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A New Class of Representations of EALA Coordinated by Quantum Tori in Two Variables

To Professor R. V. Moody on his sixtieth birthday

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Abstract. We study the representations of extended affine Lie algebras $s\ell_{\ell+1}(\mathbb{C}_q)$ where q is N-th primitive root of unity ($\mathbb{C}_q$ is the quantum torus in two variables). We first prove that $\bigoplus s\ell_{\ell+1}(\mathbb{C})$ for a suitable number of copies is a quotient of $s\ell_{\ell+1}(\mathbb{C}_q)$. Thus any finite dimensional irreducible module for $\bigoplus s\ell_{\ell+1}(\mathbb{C})$ lifts to a representation of $s\ell_{\ell+1}(\mathbb{C}_q)$. Conversely, we prove that any finite dimensional irreducible module for $s\ell_{\ell+1}(\mathbb{C}_q)$ comes from above. We then construct modules for the extended affine Lie algebras $s\ell_{\ell+1}(\mathbb{C}_q) \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2$ which is integrable and has finite dimensional weight spaces.

Introduction

Extended Affine Lie Algebras (EALA) are higher-dimensional generalizations of affine Kac-Moody Lie algebras introduced in [HKT]. They have been further studied in [AABGP], [BGK] and [ABGP]. Toroidal Lie algebras which are universal central extensions of $\mathring{\mathfrak{g}} \otimes \mathbb{C}[t_1^{\pm}, \dots, t_n^{\pm}]$ ($\mathring{\mathfrak{g}}$ is a simple finite dimensional Lie algebra) are prime examples of EALAs which are studied by [F], [W], [MEY] [Y], [EF], [EM], [BS] and [BC]. There are many EALAs which allow not only the Laurent polynomial algebra $\mathbb{C}[t_1^{\pm}, \dots, t_n^{\pm}]$ as coordinate algebra but also quantum tori, Jordon tori and the octonion tori as coordinate algebra depending on the type of Lie algebra (see [AABGP], [BGK], [BGKN], [AG] and [Y0]). For example EALAs of type A_ℓ are tied up with the Lie algebra $g\ell_{\ell+1}(\mathbb{C}) \otimes \mathbb{C}_q$ where $\mathbb{C}_q$ is a quantum torus (see Section 2). Quantum tori defined in [M] are non-commutative analogues of Laurent polynomials. In this paper we will study representations of EALA $s\ell_{\ell+1}(\mathbb{C}_q) \subseteq g\ell_{\ell+1}(\mathbb{C}) \otimes \mathbb{C}_q$ and its universal central extension (see (5.2)) where $\mathbb{C}_q$ is quantum torus in two variables. Thus $\mathbb{C}_q$ is defined as algebra $\mathbb{C}_q[s_1^{\pm}, s_2^{\pm}]$ with the relation $s_1 s_2 = q s_2 s_1$. We will also add derivations d_1 and d_2 for $s\ell_{\ell+1}(\mathbb{C}_q)$, which is our prime object of study. See [BS], [G1] and [G2] for some interesting representation theory via vertex operator theory.

We will first develop (in Section 1) a one to one correspondence between irreducible modules for $s\ell_{\ell+1}(\mathbb{C}_q)$ and $s\ell_{\ell+1}(\mathbb{C}_q) \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2$ (see Theorem 1.6) which is very general and works for universal central extension.

So we construct modules for $s\ell_{\ell+1}(\mathbb{C}_q)$ (which can be lifted to $s\ell_{\ell+1}(\mathbb{C}_q) \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2$) where q is an N-th root of unity. In this case the $s\ell_{\ell+1}(\mathbb{C}_q)$ have some very interesting ideals. In [G2], [Z], it is proved that $g\ell_N(\mathbb{C})$ is a quotient of $\mathbb{C}_q$. We

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will generalize this to prove that $\bigoplus g\ell_N(\mathbb{C})$ is a quotient of $\mathbb{C}_q$ so that the Lie algebra $g\ell_{\ell+1}(\mathbb{C}) \otimes \mathbb{C}_q$ has $\bigoplus g\ell_{\ell+1}(\mathbb{C}) \otimes g\ell_N(\mathbb{C}) \cong \bigoplus g\ell_{N(\ell+1)}(\mathbb{C})$ as a quotient (Corollary 2.17). We will also note that $g\ell_{\ell+1}(\mathbb{C}) \otimes \mathbb{C}_q = s\ell_{\ell+1}(\mathbb{C}_q) \oplus Z$ where Z is central and a direct summand (see 2.11). Now any irreducible module for $\bigoplus g\ell_{N(\ell+1)}(\mathbb{C})$ is an irreducible module for $s\ell_{\ell+1}(\mathbb{C}_q)$. Conversely we prove in Theorem 3.13 that any finite-dimensional irreducible module for $s\ell_{\ell+1}(\mathbb{C}_q)$ comes from above.

1

We will fix some notation first. All our vector spaces are over the complex numbers $\mathbb{C}$. For a fixed integer n, let $A_n = \mathbb{C}[t_1^{\pm}, \dots, t_n^{\pm}]$ be Laurent polynomials in n commuting variables. For $\underline{r} = (r_1, \dots, r_n) \in \mathbb{Z}^n$, let $t^{\underline{r}} = t_1^{r_1} \cdots t_n^{r_n} \in A_n$. For any vector space V let $v(\underline{r}) = v \otimes t^{\underline{r}} \in V \otimes A_n$. Denote $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$ for integers, positive integers and non-negative integers respectively. For a Lie algebra $\mathfrak{g}_1$ let $U(\mathfrak{g}_1)$ denote the universal enveloping Lie algebra of $\mathfrak{g}_1$.

Let $\triangle$ be a root system coming from Kac-Moody Lie algebra.

Definition 1.1 A Lie algebra $\tilde{\mathfrak{g}}$ is called graded by $(\triangle, \mathbb{Z}^n)$ if the following holds:

(1)
$$\tilde{\mathfrak{g}} = \bigoplus_{\substack{\alpha \in \triangle \cup \{0\} \\ \underline{r} \in \mathbb{Z}^n}} \mathfrak{g}(\alpha, \underline{r})$$

(2) Let $\tilde{\underline{h}} = \mathfrak{g}(0,0) = \underline{h} \oplus D$, where D is the linear span of $d_1, \dots, d_n$ and $\triangle \subseteq \underline{h}^*$.

$$\mathfrak{g}(\alpha, \underline{r}) = \{X \in \tilde{\mathfrak{g}} \mid [h, X] = \alpha(h)X \text{ and } [d_i, X] = r_i X \text{ for all } h \in \underline{h} \text{ and } 1 \leq i \leq n\}.$$

Note that $[\mathfrak{g}(\alpha, \underline{r}), \mathfrak{g}(\beta, \underline{s})] \subseteq \mathfrak{g}(\alpha + \beta, \underline{r} + \underline{s})$ and $\underline{h}$ commutes with $\mathfrak{g}(0, \underline{r})$ for all $\underline{r} \in \mathbb{Z}^n$.

Examples 1.2

(1) Let $\mathfrak{g}_1$ be a Kac-Moody Lie algebra with root system $\triangle$. Then $\mathfrak{g}_1 \otimes A_n$ is a graded by $(\triangle, \mathbb{Z}^n)$. The Lie bracket is given by $[X(\underline{m}), Y(\underline{n})] = [X, Y](\underline{m} + \underline{n})$ and $[d_i, X(\underline{m})] = m_i X(\underline{m})$.
(2) The toroidal Lie algebra τ (as defined in [EM]), the universal central extension of $\mathfrak{g}_1 \otimes A_n$ is also graded by $(\triangle, \mathbb{Z}^n)$. $\mathfrak{g}_1$ is simple finite dimensional Lie algebra.
(3) The Extended Affine Lie Algebra (EALA) coordinated by quantum torus (see Definition 2.7) as given in [G1] and [G2].

Given a graded Lie algebra $\tilde{\mathfrak{g}}$, define $\mathfrak{g} = \bigoplus_{(\alpha, \underline{r}) \neq (0,0)} \mathfrak{g}(\alpha, \underline{r}) \oplus \underline{h}$, so that $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus D$. Let $\mathfrak{g}^+ = \bigoplus_{\substack{\alpha > 0 \\ \underline{r} \in \mathbb{Z}^n}} \mathfrak{g}(\alpha, \underline{r})$, $\mathfrak{g}^- = \bigoplus_{\substack{\alpha < 0 \\ \underline{r} \in \mathbb{Z}^n}} \mathfrak{g}(\alpha, \underline{r})$ and $H = \bigoplus_{\underline{r} \neq 0} \mathfrak{g}(0, \underline{r}) \oplus \underline{h}$ and $\tilde{H} = H \oplus D$. Then clearly $\mathfrak{g} = \mathfrak{g}^- \oplus H \oplus \mathfrak{g}^+$.

The purpose of this section is to develop a correspondence between $\mathfrak{g}$ modules and $\tilde{\mathfrak{g}}$ modules which are also highest weight modules.

Most often $\tilde{\mathfrak{g}}$ is "simple" but $\mathfrak{g}$ is not. The existence of ideals of $\mathfrak{g}$ will allow us to construct modules for $\mathfrak{g}$. Then by the following theory one can lift them to $\tilde{\mathfrak{g}}$.

We will first construct induced modules for $\tilde{\mathfrak{g}}$. Let $\tilde{W}$ be irreducible $\tilde{H}$ modules which is also the weight module for $\tilde{\underline{h}}$. Let $\mathfrak{g}^+$ act trivially on $\tilde{W}$. Then consider the induced module

$$M(\tilde{W}) = U(\tilde{\mathfrak{g}}) \otimes_{\tilde{H}\oplus\mathfrak{g}^+} \tilde{W}.$$

Lemma 1.3

(1) *$M(\tilde{W})$ is a weight module for $\tilde{\underline{h}}$.*
(2) *$M(\tilde{W})$ has a unique irreducible quotient called $V(\tilde{W})$.*

Proof (1) is standard.

(2) Let W_1 and W_2 be the proper $\tilde{\mathfrak{g}}$ submodules for $M(\tilde{W})$. Since $\tilde{W}$ is irreducible it is clear that $W_i \cap \tilde{W} = \{0\}$ for $i = 1, 2$.

Claim $(W_1 \oplus W_2) \cap \tilde{W} = \{0\}$. Since $\underline{h}$ commutes with $\tilde{H}$ and $\tilde{W}$ is an $\tilde{H}$ irreducible weight module, it follows that $\underline{h}$ acts on $\tilde{W}$ by a single linear function, say λ. Further $M(W)_\lambda = W$. Let $w = w_1 + w_2$ for $w \in \tilde{W}$ and $w_i \in W_i$ for $i = 1, 2$. Then w_1 and w_2 are λ weight. But W_1 and W_2 have no λ weights. Thus $w_1 = 0 = w_2$. This proves the claim.

Hence we proved that sum of two proper submodules is also proper. So $M(\tilde{W})$ has a unique maximal submodule, and hence a unique irreducible quotient. ■

We will now define a similar notion for $\mathfrak{g}$. Let W be an irreducible module for H. Then define an inducible $\mathfrak{g}$ module $M(W)$, where $\mathfrak{g}^+$ acts trivially in W. As in the earlier case, $M(W)$ has a unique irreducible quotient, say $V(W)$.

For an irreducible module W of H, define an $\tilde{H}$ module on $W \otimes A_n$ in the following way:

$$X \cdot v(\underline{s}) = (Xv)(\underline{r} + \underline{s}) \tag{1.4}$$

for all $X \in \mathfrak{g}(0, \underline{r})$, $\underline{r} \neq 0$ and for all $X \in \underline{h}$. $d_i v(\underline{r}) = r_i v(\underline{r})$. Note $\cdot$ denotes the $\sim$ action.

Assumption 1.5 We will make the following assumptions on the $\tilde{H}$ module $W \otimes A$ throughout this section.

(1) $W \otimes A = \bigoplus_{i=1}^k W_i$ as $\tilde{H}$ modules.
(2) Each W_i is an irreducible $\tilde{H}$ module.
(3) The sum in (1) is direct.

See [E1] for example where W is one dimensional.

Let us start with an irreducible module W of H which is a weight module for $\underline{h}$. Let $V(W)$ be the irreducible module for $\mathfrak{g}$ considered earlier. Define $\tilde{\mathfrak{g}}$ module structure on $V(W) \otimes A_n$ similar to (1.4). Now we will state our main theorem of this section.

Theorem 1.6

(1) $V(W) \otimes A_n = \bigoplus_{i=1}^{k} U(\tilde{\mathfrak{g}})W_i$ *as* $\tilde{\mathfrak{g}}$ *modules.*
(2) *Each* $U(\tilde{\mathfrak{g}})W_j$ *is an irreducible* $\tilde{\mathfrak{g}}$ *module.*
(3) *The sum in (1) is direct.*
(4) $V(W_j) \cong U(\tilde{\mathfrak{g}})W_j$ *as* $\tilde{\mathfrak{g}}$ *modules.*

Proof First note that $U(\tilde{\mathfrak{g}})W_j = U(\mathfrak{g}^-)W_j$, which is clear because $\mathfrak{g}^+$ kills W_j and $\tilde{H}$ leaves W_j invariant.

1.7 Note that $U(\tilde{\mathfrak{g}})W_j \cap W \otimes A_n = W_j$.

(1) Let $v(\underline{r}) \in V(W) \otimes A_n$, $v \in V(W)$, $\underline{r} \in \mathbb{Z}^n$. Let $w \in W$. Then there exists $X \in U(\mathfrak{g})$ such that $Xw = v$. Let $X = \sum X_{\underline{r}_i}$ such that $[d_j, X_{\underline{r}_i}] = (\underline{r}_i)_j X_{\underline{r}_i}$. Consider $\sum X_{\underline{r}_i} w(-\underline{r}_i + \underline{r}) = \sum (X_{\underline{r}_i} w)(\underline{r}) = v(\underline{r})$. Thus $v(\underline{r}) \in \sum_{j=1}^{k} U(\tilde{\mathfrak{g}})W_j$. This proves (1).

(2) Let $Xw(\underline{r}) \in U(\tilde{\mathfrak{g}})W_j$ where $w(\underline{r}) \in W_j$ and $X \in U(\mathfrak{g})_{(\beta,\underline{s})}$. It is sufficient to prove that there exists $Y \in U(\tilde{\mathfrak{g}})$ such that $YX \cdot w(\underline{r}) \in W_j$. Since $V(W)$ is an irreducible $\mathfrak{g}$ module there exists $Y \in U(\mathfrak{g})$ such that $YXw \in W$. Let $Y = \sum Y_{\underline{r}_i}$ where $[d_\ell, Y_{\underline{r}_i}] = (\underline{r}_i)_\ell Y_{\underline{r}_i}$. Note that for weight reasons each $Y_{\underline{r}_i} w \in W$. Consider $Y_{\underline{r}_i} X \cdot w(\underline{r}) = (Y_{\underline{r}} Xw)(\underline{r} + \underline{s} + \underline{r}_i)$ belongs to $W \otimes A$. At the same time $Y_{\underline{r}_i} X \cdot w(\underline{r}) \in U(\tilde{\mathfrak{g}})W_j$. Hence by 1.7 $Y_{\underline{r}_i} X \cdot w(\underline{r}) \in W_j$. So $YX \cdot w(\underline{r}) = \sum Y_{\underline{r}_i} X \cdot w(\underline{r}) \in W_j$ which proves (2).

(3) Suppose $U(\tilde{\mathfrak{g}})W_i \cap \sum_{j \neq i} U(\tilde{\mathfrak{g}})W_j \neq \{0\}$. Since $U(\tilde{\mathfrak{g}})W_i$ is irreducible, it follows that

$$U(\tilde{\mathfrak{g}})W_i \subseteq \sum_{j \neq i} U(\tilde{\mathfrak{g}})W_j.$$

By weight reasons it follows that $W_i \subseteq \sum_{j \neq i} W_j$, a contradiction to Assumption 1.5 (3).

(4) Now each W_j is an irreducible $\tilde{H}$ module. Both $V(W_j)$ and $U(\tilde{\mathfrak{g}})W_j$ are irreducible $\tilde{\mathfrak{g}}$ modules with the same top and hence they are isomorphic.

Theorem 1.8 *$V(W_j)$ has finite dimensional weight spaces with respect to $\underline{\tilde{h}}$ if and only if $V(W)$ has finite dimensional weight spaces with respect to $\underline{h}$.*

Proof Suppose $V(W)$ has finite dimensional weight spaces with respect to $\underline{h}$. Then clearly $V(W) \otimes A_n$ has finite dimensional weight spaces with respect to $\underline{\tilde{h}}$. Now by Theorem 1.6 (1) and (4) it follows that $V(W_j)$ has finite dimensional weight spaces with respect to $\underline{\tilde{h}}$.

For the converse, consider the $\mathfrak{g}$ module map φ from $V(W) \otimes A_n \to V$ given by $\varphi\big(v(\underline{r})\big) = v$. Note that $\varphi_{\underline{r}} := \varphi|_{V(W) \otimes t^{\underline{r}}}$ is a one to one and onto map and

$$V(W) \otimes t^{\underline{r}} = \Big(\bigoplus_{j=1}^{k} U(\tilde{\mathfrak{g}})W_j\Big)_{\underline{r}}. \tag{1.9}$$

Suppose $U(\tilde{\mathfrak{g}})W_j$ has finite dimensional weight space for all j. If $V(W)_\lambda$ is infinite dimensional, then $\Phi_r^{-1}\big(V(W)_\lambda\big) = \big(V(\psi) \otimes t^{\underline{r}}\big)_\lambda$ is infinite dimensional. Then by

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(1.9) at least for one j, λ weight space of $\big(U(\tilde{\mathfrak{g}})W_j\big)_{\underline{r}}$ is infinite dimensional which contradicts our supposition.

1.10 Thus to complete our theorem it is sufficient to prove that, if $U(\tilde{\mathfrak{g}})W_j$ has finite dimensional weight space for some j, so it is for all j.

Consider the H module map from $\overline{\varphi}\colon W \otimes A_n \to W$ given by $\overline{\varphi}\big(w(\underline{r})\big) = w$. Then clearly $\overline{\Phi}(W_j)$ is a non-zero submodule of W and hence equal to W. Now fix a non-zero vector w in W. Then there exists $\underline{r}_j \in \mathbb{Z}^n$ such that $w(\underline{r}_j) \in W_j$ and $\overline{\varphi}\big(w(\underline{r}_j)\big) = w$. Clearly $U(\tilde{H})w(r_j) = W_j$. Consider the H module map from $W_1 \to W_2$ by sending $w(r_1)$ to $w(r_2)$. This induces an isomorphism H module map. It need not be an $\tilde{H}$ module map. But it sends homogeneous spaces to homogeneous spaces. For example $(W_1)_{\underline{r}}$ goes to $(W)_{\underline{r}+\underline{r}_2-\underline{r}_1}$ injectively. Now the same is true for $U(\tilde{\mathfrak{g}})W_1$ and $U(\tilde{\mathfrak{g}})W_2$ by Theorem 1.6. This proves 1.10. Thus the proof of our theorem is completed.

We need the following lemma. The notation is as above.

Lemma 1.11 *Suppose V is an irreducible $\mathfrak{g}$ (respectively $\tilde{\mathfrak{g}}$) module generated by $\underline{h}$ (resp. $\underline{\tilde{h}}$) weight vector v such that $\mathfrak{g}^+ v = 0$. Then $U(H)v$ (resp. $U(\tilde{H})v$) is irreducible H (resp. $\tilde{H}$) module.*

Proof The proof is similar for $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$. Thus we will only prove the lemma for $\mathfrak{g}$. Let $w \in U(H)v$. First note that $\mathfrak{g}^+ w = 0$ for weight reasons. Since V is irreducible there exists $X \in U(\mathfrak{g})$ such that $Xw = v$. Write $X = X_- h_1 X_+$ where $X_\pm \in U(\mathfrak{g}^\pm)$ and $h_1 \in U(H)$. Then X_+ has to be scalar since otherwise it kills w. Then by weight reasons X_- has to be scalar. Thus $X = h_1$, which completes the lemma.

2

We first recall the definition of quantum torus from [BGK]. Fix a positive integer $n \geq 2$. Let $q = (q_{ij})$ be a matrix of $n \times n$ order where q_{ij} are non-zero complex numbers and $q_{ii} = 1$ and $q_{ij} = q_{ji}^{-1}$. Let J_q denote the ideal of the non-commutative Laurent polynomials ring $S_{[n]} = \mathbb{C}[s_1^{\pm}, \dots, s_n^{\pm}]$ generated by the elements $s_i s_j - q_{ij} s_j s_i$. We let $\mathbb{C}_q$ be the factor ring $S_{[n]}/J_q$. We again write $s_i \in \mathbb{C}_q$ for the image of s_i in $S_{[n]}$. So we have $s_i s_j = q_{ij} s_j s_i$. Then $\mathbb{C}_q$ is called the quantum torus associated to q. We define the following maps $\sigma, f\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{C}^*$ by

$$\sigma(\underline{a}, \underline{b}) = \prod_{i \leq i < j \leq n} q_{ji}^{a_j b_i} \tag{2.1}$$

$$f(\underline{a}, \underline{b}) = \sigma(\underline{a}, \underline{b})\sigma(\underline{b}, \underline{a})^{-1} \quad \text{for all } \underline{a}, \underline{b} \in \mathbb{Z}^n. \tag{2.2}$$

Then it is easy to check that

$$f(\underline{a}, \underline{b}) = \prod_{i,j=1}^{n} q_{ji}^{a_j b_i} f(\underline{a}, \underline{b}) = f(\underline{b}, \underline{a})^{-1} \quad \text{and} \quad f(\underline{a}, \underline{a}) = f(\underline{a}, -\underline{a}) = 1.$$

We also have the following.

$$\sigma(\underline{a}+\underline{b},\underline{c}) = \sigma(\underline{a},\underline{b})\sigma(\underline{a},\underline{c})\sigma(\underline{a},\underline{b}+\underline{c}) = \sigma(\underline{a},\underline{b})\sigma(\underline{a},\underline{c})f(\underline{a}+\underline{b},\underline{c})$$
$$= f(\underline{a},\underline{c})f(\underline{b},\underline{c})f(\underline{a},\underline{b}+\underline{c}) = f(\underline{a},\underline{b})f(\underline{a},\underline{c}).$$

Further

$$t^{\underline{a}}t^{\underline{b}} = \sigma(\underline{a},\underline{b})t^{\underline{a}+\underline{b}}t^{\underline{a}}t^{\underline{b}}(t^{\underline{a}})^{-1}(t^{\underline{b}})^{-1} = f(\underline{a},\underline{b}). \tag{2.4}$$

We define the radical f denoted by

$$\operatorname{rad}(f) = \{\underline{a} \in \mathbb{Z}^n \mid f(\underline{a},\underline{b}) = 1, \forall b \in \mathbb{Z}^n\}. \tag{2.5}$$

Note that $\operatorname{rad}(f)$ is a subgroup of $\mathbb{Z}^n$.

Proposition 2.6 **[BGK]** *Let $\mathbb{C}_q$ be as above.*

(1) *The center $\mathbb{Z}(\mathbb{C}_q)$ has basis consisting of monomials $t^{\underline{a}}, a \in \operatorname{rad} f$.*
(2) *The Lie algebra $[\mathbb{C}_q,\mathbb{C}_q]$ has basis consisting of monomial $t^{\underline{a}}, a \notin \operatorname{Rad}(f)$.*
(3) $\mathbb{C}_q = [\mathbb{C}_q,\mathbb{C}_q] \oplus \mathbb{Z}(\mathbb{C}_q)$.

Clearly $\mathbb{C}_q$ is $\mathbb{Z}^n$ graded with each graded component to be one dimensional. Let $M_{\ell+1}(\mathbb{C})$ be the matrix algebra with basis E_{ij} and multiplication $E_{ij}E_{k\ell} = \delta_{jk}E_{i\ell}$. We denote the corresponding Lie algebra as $g\ell_{\ell+1}(\mathbb{C})$. Let $s\ell_{\ell+1}(\mathbb{C})$ be the simple subalgebra of trace zero matrices.

Define the Extended Affine Lie Algebra (EALA) coordinated by the quantum torus:

$$s\ell_{\ell+1}(\mathbb{C}_q) = \{X \in M_{\ell+1}(\mathbb{C}_q) \mid \operatorname{trace} X \in [\mathbb{C}_q,\mathbb{C}_q]\}. \tag{2.7}$$

Here $M_{\ell+1}(\mathbb{C}_q)$ is the full matrix algebra with entries in $\mathbb{C}_q$. Trace $X = \sum X_{ii}$ where $X = (X_{ij})$.

Consider a new Lie algebra $I[\mathbb{C}_q,\mathbb{C}_q] \oplus s\ell_{\ell+1}(\mathbb{C}) \otimes \mathbb{C}_q$ where $I(s) = \sum_{i=1}^{\ell+1} E_{ii}(s)$ and $E_{ij}(s)$ is the matrix with (i,j) entry as s and zero every where. The Lie bracket is given by

$$[X \otimes t^{\underline{a}}, Y \otimes t^{\underline{b}}] = B(X,Y)I[t^{\underline{a}},t^{\underline{b}}] + [X,Y] \otimes \frac{t^{\underline{a}} o t^{\underline{b}}}{2} + (XoY) \otimes \left[\frac{t^{\underline{a}},t^{\underline{b}}}{2}\right] \tag{2.8}$$

where

$$[X,Y] = XY - YX$$
$$XoY = XY + YX - \frac{2}{(\ell+1)}\operatorname{Tr}(XY)I(1)$$
$$[t^{\underline{a}},t^{\underline{b}}] = t^{\underline{a}}t^{\underline{b}} - t^{\underline{b}}t^{\underline{a}}$$
$$t^{\underline{a}}0t^{\underline{b}} = t^{\underline{a}}\cdot t^{\underline{b}} + t^{\underline{b}}\cdot t^{\underline{a}} B(X,Y) = \frac{1}{\ell+1}\operatorname{Tr}(XY)$$
$$\big[I[t^{\underline{a}},t^{\underline{b}}], X \otimes t^{\underline{c}}\big] = X \otimes \big[[t^{\underline{a}},t^{\underline{b}}],t^{\underline{c}}\big]$$
$$\big[I[t^{\underline{a}},t^{\underline{b}}], I[t^{\underline{c}},t^{\underline{d}}]\big] = I\big[[t^{\underline{a}},t^{\underline{b}}],[t^{\underline{c}},t^{\underline{d}}]\big].$$

One can check that the above defines a Lie algebra.

Lemma 2.9 **[BGK]** $s\ell_{\ell+1}(\mathbb{C}_q) \cong I[\mathbb{C}_q,\mathbb{C}_q] \oplus s\ell_{\ell+1}(\mathbb{C}) \otimes \mathbb{C}_q$.

Now consider the $M_{\ell+1}(\mathbb{C}) \otimes \mathbb{C}_q$ as an associative algebra with associative product

$$X \otimes t^{\underline{a}} \cdot Y \otimes t^{\underline{b}} = XY \otimes t^{\underline{a}} \cdot t^{\underline{b}}. \tag{2.10}$$

The corresponding Lie algebra is denoted by $g\ell_{\ell+1}(\mathbb{C}) \otimes \mathbb{C}_q$.

Lemma 2.10 *$I[\mathbb{C}_q, \mathbb{C}_q] \oplus s\ell_{\ell+1}(\mathbb{C}) \otimes \mathbb{C}_q$ is a Lie subalgebra of $g\ell_{\ell+1}(\mathbb{C}) \otimes \mathbb{C}_q$ by the map $I(t^{\underline{a}}) = I \otimes t^{\underline{a}}$, $X \otimes t^{\underline{a}} = X \otimes t^{\underline{a}}$, $\underline{a} \in \mathbb{Z}^n$, $X \in s\ell_{\ell+1}(\mathbb{C})$.*

Proof By direct check the Lie brackets are compatible.

2.11 Also note the following $g\ell_{\ell+1} \otimes \mathbb{C}_q = s\ell_{\ell+1}(\mathbb{C}_q) \oplus I \otimes Z(\mathbb{C}_q)$ and $I \otimes Z(\mathbb{C}_q)$ is a direct summand and central.

2.12 Our interest is to construct and classify finite dimensional irreducible modules for $s\ell_{\ell+1}(\mathbb{C}_q)$. But we work with bigger Lie algebra $g\ell_{\ell+1} \otimes \mathbb{C}_q$. The additional central space acts as scalars and does not interfere with $s\ell_{\ell+1}(\mathbb{C}_q)$ modules.

2.13 From this point onwards we will assume that $n = 2$ and $q_{12} = q$. We will also assume that q is an N-th primitive root of unity.

We will first recall a certain isomorphism from [G2] and then generalize it.

Let E_{ij} be the matrix of order $N \times N$ such that the (i, j) entry is one and zero elsewhere. Let $E = E_{12} + E_{23} + \cdots + E_{N-1,N} + E_{N1}$.

$$F = \operatorname{diag}\{1, q, q^2, \ldots, q^{N-1}\}.$$

It is easy to verify the following: $EF = qFE$, $E^N = \mathrm{Id}$ and $F^N = \mathrm{Id}$. Let I be the ideal generated by $s_1^N - 1$, $s_2^N - 1$ (as associative algebras) inside $\mathbb{C}_q$.

Lemma 2.14 **[G2]** *The map $s_1^{i_1} s_2^{i_2} \to E^{i_1} F^{i_2}$ is isomorphic as associative algebra from $\mathbb{C}_q/I$ and $M_N(\mathbb{C})$.*

We will now generalize this. Let $a_1, \ldots, a_k, b_1, \ldots, b_\ell$ be distinct complex numbers such that $a_i^N \neq a_j^N$ and $b_i^N \neq b_j^N$ for all $i \neq j$. Consider the associative algebra homomorphism.

$$\pi \colon \mathbb{C}_q \longrightarrow \bigoplus M_N(\mathbb{C}) \quad (k\ell \text{ copies})$$
$$s_1^{i_1} s_2^{j_1} \longrightarrow (E_i^{i_1} F_j^{j_1})_{\substack{1 \le i \le k \\ 1 \le j \le \ell}}.$$

where $E_i = a_i E$ and $F_j = b_j F$. Let $P_1(s_1) = \prod_{i=1}^{k} (s_1^N - a_i^N)$ and $P_2(s_2) = \prod_{j=1}^{\ell} (s_2^N - b_j^N)$. Let J be the two sided ideal generated by $P_1(s_1)$ and $P_2(s_2)$ inside the $\mathbb{C}_q$ as associative algebras.

Proposition 2.15 *The homomorphism π induces an isomorphism*

$$\mathbb{C}_q / J \cong \bigoplus M_N(\mathbb{C}) \quad (k\ell \text{ copies}).$$

Proof It is easy to check that $\ker\pi \supseteq J$. Let F be the linear span of $\{s_1^{i_1}s_2^{j_1}, 1 \le i_1 \le kN, 1 \le j_1 \le kN\}$. This is clearly a spanning set for $\mathbb{C}_q/J$. To prove the proposition it is sufficient to prove that π is injective on F. (The subjectivity follows from dimensional reasons). Consider

$$\pi\Big(\sum_{\substack{1\le i_1\le kN\\ 1\le j_1\le kN}} a_{i_1 j_1} s_1^{i_1} s_2^{j_1}\Big) = 0.$$

Then clearly we have:

2.16 $\sum a_{i_1 j_1} a_i^{i_1} b_j^{j_1} E^{i_1} F^{j_1} = 0.$

Let p be an integer such that $1 \le p \le N$. First note that $F^{j_1} e_p = e_{\overline{p-j_1}}$. ($^{-}$ denotes the unique positive integer $\le N$ modulo N) e_p is the unit column with one at the p-th place and zero elsewhere. Fix integers m, n such that $1 \le m, n \le N$. Consider

$$E^{i_1} F^{j_1} e_p = q^{\overline{(p-n)}} e_{\overline{p-n}}$$

for $i_1 \equiv m(N)$ and $j_1 \equiv n(N)$. Then 2.16 becomes

$$\sum_{m=1}^{N}\Big(\sum_{\substack{i_1\equiv m(N)\\ j_1\equiv n(N)}} a_{i_1 j_1} a_i^{i_1} b_j^{j_1}\Big) q^{\overline{(p-n)m}} = 0.$$

The matrix $(q^{\overline{(p-n)m}})_{\substack{1\le n\le N\\ 1\le p\le N}}$ is invertible. Hence

$$\sum_{i_1=0}^{\ell-1}\sum_{j_1=0}^{k-1} a_{m+Ni_1, n+Nj_1} a_i^m b_j^n (a_i^N)^{i_1} (b_j^N)^{j_1} = 0.$$

By Lemma 3.11 of [E3] (see the Proof of the Lemma), $\big((a_i^N)^{i_1}(b_j^N)^{j_1}\big)_{\substack{0\le i_1\le N-1\\ 0\le j_1\le N-1}}$ is invertible. Hence

$$a_{m+Ni_1, m+Nj_1} = 0 \quad \forall m, n, i_1, j_1.$$

Thus the map π is injective on F.

Corollary 2.17 *$M_{\ell+1}(\mathbb{C})\otimes\mathbb{C}_q/J \cong \bigoplus M_{(\ell+1)N}(\mathbb{C})$ ($k\ell$ copies) as an associative algebra.*

Proof By Proposition 2.15 it is sufficient to prove that

$$g\ell_{\ell+1}(\mathbb{C}) \otimes M_N(\mathbb{C}) \cong M_{(\ell+1)N}(\mathbb{C}).$$

But this is a well known fact [RP, Corollary 9.3 (b)].

Corollary 2.18 *$\pi(I \otimes Z_q) = (A_1, \dots, A_{k\ell})$ where each matrix A_i is of order $(\ell+1)N$ and scalar.*

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Proof Note that from Proposition 2.6 we have $Z(\mathbb{C}_q) = \{s_1^{aN} s_2^{bN}, a, b \in \mathbb{Z}\}$. Consider $\pi(I \otimes s_1^{aN} s_2^{bN}) = (a_i^a b_j^b I_{\ell+1} \otimes I_N)_{\substack{1 \le i \le \ell \\ 1 \le j \le k}}$. Note that $I_{\ell+1} \otimes I_N = I_{(\ell+1)N}$, where I_j is the identity matrix of order j, so that one inclusion follows. The other inclusion follows from the fact that the matrix $(a_i^a b_j^b)_{\substack{1 \le i, a \le \ell \\ 1 \le j, b \le k}}$ is invertible. (See the proof of Lemma 3.11 of [E3]).

Corollary 2.19 *$\pi\big(s\ell_{\ell+1}(\mathbb{C}_q)\big) = \bigoplus s\ell_{N(\ell+1)}(\mathbb{C})$, ($k\ell$ copies) follows from 2.11 and Corollary 2.17.*

2.20 Thus we have a surjective map π (again denoted by π) from $s\ell_{\ell+1}(\mathbb{C}_q)$ to $\bigoplus s\ell_{(\ell+1)N}(\mathbb{C})$ such that $\ker \pi = s\ell_{\ell+1}(\mathbb{C}_q) \cap J$ where J is the ideal generated by $P_1(s_1)$ and $P_2(s_2)$ inside $\mathbb{C}_q$. Let $\mathfrak{g}_1 = \bigoplus s\ell_{(\ell+1)N}(\mathbb{C})$ ($k\ell$ copies), which is a finite dimensional semisimple Lie algebra. Let $\bigoplus h_1$ be the cartan subalgebra which is direct sum of cartan subalgebra h_1 of $s\ell_{(\ell+1)N}(\mathbb{C})$.

2.21 Any $h_{(\ell+1)N}$ weight module V of $\mathfrak{g}_1$ can be lifted to $s\ell_{\ell+1}(\mathbb{C}_q)$ via π given in 2.20. By letting the center $I \otimes Z(\mathbb{C}_q)$ act as scalars on V, we get a module for $g\ell_{\ell+1}(\mathbb{C}) \otimes \mathbb{C}_q$.

3

As earlier q is an N-th primitive root of unity and $\mathbb{C}_q[s_1^{\pm}, s_2^{\pm}] = \mathbb{C}_q$.

Fix a cartan subalgebra h_1 of $s\ell_{\ell+1}(\mathbb{C})$. Then $h_1 \otimes 1$ can be treated as Cartan subalgebra of $s\ell_{\ell+1}(\mathbb{C}_q)$. We are interested in studying irreducible modules for $s\ell_{\ell+1}(\mathbb{C}_q)$ which are finite dimensional weight spaces for $h_1 \otimes 1$.

$$\text{Let } Q_1 = \prod_{i=1}^{k} (s_1^N - a_i^N) \quad \text{and}$$

$$Q_2 = \prod_{j=1}^{\ell} (s_2^N - b_j^N)$$

where $a_1, \dots, a_k, b_1, \dots, b_\ell$ are non-zero complex numbers. (No condition on N-th powers being distinct). Let $j(Q_1, Q_2)$ be a two-sided ideal generated by Q_1 and Q_2 inside $\mathbb{C}_q$. Let $j_1(Q_1, Q_2) = s\ell_{\ell+1}(\mathbb{C}_q) \cap g\ell_{\ell+1} \otimes j(Q_1, Q_2)$.

Proposition 3.1 *Let V be an irreducible module for $s\ell_{\ell+1}(\mathbb{C}_q)$ with finite dimensional weight spaces for $h_1 \otimes 1$. Then there exist polynomials Q_1, Q_2 in s_1^N and s_2^N such that V is a module for $s\ell_{\ell+1}(\mathbb{C}_q)/J_1(Q_1, Q_2)$.*

Proof Let V_λ be a finite dimensional weight space. Consider $H = h_1 \otimes \mathbb{C}_q[s_1^{\pm N}, s_2^{\pm N}]$ which is an abelian subalgebra and leaves V_λ invariant. Thus there exists a common eigenvector v for H. That is:

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3.2 $h_1 \otimes s_1^{aN} s_2^{bN} v = \lambda_{a,b}(h_1)v$, for all $a, b \in \mathbb{Z}$, $\forall h_1 \in h_1$. Now fix integers ℓ_1, ℓ_2 such that $0 \leq \ell_1 < N$ and $0 \leq \ell_2 < N$. Consider the vectors for $i \neq j$.

$$E_{ij} s_1^{\ell_1} s_2^{\ell_2} s_2^{Ns} v, \quad \text{for } s \in \mathbb{Z}.$$

Let α be the root of $s\ell_{\ell+1}(\mathbb{C})$ corresponding to root vector E_{ij}. Then the above vectors belong to the weight space $V_{\lambda+\alpha}$ which is finite dimensional. Thus there exist polynomials $P_{ij}^{\ell_1 \ell_2}$ in s_2^N such that

$$E_{ij} s_1^{\ell_1} s_2^{\ell_2} P_{ij}^{\ell_1 \ell_2}(s_2^N) v = 0.$$

Now apply $(E_{ii} - E_{jj}) s_1^{Ns} s_2^{Nt}$ to the above vector. From 3.2 we get the following:

3.3 $E_{ij} s_1^{\ell+1Ns} s_2^{\ell+2Nt} P_{ij}^{\ell_1 \ell_2}(s_2^N) v = 0$.

Let $P(s_2^N) = \prod_{\substack{i \neq j \\ 0 \leq \ell_1 < N \\ 0 \leq \ell_2 < N}} P_{ij}^{\ell_1 \ell_2}(s_2^N)$ which is a polynomial in s_2^N. Now from 3.3 we have the following:

3.4 $E_{ij} s_1^k s_2^\ell P(s_2^N) v = 0$ for all $k, \ell \in \mathbb{Z}$ and $i \neq j$. For any integer k, let $\overline{k_1}$ be such that $k \equiv \overline{k}(N)$ and $1 \leq \overline{k} \leq N$.

Case 1 Let k, ℓ be integers such that $(\overline{k}, \overline{\ell}) \neq (N, N)$. Let k_1, ℓ_1, k_2, ℓ_2 be integers such that $k_1 + k_2 = k$ and $\ell_1 + \ell_2 = \ell$. Let $Q^1 = \prod_{(s,t,i_1,j_1) \neq (\overline{k_1}, \overline{\ell_1}, i, j)} P_{\ell_1, j_1}$. From 3.3 we have $E_{ij} s_1^{k_1} s_2^{ii} P_{ij}^{\overline{k_1}\,\overline{\ell_1}} v = 0$ and $E_{ji} s_1^{k_1} s_2^{\ell 2} Q^1 v = 0$. Hence their commutation on v is also zero. This will give:

$$q^{\ell_1 k_2} E_{ii} s_1^k s_2^\ell P v - q^{\ell_2 k_1} E_{jj} s_1^k s_2^\ell P v = 0. \tag{3.5}$$

Choose $(k_1, \ell_1) = (0, 0)$ in 3.5. So that we have:

$$(E_{ii} s_1^k t_2^\ell P - E_{jj} s_1^k s_2^\ell P) v = 0. \tag{3.6}$$

Now multiply 3.6 by $q^{\ell_1 k_2}$ and subtract from 3.5 to get:

$$E_{jj} s_1^k s_2^\ell P (1 - q^{\ell_1 k_2 - \ell_2 k_1}) q^{\ell_2 k_1} v = 0. \tag{3.7}$$

Claim There exists k_1, k_2, ℓ_1, ℓ_2 such that $k_2 \ell_1 - k_1 \ell_2 \not\equiv 0(N)$. Suppose not. Take $(k_1, k_2) = (1, 0)$ and $(0, 1)$ to conclude that $k \equiv 0(N)$ and $\ell \equiv 0(N)$ a contradiction to our assumption that $(\overline{k}, \overline{\ell}) \neq (N, N)$. Thus the claim is true.

Hence from 3.7 and the claim we conclude that:

$$E_{jj} s_1^k s_2^\ell P v = 0 \quad \text{for all } \ell, k \text{ such that } (\overline{\ell}, \overline{k}) \neq (N, N) \text{ and for all } j. \tag{3.8}$$

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Case 2 $(k,\ell)=(sN,tN)$. Then by an argument similar to above we have:

$$(E_{ii}-E_{jj})s_1^k s_2^\ell P v = 0 \quad \forall \ell \text{ and } k. \tag{3.9}$$

But we cannot get 3.8 this way.

Let $\mathfrak{g}_p$ be the liner span of $E_{ij}s_1^\ell s_2^k P$, $E_{ii}s_1^\ell s_2^k P - E_{jj}s_1^\ell s_2^k P$ (for $i \neq j$ and for all $k,\ell \in \mathbb{Z}$) and $E_{jj}s_1^k s_2^\ell \big((\bar{k},\bar{\ell}) \neq (N,N)\big)$ for all j. Then from 3.4, 3.6 and 3.8 we get

$$\mathfrak{g}_p v = 0.$$

The same argument will produce a polynomial Q in s_1^N such that

$$\mathfrak{g}_Q v = 0.$$

Note that each $\mathfrak{g}_p$ and $\mathfrak{g}_Q$ are ideals in $s\ell_{\ell+1}(\mathbb{C}_q)$. Now consider $W=\{w \in V; \mathfrak{g}_p w = 0 = \mathfrak{g}_Q w\}$ which is a non-zero submodule of V and hence $V=W$. This proves the proposition. ■

Continuing with the notation in the above proof let

$$P=\prod_{i=1}^{k}(s_2^N-b_i)^{k_i}, \qquad Q=\prod_{j=1}^{\ell}(s_1^N-a_j)^{\ell_i}$$

where $a_1\cdots a_\ell$ and $b_1\cdots b_k$ are distinct complex numbers. Consider

$$P^1=\prod_{i=1}^{k}(s_2^N-b_i), \qquad Q^1=\prod_{j=1}^{\ell}(s_1^N-a_j).$$

3.10 Consider the quotient map

$$\Phi\colon s\ell_{\ell+1}(\mathbb{C}_q)/\mathfrak{g}_p \oplus \mathfrak{g}_Q \longrightarrow \frac{s\ell_{\ell+1}(\mathbb{C}_q)}{\mathfrak{g}_{P^1}\oplus \mathfrak{g}_{Q^1}}.$$

Lemma 3.11 $\ker\Phi$ *is solvable.*

Trivial checking is all that is necessary.

Proposition 3.12 *Let V be* finite *dimensional module for $s\ell_{\ell+1}(\mathbb{C}_q)$. Then there exist polynomials P^1 and Q^1 in s_2^N and s_1^N with distinct roots, such that V is a module for $s\ell_{\ell+1}(\mathbb{C}_q)/\mathfrak{g}_{P^1}\oplus\mathfrak{g}_{Q^1}$.*

Proof In view of Proposition 3.1 and the map at 3.10, it is sufficient to prove that the solvable ideal $\ker\varphi$ at 3.10 acts trivially on V. Since $\ker\varphi$ is solvable and V is finite dimensional there exists a vector v in V such that $\ker\varphi$ acts as scalar. By the argument similar to Proposition 2.1 of [E2] we conclude that $\ker\varphi v=0$. In any case, the non-zero roots act trivially as can be seen from dimensional reasons.

Consider $W=\{v\in V; \ker\varphi v=0\}$ a non-zero submodule of V. Since V is irreducible, $V=W$.

Theorem 3.13 *Let V be a finite dimensional irreducible module for $s\ell_{\ell+1}(\mathbb{C}_q)$. Then V comes from a lift of π as in 2.21. That is, V is a module for $\bigoplus s\ell_{(\ell+1)N}(\mathbb{C})$.*

Proof Follows from Proposition 3.12 and Corollary 2.19.

The same can be said for an irreducible module of $g\ell_{\ell+1}(\mathbb{C}) \otimes \mathbb{C}_q$ as the additional central element acts as a scalar. The restriction to $s\ell_{\ell+1}(\mathbb{C}_q)$ is still irreducible. The following remarks can be made for the modules factoring through π.

Remark 3.14 (1) A highest weight module for $\bigoplus s\ell_{(\ell+1)N}(\mathbb{C})$ is certainly a highest weight module for $s\ell_{\ell+1}(\mathbb{C}_q)$. The converse need not be true.

(2) A weight module for $\bigoplus s\ell_{(\ell+1)N}(\mathbb{C})$ with finite dimensional weight spaces need not be a finite dimensional weight module for $s\ell_{\ell+1}(\mathbb{C}_q)$. The cartan of $s\ell_{\ell+1}(\mathbb{C}_q)$ is too small.

We will now apply our Theorem 1.6 to the Lie algebra $\tilde{\mathfrak{g}} = s\ell_{\ell+1}(\mathbb{C}_q) \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2$ which is the main object of study. Let $\mathfrak{g} = s\ell_{\ell+1}(\mathbb{C}_q)$. $\underline{h} = h_1 \otimes 1$ sitting inside $s\ell_{\ell+1}(\mathbb{C}_q)$ where h_1 is the diagonal matrices with trace zero inside $s\ell_{\ell+1}(\mathbb{C})$.

$$H = \bigoplus_{i \neq j} (E_{ii} - E_{jj}) s_1^{i_1} s_2^{i_2} \bigoplus_{\substack{i_1 \equiv 0(N) \text{ or} \\ i_2 \equiv 0(N)}} I \otimes s_1^{i_1} s_2^{i_2}.$$

Let π as defined in Corollary 2.19. Then $\pi(h_1 \otimes s_1^{i_1} s_2^{Ns}) = \sum a_i^{i_1} b_j^{N_s} E^{i_1} Id$, which is an element of $\bigoplus h_{(\ell+1)N}$. From [G2] it follows that

$$\pi^{-1}\Big(\bigoplus h_{(\ell+1)N}\Big) = \sum_{\substack{i_1 \in \mathbb{Z} \\ s \in \mathbb{Z}}} h \otimes s_1^{i_1} s_2^{Ns} \bigoplus_{i_1 \not\equiv 0(N)} I \otimes s_1^{i_1} s_2^{N_s}.$$

Let V be a finite dimensional irreducible module for $s\ell_{\ell+1}(\mathbb{C}_q)$. Then V is a module for $\bigoplus s\ell_{(\ell+1)N}(\mathbb{C})$ (Theorem 3.13 via the map π). Then V has a highest weight vector in V. The Cartan subalgebra $\bigoplus h_{(\ell+1)N}$ acts as by a linear function λ. Consider the H submodule $W = U(H)v$ which is irreducible from Lemma 1.11. Let $W \otimes A$ be an $\tilde{H}$ module as defined in (1.4). Define

$$S_1 = \{i_1 \in \mathbb{Z} \mid h_1 \otimes e^{i_1} v \neq 0 \text{ for some } h_1 \text{ or } I \otimes E^{i_1} v \neq 0 \text{ for } i_1 \equiv 0(N)\}.$$

Note that $i_1 \in S_1$ implies $i_1 + kN \in S_1$ for all k. Let G be the semigroup generated by S_1.

Lemma 3.15 *G is a sub group of $\mathbb{Z}$.*

Proof We want to show that for each i_1 in G there is inverse in G. By the note we can assume that $0 < i_1 < N$. Let i_1 be the minimal with that property. Let k be a positive integer such that $ki_1 < N$ and $(k+1)i_1 \geq N$. Then $0 \leq (k+1)i_1 - N$ belongs to G. Suppose $(k+1)i_1 - N = 0$ then $ki_1 - N$ is in the inverse. Suppose $(k+1)i_1 - N > 0$. Then, by the minimality of i_1, $(k+1)i_1 - N \geq i_1$, we have $ki_1 - N \geq 0$, a contradiction.

Now take $j_1 \in G$ such that $0 < j_1 < N$. Then we can see that j_1 is a multiple of i_1 and hence has an inverse.

Let $\tilde{G} = G \times N\mathbb{Z}$.

Lemma 3.16 *$v(\underline{r}) \in U(\tilde{H})v(\underline{s})$ iff $\underline{r} - \underline{s} \in \tilde{G}$.*

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Proof Clear from the definition of G and s_1.

Lemma 3.17 *Let $\underline{r}_1, \dots, \underline{r}_k$ be the coset representations of $\tilde{G}$ inside $\mathbb{Z}^2$ then*

(1) $V \otimes W = \bigoplus_{i=1}^k U(\tilde{H})v(\underline{r}_i)$;
(2) *each $U(\tilde{H})v(\underline{r}_i)$ is irreducible;*
(3) *the sum in (1) is direct.*

Proof Let $w(\underline{r}) \in U(\tilde{H})v(\underline{r}_i)$. Suppose w is a multiple of v. Then $\underline{r} - \underline{r}_i \in \tilde{G}$. As $\tilde{G}$ is group, $\underline{r}_i - \underline{r} \in \tilde{G}$. Then again by Lemma 3.16 there exists $X \in U(\tilde{H})$ such that $X(\underline{r}) = v(\underline{r}_i)$. Now suppose w is a weight vector of weight $\lambda - \beta$, $\beta \neq 0$. Then there exists $Y \ni Yw = v$, $Y = \sum Y_{\underline{s}_i}$ where each $Y_{\underline{s}_i}$ is of degree $\underline{s}_i$. By weight argument each $Y_{\underline{s}_i} w$ is a multiple of v. By the earlier case (2) is proved.

(1) Let $w(\underline{r}) \in V \otimes W$. There exist $X \ni Xv = w$. Write $X = \sum X_{\underline{s}_i}$ where degree $X_{\underline{s}_i}$ is $\underline{s}_i$. Then

$$\sum X_{s_1} v(-s_i + \underline{r}) = \sum (X_{s_i} v)(\underline{r}) = w(\underline{r})$$

which proves (2).

(3) Suppose $U(\tilde{H})v(\underline{r}_i) \cap \sum_{i \neq j} U(\tilde{H})v(\underline{r}_j)$. Then by irreducibility we have

$$U(\tilde{H})v(\underline{r}_i) \subseteq \sum_{j \neq i} U(\tilde{H})v(\underline{r}_j).$$

By a weight argument $v(\underline{r}_i)$ is a linear confirmation of $v(s)$ such that $\underline{s} - r_i \in \tilde{G}$. But by the choice of $\underline{r}_i$, this is not possible. ■

Now Assumption 1.5 for $W \otimes A$ is satisfied.

Theorem 3.18 *Let V be a finite dimensional irreducible module for $\bigoplus s\ell_{(\ell+1)N}(\mathbb{C})$. Then $V \otimes W$ as $s\ell_{\ell+1}(\mathbb{C}_q) \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_i$ decomposes into finitely many irreducible modules with finite dimensional weight spaces.*

Proof We will apply Theorem 1.6 for $\tilde{\mathfrak{g}} = s\ell_{\ell+1}(\mathbb{C}_q) \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2$. By Lemma 3.17, Assumption 1.5 is satisfied. By Theorem 1.8 each component of $V \otimes W$ has finite dimensional weight spaces.

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