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Small Prime Solutions to Cubic Diophantine Equations II

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Abstract. Let $a_1, \ldots, a_9$ be non-zero integers and n any integer. Suppose that $a_1 + \cdots + a_9 \equiv n \pmod 2$ and $(a_i, a_j) = 1$ for $1 \le i < j \le 9$. In this paper we prove that

(i) if a_j are not all of the same sign, then the cubic equation $a_1 p_1^3 + \cdots + a_9 p_9^3 = n$ has prime solutions satisfying $p_j \ll |n|^{1/3} + \max\{|a_j|\}^{8+\varepsilon}$;

(ii) if all a_j are positive and $n \gg \max\{|a_j|\}^{25+\varepsilon}$, then $a_1 p_1^3 + \cdots + a_9 p_9^3 = n$ is soluble in primes p_j.

These results improve our previous results with the bounds $\max\{|a_j|\}^{14+\varepsilon}$ and $\max\{|a_j|\}^{43+\varepsilon}$ in place of $\max\{|a_j|\}^{8+\varepsilon}$ and $\max\{|a_j|\}^{25+\varepsilon}$ above, respectively.

1 Introduction

Let n be an integer, and let $a_1, \ldots, a_9$ be non-zero integers. We consider cubic equations in the form

$$a_1 p_1^3 + \cdots + a_9 p_9^3 = n, \tag{1.1}$$

where p_j are prime variables. A necessary condition for the solubility of (1.1) is

$$a_1 + \cdots + a_9 \equiv n \pmod 2. \tag{1.2}$$

We also suppose

$$(a_i, a_j) = 1, \quad 1 \le i < j \le 9, \tag{1.3}$$

and write $A = \max\{2, |a_1|, \ldots, |a_9|\}$. The main results in this paper are the following two theorems.

Theorem 1.1 *Suppose (1.2) and (1.3). If $a_1, \ldots, a_9$ are not all of the same sign, then (1.1) has solutions in primes p_j satisfying $p_j \ll |n|^{1/3} + A^{8+\varepsilon}$, where the implied constant depends only on ε.*

Theorem 1.2 *Suppose (1.2) and (1.3). If $a_1, \ldots, a_9$ are all positive, then (1.1) is soluble whenever $n \gg A^{25+\varepsilon}$, where the implied constant depends only on ε.*

Theorem 1.2 with $a_1 = \cdots = a_9 = 1$ is a classical result of Hua [3] in 1938. Theorems 1.1 and 1.2 improve our previous results in [4] with the bounds $A^{14+\varepsilon}$ and $A^{43} + \varepsilon$ in the place of $A^{8+\varepsilon}$ and $A^{25+\varepsilon}$, respectively. Our investigation on (1.1) is also motivated

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by the linear and quadratic relative problems. (See [1] and [2] and their references for the linear and quadratic relative problems, respectively).

Most of the arguments are similar to those in [4] and we therefore only sketch the proof in this note. We refer the reader to [4] for all the details and only emphasize the main difference between the arguments.

2 Outline of the Method

As in [4], we denote by $r(n)$ the weighted number of solutions of (1.1), *i.e.*,

$$r(n) = \sum_{\substack{n=a_1p_1^3+\cdots+a_9p_9^3 \\ M<|a_j|p_j^3\le N}} (\log p_1)\cdots(\log p_9),$$

where $M = N/200$. We will investigate $r(n)$ by the circle method. To this end, we set $N_j = (N/a_j)^{1/3}$, and

$$P = (N/A)^{3/13-\varepsilon}, \quad Q = N^{1-2\varepsilon}P^{-1}, \quad \text{and} \quad L = \log N. \tag{2.1}$$

By Dirichlet's lemma on rational approximation, each $\alpha \in [1/Q, 1+1/Q]$ may be written in the form

$$\alpha = a/q + \lambda, \quad |\lambda| \le 1/(qQ) \tag{2.2}$$

for some integers a, q with $1 \le a \le q \le Q$, and $(a,q) = 1$. We denote by $\mathfrak{M}(a,q)$ the set of α satisfying (2.2), and define the major arcs $\mathfrak{M}$ and the minor arcs $\mathfrak{m}$ as follows:

$$\mathfrak{M} = \mathfrak{M}(P) = \bigcup_{q\le P} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathfrak{M}(a,q), \quad \mathfrak{m} = \Big[\frac{1}{Q}, 1+\frac{1}{Q}\Big] \setminus \mathfrak{M}. \tag{2.3}$$

It follows from $2P \le Q$ that the major arcs $\mathfrak{M}(a,q)$ are mutually disjoint. Let

$$S_j(\alpha) = \sum_{M<|a_j|p^3\le N} (\log p)e(a_jp^3\alpha).$$

Then we have $r(n) = \int_0^1 S_1(\alpha)\cdots S_9(\alpha)e(-n\alpha)\,d\alpha = \int_{\mathfrak{M}} + \int_{\mathfrak{m}}$.

The integral on the major arcs $\mathfrak{M}$ causes the main difficulty, which is solved by Theorem 2.1 and Lemmas 2.3–2.4 in [4]. We state these here.

Theorem 2.1 *Assume (1.3). Let $\mathfrak{M}$ be as in (2.3) with P, Q determined by (2.1). Then we have*

$$\int_{\mathfrak{M}} S_1(\alpha)\cdots S_9(\alpha)e(-n\alpha)d\alpha = \frac{1}{3^9}\mathfrak{S}(n,P)\mathfrak{J}(n) + O\Big(\frac{N^2}{|a_1\cdots a_9|^{1/3}L}\Big),$$

where $\mathfrak{S}(n,P)$ and $\mathfrak{J}(n)$ are defined in (2.4) and (2.5), respectively.

To derive Theorem 2.1, we need to bound $\mathfrak{S}(n,P)$ and $\mathfrak{J}(n)$ from below. For χ mod q, we define

$$C(\chi,a) = \sum_{h=1}^{q} \overline{\chi}(h)e\Big(\frac{ah^3}{q}\Big), \quad C(q,a) = C(\chi^0,a).$$

If $\chi_1, \dots, \chi_9$ are characters $\mod q$, then we write

$$B(n, q, \chi_1, \dots, \chi_9) = \sum_{\substack{h=1 \\ (h,q)=1}}^{q} e\Big(-\frac{hn}{q}\Big) C(\chi_1, a_1 h) \cdots C(\chi_9, a_9 h),$$

$$B(n, q) = B(n, q, \chi^0, \dots, \chi^0), \quad A(n, q) = \frac{B(n, q)}{\varphi^9(q)},$$

and

$$\mathfrak{S}(n, P) = \sum_{q \le P} A(n, q). \tag{2.4}$$

Lemma 2.2 *Assuming (1.2), we have* $\mathfrak{S}(n, P) \gg (\log\log A)^{-c}$ *for some constant* $c > 0$.

Lemma 2.3 *Suppose (1.3) and*

(i) $a_1, \dots, a_9$ *are not all of the same sign and* $N \ge 27|n|$; *or*
(ii) $a_1, \cdots, a_9$ *are positive and* $n = N$.

Then we have

$$\mathfrak{J}(n) := \sum_{\substack{a_1 m_1 + \cdots + a_9 m_9 = n \\ M < |a_j| m_j \le N}} (m_1 \cdots m_9)^{-2/3} \asymp \frac{N^2}{|a_1 \cdots a_9|^{1/3}}. \tag{2.5}$$

Now we turn to the estimation of $\int_{\mathfrak{m}}$. In section 4, we will prove

$$\int_{\mathfrak{m}} |S_1(\alpha) \cdots S_9(\alpha)| d\alpha \ll \frac{N^{47/24+\varepsilon}}{|a_1 \cdots a_9|^{47/216}}.$$

Thus,

$$r(n) = \frac{1}{3^9} \mathfrak{S}(n, P) \mathfrak{J}(n) + O\left(\frac{N^2}{|a_1 \cdots a_9|^{1/3} L}\right) + O\Big(\frac{N^{47/24+\varepsilon}}{|a_1 \cdots a_9|^{47/216}}\Big).$$

Then we conclude that $r(n) \gg |a_1 \cdots a_9|^{-1/3} N^2 (\log\log N)^{-c}$, provided that

$$\frac{N^{47/24+\varepsilon}}{|a_1 \cdots a_9|^{47/216}} \ll \frac{N^2}{|a_1 \cdots a_9|^{1/3} L},$$

or equivalently $N \gg A^{25+\varepsilon}$. Theorems 1.1 and 1.2 follow from this and the argument leading in [4]. Details are therefore omitted.

3 Some Lemmas

We derive estimates for the generating functions appearing in the proof from estimates for the exponential sum

$$S(\alpha) = \sum_{X < p \le 2X} (\log p) e(\alpha p^3), \tag{3.1}$$

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which are given in terms of the rational approximation

$$\alpha = \frac{a}{q} + \lambda, \quad \text{with } 1 \le a \le q, \quad (a,q) = 1.$$

We start by quoting the result of Zhao [6].

Lemma 3.1 *Suppose that $\alpha \in \mathbb{R}$ and that exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying*

$$1 \le q \le Q, \quad (a,q) = 1, \quad |q\alpha - a| < Q^{-1}$$

with $X^{1/2} \le Q \le X^{5/2}$. Then for any fixed $\varepsilon > 0$,

$$S(\alpha) \ll X^{11/12+\varepsilon} + \frac{X^{1+\varepsilon}}{q^{1/6}\sqrt{(1 + X^3|\alpha - a/q|)}},$$

where the implied constant depends at most on k and ε.

The next lemma generalizes Lemma 3.1 to $S(b\alpha)$, with b a non-zero integer.

Lemma 3.2 *Let b be a non-zero integer and let $S(\alpha)$ be defined by (3.1). Suppose that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying*

$$1 \le q \le |b|X^3P^{-1}, \quad (a,q) = 1, \quad |q\alpha - a| < P/(|b|X^3), \tag{3.2}$$

with P subject to

$$2|b|X^{1/6} < P \le X. \tag{3.3}$$

Then for any fixed $\varepsilon > 0$, we have

$$S(b\alpha) \ll X^{11/12+\varepsilon} + X^{1+\varepsilon}q_1^{-1/6}\Phi(\alpha)^{-1/2}, \tag{3.4}$$

where $\Phi(\alpha) = 1 + |b|X^3|\alpha - a/q|$ and $q_1 = q/(b,q)$.

Proof By Dirichlet's theorem, there exist integers a_1 and q_1 such that

$$1 \le q_1 \le Q, \quad (a_1, q_1) = 1, \quad |q_1 b\alpha - a_1| < Q^{-1},$$

with some Q satisfying $X^{1/2} \le Q \le X^{5/2}$. Hence, by Lemma 3.1 with $\alpha = b\alpha$, $q = q_1$, and $a = a_1$,

$$S(b\alpha) \ll X^{11/12+\varepsilon} + \frac{X^{1+\varepsilon}}{q_1^{1/6}\sqrt{1 + X^3|q_1 b\alpha - a_1|}}. \tag{3.5}$$

If $q_1 > X^{1/2}$ or $|q_1 b\alpha - a_1| > X^{-17/6}$, the first term on the right-hand side of (3.5) dominates the second and (3.4) follows. Otherwise, recalling (3.2) and (3.3), we get

$$\begin{aligned} |q_1 ba - qa_1| &\le q_1|b||q\alpha - a| + q|q_1 b\alpha - a_1| \\ &\le PX^{-3/2} + |b|X^{1/6}P^{-1} < 1. \end{aligned}$$

Thus $\frac{a_1}{q_1} = \frac{ab}{q}$ and $q_1 = \frac{q}{(q,b)}$, and (3.5) turns into (3.4). ∎

The following lemma is Lemma 3.3 in [4] which generalizes Theorem 1.1 in [5].

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Lemma 3.3 *Let b be a non-zero integer and let $S(\alpha)$ be defined by (3.1). Suppose that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying*

$$1 \le q \le P, \quad (a,q) = 1, \quad |q\alpha - a| < P/(|b|X^3),$$

with $P < X/2$. Then for any fixed $\varepsilon > 0$, we have

$$S(b\alpha) \ll (X^{1/2}\Phi(\alpha)^{1/2} + X^{4/5} + X\Phi(\alpha)^{-1/2})q^{\varepsilon}\log^c X,$$

where $\Phi(\alpha) = q_1(1 + |b|X^3|\alpha - a/q|)$ and $q_1 = q/(b,q)$.

4 The Estimation of $\int_{\mathfrak{m}}$

Let N be a parameter with $N \ge A^{25+\varepsilon}$ that also satisfies hypothesis (i) or (ii) of Lemma 2.3 according as $a_1, \dots, a_9$ are all positive or not. Now we turn to the estimation of $\int_{\mathfrak{m}}$.

By Dirichlet's approximation theorem, when $\alpha \in \mathfrak{m}$, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying (3.2) with $b = a_9$ and $X = N_9$ such that $q + N_9|q\alpha - a| \ge P$.

We decompose the minor arcs into three parts, $\mathfrak{m} = \mathfrak{m}_1 \cup \mathfrak{m}_2 \cup \mathfrak{m}_3$, where

$$\begin{aligned}
\mathfrak{m}_1 &= \mathfrak{m} \cup \{q \le N_9^{1/2}|a_9| \text{ and } |\alpha - a/q| \le 1/(qN_9^{5/2})\},\\
\mathfrak{m}_2 &= \mathfrak{m} \cup \{q \ge N_9^{1/2}|a_9|\},\\
\mathfrak{m}_3 &= \mathfrak{m} \cup \{q \le N_9^{1/2}|a_9| \text{ and } |\alpha - a/q| \ge 1/(qN_9^{5/2})\}.
\end{aligned}$$

When $\alpha \in \mathfrak{m}_1$, using Lemma 3.3, we have

$$\begin{aligned}
S_9(\alpha) &\ll \Big(N_9^{1/2}\sqrt{q_1(1 + |a_9|N_9^3|\alpha - a/q|)} + N_9^{4/5}\\
&\qquad + \frac{N_9}{\sqrt{q_1(1 + |a_9|N_9^3|\alpha - a/q|)}}\Big) q^{\varepsilon}\log^c X\\
&\ll N_9^{1/2}q_1^{1/2} + N_9^{1/2} + N_9^{4/5} + \frac{N_9(q,|a_9|)^{1/2}}{\sqrt{q(1 + N|\alpha - a/q|)}}\\
&\ll N_9^{3/4}|a_9|^{1/2} + \frac{N_9(q,|a_9|)^{1/2}}{\sqrt{P}}\\
&\ll N_9^{11/12+\varepsilon}.
\end{aligned}$$

We apply Lemma 3.2 for $\alpha \in \mathfrak{m}_2$ and $\alpha \in \mathfrak{m}_3$,

$$\begin{aligned}
S_9(\alpha) &\ll N_9^{11/12+\varepsilon} + \frac{N_9^{1+\varepsilon}}{q_1^{1/6}\sqrt{1 + |a_9|N_9^3|\alpha - a/q|}}\\
&\ll N_9^{11/12+\varepsilon} + \frac{N_9^{1+\varepsilon}}{q_1^{1/6}}\\
&\ll N_9^{11/12+\varepsilon} + \frac{N_9^{1+\varepsilon}|a_9|^{1/6}}{q^{1/6}}\\
&\ll N_9^{11/12+\varepsilon},
\end{aligned}$$

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and

$$\begin{aligned} S_9(\alpha) &\ll N_9^{11/12+\varepsilon} + \frac{N_9^{1+\varepsilon}}{q_1^{1/6}\sqrt{1+|a_9|N_9^3|\alpha - a/q|}} \\ &\ll N_9^{11/12+\varepsilon} + \frac{N_9^{1+\varepsilon} q^{1/2} N_9^{5/4}}{q_1^{1/6}|a_9|^{1/2} N_9^{3/2}} \\ &\ll N_9^{11/12+\varepsilon} + \frac{N_9^{3/4+\varepsilon}(q,|a_9|)^{1/6} q^{1/2}}{q^{1/6}|a_9|^{1/2}} \\ &\ll N_9^{11/12+\varepsilon} + \frac{N_9^{3/4+\varepsilon}(q,|a_9|)^{1/6} q^{1/2}}{q^{1/6}|a_9|^{1/2}} \\ &\ll N_9^{11/12+\varepsilon} + \frac{N_9^{3/4+\varepsilon}|a_9|^{1/6} q^{1/2}}{q^{1/6}|a_9|^{1/2}} \\ &\ll N_9^{11/12+\varepsilon}. \end{aligned}$$

Thus, we have

$$\max_{\alpha\in\mathfrak{m}} |S_9(\alpha)| \ll N_9^{11/12+\varepsilon}. \tag{4.1}$$

We introduce the following notation: $T(t) = \int_{\mathfrak{m}} |S_9(\alpha)|^t d\alpha$ for $t \geq 1$. On considering the underlying equation and applying Hua's lemma (see [3])

$$T(8) \ll \int_0^1 |S_9(\alpha)|^8 d\alpha \ll L^8 \sum_{\substack{m_1^3+\cdots+m_4^3=m_5^3+\cdots+m_8^3 \\ m_\nu \leq N_9, \nu=1,\ldots,8}} 1 \ll N_9^{5+\varepsilon}.$$

Then by Schwartz's inequality,

$$T(9) \ll N_9^{5/2+\varepsilon} T(10)^{1/2}. \tag{4.2}$$

By applying Lemmas 2.2 and 3.1 in [6], we obtain

$$T(10) \ll N_9^{3/4+\varepsilon} T(16)^{1/4} T(9)^{1/2} + N_9^{7/8+\varepsilon} T(9). \tag{4.3}$$

We deduce from (4.1) that

$$T(16) \ll (N_9^{11/12+\varepsilon})^6 T(10). \tag{4.4}$$

Inserting (4.2) and (4.4) into (4.3), we have $T(10) \ll N_9^{27/8+\varepsilon} T(10)^{1/2}$, which implies

$$T(10) \ll N_9^{27/4+\varepsilon}.$$

This together with (4.2), we have $T(9) \ll N_9^{47/8+\varepsilon}$. Therefore,

$$\int_{\mathfrak{m}} |S_9(\alpha)|^9 \, d\alpha \ll N_9^{47/8+\varepsilon}.$$

Similarly, we have $\int_{\mathfrak{m}} |S_i(\alpha)|^9 d\alpha \ll N_i^{47/8+\varepsilon}, 1 \leq i \leq 8$.

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Therefore,

$$\begin{aligned}\int_{\mathfrak{m}} |S_1(\alpha)\cdots S_9(\alpha)|\,d\alpha &\ll \Big(\int_{\mathfrak{m}} |S_1(\alpha)|^9\,d\alpha\Big)^{1/9}\cdots\Big(\int_{\mathfrak{m}} |S_9(\alpha)|^9\,d\alpha\Big)^{1/9}\\ &\ll (N_1\cdots N_9)^{47/72+\varepsilon}\\ &\ll \frac{N^{47/24+\varepsilon}}{|a_1\cdots a_9|^{47/216}}.\end{aligned}$$

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