\, \mathrm{d}r - (h(B) - \mathrm{char}(B^*)) m(B^*) \right\}\\ &\quad + \frac{\mu}{d_0 \ldots d_{k-1} d} m(B^*).\end{aligned}$$

It follows that $\hat{J}(B) \subset A(f)$, and every number of $\hat{J}(B)$ can be realised in this way.

Now suppose that $\mathrm{O}(\gamma, \beta_j) = h(B)$ for all $\beta_j \in B$. If $\beta_i \notin B^*$, then we can replace it by some $\beta_j \in B$. The proof is therefore reduced to the previous case.

Finally, suppose that $\beta_i \notin B^*$ and $\mathrm{O}(\gamma, \beta_i) > h(B)$. This case is similar to Cases 1 and 2, γ leads to the numbers in $\hat{I}(B)$.

Acknowledgements

This work was done in the period when the first and second authors were visiting Sydney. They would like to thank the University of Sydney for its support and hospitality. All three authors are partially supported by an ARC Grant.

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Addendum. Recently we have noticed the striking works of Denef and Loeser (See [D-L] below) on the structure of the space of truncated arcs which are in contact with a complex algebraic variety. The invariants introduced by them are finer than that of Fukui. However, it is not yet clear to us whether or not they can be used to distinguish blow-analytic equivalence classes.

[D-L] Denef, J. and Loeser, F.: Geometry on arc spaces of algebraic varieties, NATO ASI/EC Summer School, *New Developements in Singularity Theory* (URL: http://www.newton.cam.ac.uk/programs/sgt_ws.html).

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