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APPLICATION OF THE τ-FUNCTION THEORY OF PAINLEVÉ EQUATIONS TO RANDOM MATRICES: $\mathrm{P_{VI}}$, THE JUE, CyUE, cJUE AND SCALED LIMITS

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Abstract. Okamoto has obtained a sequence of τ-functions for the $\mathrm{P_{VI}}$ system expressed as a double Wronskian determinant based on a solution of the Gauss hypergeometric equation. Starting with integral solutions of the Gauss hypergeometric equation, we show that the determinant can be re-expressed as multidimensional integrals, and these in turn can be identified with averages over the eigenvalue probability density function for the Jacobi unitary ensemble (JUE), and the Cauchy unitary ensemble (CyUE) (the latter being equivalent to the circular Jacobi unitary ensemble (cJUE)). Hence these averages, which depend on four continuous parameters and the discrete parameter N, can be characterised as the solution of the second order second degree equation satisfied by the Hamiltonian in the $\mathrm{P_{VI}}$ theory. We show that the Hamiltonian also satisfies an equation related to the discrete $\mathrm{P_V}$ equation, thus providing an alternative characterisation in terms of a difference equation. In the case of the cJUE, the spectrum singularity scaled limit is considered, and the evaluation of a certain four parameter average is given in terms of the general $\mathrm{P_V}$ transcendent in σ form. Applications are given to the evaluation of the spacing distribution for the circular unitary ensemble (CUE) and its scaled counterpart, giving formulas more succinct than those known previously; to expressions for the hard edge gap probability in the scaled Laguerre orthogonal ensemble (LOE) (parameter a a non-negative integer) and Laguerre symplectic ensemble (LSE) (parameter a an even non-negative integer) as finite dimensional combinatorial integrals over the symplectic and orthogonal groups respectively; to the evaluation of the cumulative distribution function for the last passage time in certain models of directed percolation; to the τ-function evaluation of the largest eigenvalue in the finite LOE and LSE with parameter $a = 0$; and to the characterisation of the diagonal-diagonal spin-spin correlation in the two-dimensional Ising model.

§1. Introduction and summary

1.1. Setting and objectives

This paper is the last in a series devoted to a systematic account of the application of the Okamoto τ-function theory of Painlevé equations to

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the characterisation of certain averages in random matrix theory. The τ-function theory applies directly to random matrix ensembles defined by a probability density function (PDF) of the form

$$\frac{1}{C}\prod_{j=1}^{N} w(x_j) \prod_{1\le j<k\le N}(x_k - x_j)^2, \tag{1.1}$$

where the weight function $w(x)$ is one of the classical forms

$$w(x) = \begin{cases} e^{-x^2}, & \text{Gaussian} \\ x^a e^{-x} \quad (x>0), & \text{Laguerre} \\ x^a(1-x)^b \quad (0<x<1), & \text{Jacobi} \\ (1+x^2)^{-\eta}, & \text{Cauchy} \end{cases} \tag{1.2}$$

The symbol C, which in (1.1) denotes the normalisation, will be used throughout to denote *some* constant (i.e. quantity independent of the primary variables of the equation). The PDFs (1.1) can be realised as the joint eigenvalue distribution of Hermitian matrices with independent complex Gaussian entries for the first three weights of (1.2), and by a stereographic projection of random unitary matrices for the Cauchy weight (with $\eta = N$) (see (1.16) below). The underlying matrix distributions giving rise to (1.1) are invariant with respect to similarity transformations involving unitary matrices, and for this reason are termed matrix ensembles with a unitary symmetry or simply unitary ensembles (not to be confused with unitary matrices). The name of the weight function is then prefixed to the term unitary ensemble. In this work our interest is in the Jacobi unitary ensemble (JUE) and the Cauchy unitary ensemble (CyUE). We will see that the Cauchy unitary ensemble is equivalent to the circular Jacobi unitary ensemble (cJUE) in which the eigenvalues are on the unit circle in the complex plane. The averages of interest are

$$\widetilde{E}_N(s;\mu) := \Big\langle \prod_{l=1}^{N} \chi^{(l)}_{(-\infty,s]}(s-x_l)^{\mu} \Big\rangle, \quad \chi^{(l)}_{(-\infty,s]} = \begin{cases} 1, & x_l \in (-\infty,s] \\ 0, & \text{otherwise} \end{cases} \tag{1.3}$$

and

$$F_N(s;\mu) := \Big\langle \prod_{l=1}^{N}(s-x_l)^{\mu} \Big\rangle \tag{1.4}$$

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(in the latter, for $\mu \notin \mathbb{Z}$ a suitable branch must be specified).

For $N = 1$ (1.3) and (1.4) read

$$\widetilde{E}_1(s;\mu) = \int_{-\infty}^{s} (s-x)^{\mu} w(x)\,dx, \quad F_1(s;\mu) = \int_{-\infty}^{\infty} (s-x)^{\mu} w(x)\,dx. \tag{1.5}$$

A fundamental fact is that as a function of s, and after multiplication by a suitable elementary function of s, these functions satisfy a classical second order linear differential equation (Hermite-Weber equation in the Gaussian case [29], confluent hypergeometric equation in the Laguerre case [30], and, as will be shown below, Gauss hypergeometric equation in the Jacobi and Cauchy cases). In the Okamoto τ-function theory of $\mathrm{P_{IV}}$ [53], $\mathrm{P_{V}}$ [55] and $\mathrm{P_{VI}}$ [54], these same linear differential equations respectively characterise the first member of an infinite sequence of τ-functions (the zeroth member, as with the random matrix averages (1.3) and (1.4), is unity). The general Nth member of the τ-function sequence is characterised by the fact that its logarithmic derivative satisfies a second order second degree differential equation. Furthermore the Nth member can be written explicitly as a Wronskian determinant and we have shown in the Gaussian case in [29], and in the Laguerre case in [30], that the Wronskian determinant is just a rewrite of the average (1.3) (or (1.4) as appropriate). This way we have been able to characterise (1.3) and (1.4) in terms of the solution of second order second degree differential equations from the Painlevé theory. Bäcklund transformations of quantities associated with the Hamiltonian formalism of the particular Painlevé system have also allowed us to characterise these quantities in terms of difference equations related to discrete Painlevé equations. In this work we will show that the same strategy allows (1.3) and (1.4) in the Jacobi and Cauchy cases to be characterised similarly. Scaled limits of these averages can then be characterised as solutions of limiting forms of the differential equations. Applications are given to the exact evaluation of eigenvalue spacing distribution functions, probabilities of last passage times in directed percolation models, and the diagonal spin-spin correlation of the two-dimensional Ising model.

1.2. Definition of averages in the various ensembles

In the Jacobi case the explicit form of (1.3) and (1.4) is

$$\widetilde{E}_N^{\mathrm{J}}(s;a,b,\mu) = \frac{1}{C}\int_0^s dx_1\, x_1^a (1-x_1)^b (s-x_1)^{\mu} \cdots \tag{1.6}$$

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$$\times \int_0^s dx_N\, x_N^a (1-x_N)^b (s-x_N)^\mu \prod_{1\le j<k\le N} (x_k-x_j)^2$$

$$= s^{N(a+\mu+N)} \frac{1}{C} \int_0^1 dx_1\, x_1^a (1-sx_1)^b (1-x_1)^\mu \cdots$$

$$\times \int_0^1 dx_N\, x_N^a (1-sx_N)^b (1-x_N)^\mu \prod_{1\le j<k\le N} (x_k-x_j)^2$$

$$F_N^{\mathrm{J}}(s;a,b,\mu) = \frac{1}{C} \int_0^1 dx_1\, x_1^a (1-x_1)^b (s-x_1)^\mu \cdots \tag{1.7}$$

$$\times \int_0^1 dx_N\, x_N^a (1-x_N)^b (s-x_N)^\mu \prod_{1\le j<k\le N} (x_k-x_j)^2,$$

while the explicit form of (1.3) and (1.4) in the Cauchy case is

$$\widetilde{E}_N^{\mathrm{Cy}}(s;\eta,\mu) = \frac{1}{C} \int_{-\infty}^s dx_1\, \frac{(s-x_1)^\mu}{(1+x_1^2)^\eta} \cdots \int_{-\infty}^s dx_N\, \frac{(s-x_N)^\mu}{(1+x_N^2)^\eta} \times \prod_{1\le j<k\le N} (x_k-x_j)^2 \tag{1.8}$$

$$F_N^{\mathrm{Cy}}(s;\eta,\mu) = \frac{1}{C} \int_{-\infty}^\infty dx_1\, \frac{(s-x_1)^\mu}{(1+x_1^2)^\eta} \cdots \int_{-\infty}^\infty dx_N\, \frac{(s-x_N)^\mu}{(1+x_N^2)^\eta} \times \prod_{1\le j<k\le N} (x_k-x_j)^2. \tag{1.9}$$

A single quantity which combines both (1.6) and (1.7) is

$$\widetilde{E}_N^{\mathrm{J}}(s;a,b,\mu;\xi) := \frac{1}{C} \Big(\int_0^1 -\xi \int_s^1 \Big) dx_1\, x_1^a (1-x_1)^b (s-x_1)^\mu \cdots \tag{1.10}$$

$$\times \Big(\int_0^1 -\xi \int_s^1 \Big) dx_N\, x_N^a (1-x_N)^b (s-x_N)^\mu \prod_{1\le j<k\le N} (x_k-x_j)^2.$$

Similarly, a single quantity which combines both (1.8) and (1.9) is

$$\widetilde{E}_N^{\mathrm{Cy}}(s;\eta,\mu;\xi) := \frac{1}{C} \Big(\int_{-\infty}^\infty -\xi \int_s^\infty \Big) dx_1\, \frac{(s-x_1)^\mu}{(1+x_1^2)^\eta} \cdots \tag{1.11}$$

$$\times \Big(\int_{-\infty}^\infty -\xi \int_s^\infty \Big) dx_N\, \frac{(s-x_N)^\mu}{(1+x_N^2)^\eta} \prod_{1\le j<k\le N} (x_k-x_j)^2.$$

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As will be revised in Section 5.1, the power series expansions in ξ of (1.10) and (1.11) are quantities of relevance to conditional spacing distributions in the corresponding random matrix ensembles.

The integrand in (1.11) requires η to be real for itself to be real. However, if we write

$$(1+x^2)^{-\eta} \longmapsto (1+ix)^{-\eta}(1-ix)^{-\bar{\eta}},$$

where $\bar{\eta}$ denotes the complex conjugate of η, then the integrand remains real for η complex, giving a meaningful generalisation of the original Cauchy ensemble [14]. Doing this, and setting $\eta = \eta_1 + i\eta_2$ we can generalise (1.11) to read

$$(1.12)\quad \begin{aligned} &\widetilde{E}_N^{\mathrm{Cy}}(s;(\eta_1,\eta_2),\mu;\xi) \\ &\quad := \frac{1}{C}\Big(\int_{-\infty}^{\infty} -\xi\int_s^{\infty}\Big)dx_1\,\frac{(s-x_1)^{\mu}}{(1+ix_1)^{\eta_1+i\eta_2}(1-ix_1)^{\eta_1-i\eta_2}}\cdots \\ &\times\Big(\int_{-\infty}^{\infty} -\xi\int_s^{\infty}\Big)dx_N\,\frac{(s-x_N)^{\mu}}{(1+ix_N)^{\eta_1+i\eta_2}(1-ix_N)^{\eta_1-i\eta_2}}\prod_{1\le j<k\le N}(x_k-x_j)^2. \end{aligned}$$

It was remarked below (1.4) that for $(s-x_l)^{\mu}$ to be well defined for $\mu \notin \mathbb{Z}$ a definite branch must be specified. For s real a natural choice is that $(s-x_l)^{\mu}$ is real for $x_l < s$ and $(s-x_l)^{\mu} = e^{-\pi i\mu}|x_l - s|^{\mu}$ for $x_l > s$. In particular this shows

$$(1.13)\quad \begin{aligned} \widetilde{E}_N^{\mathrm{J}}(s;a,b,\mu;\xi) &:= \frac{1}{C}\Big(\int_0^1 -\xi^*\int_s^1\Big)dx_1\,x_1^a(1-x_1)^b|s-x_1|^{\mu}\cdots \\ &\times\Big(\int_0^1 -\xi^*\int_s^1\Big)dx_N\,x_N^a(1-x_N)^b|s-x_N|^{\mu}\prod_{1\le j<k\le N}(x_k-x_j)^2, \end{aligned}$$

where $\xi^* := 1-(1-\xi)e^{-\pi i\mu}$ and similarly for (1.11) and (1.12).

In the opening paragraph it was noted that the Cauchy ensemble results from a stereographic projection of the eigenvalue PDF for random unitary matrices. To be more explicit, following [71], consider the ensemble of random unitary matrices specified by the eigenvalue PDF

$$(1.14)\qquad \frac{1}{C}\prod_{l=1}^{N}|1+z_l|^{2\omega_1}\prod_{1\le j<k\le N}|z_k-z_j|^2, \quad z_l = e^{i\theta_l},\quad \theta_l\in[-\pi,\pi].$$

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In [71] this was referred to as the circular Jacobi ensemble with unitary symmetry, and denoted cJUE. When $\omega_1 = 0$ (1.14) is realised by random unitary matrices with the uniform (Haar) measure and is then referred to as the circular unitary ensemble (CUE), while (1.14) with $\omega_1 = 1$ gives the eigenvalue PDF of $(N+1)\times(N+1)$ CUE matrices with all angles θ measured from any one eigenvalue (taken to be $\theta = \pi$). Making the change of variable

$$e^{i\theta} = \frac{1+ix}{1-ix}, \quad x = \tan\frac{\theta}{2} \tag{1.15}$$

(note that $\theta = \pm\pi$ corresponds to $x \to \pm\infty$) shows

$$\begin{aligned}\prod_{l=1}^{N} |1+z_l|^{2\omega_1} \prod_{1\le j<k\le N} |z_k - z_j|^2 \, d\theta_1 \cdots d\theta_N \\ = 2^{N(N+2\omega_1)} \prod_{l=1}^{N} \frac{1}{(1+x_l^2)^{N+\omega_1}} \prod_{1\le j<k\le N} |x_j - x_k|^2 \, dx_1 \cdots dx_N,\end{aligned} \tag{1.16}$$

thus specifying the relation with the Cauchy unitary ensemble. A generalisation of the cJUE eigenvalue PDF (1.14) is the PDF

$$\frac{1}{C}\prod_{l=1}^{N} e^{\omega_2\theta_l}|1+z_l|^{2\omega_1} \prod_{1\le j<k\le N} |z_k - z_j|^2, \quad z_l = e^{i\theta_l}, \quad \theta_l \in [-\pi,\pi]. \tag{1.17}$$

Under the change of variable (1.15) this transforms into the PDF

$$\frac{1}{C}\prod_{l=1}^{N} \frac{1}{(1+ix_l)^{\omega_1+i\omega_2+N}(1-ix_l)^{\omega_1-i\omega_2+N}} \prod_{1\le j<k\le N} |x_k - x_j|^2$$

which corresponds to the generalised Cauchy probability density in (1.12). Consequently, changing variables according to (1.15) in

$$\widetilde{E}_N^{\mathrm{cJ}}(\phi;(\omega_1,\omega_2),\mu;\xi) := \Big\langle \prod_{l=1}^{N} (1-\xi\chi^{(l)}_{(\pi-\phi,\pi)}) e^{\omega_2\theta_l} |e^{i(\pi-\phi)} - e^{i\theta_l}|^{\mu} \Big\rangle_{\mathrm{cJUE}}. \tag{1.18}$$

and making use of (1.16) and the analogue of (1.13) for $\widetilde{E}_N^{\mathrm{cJ}}$ shows

$$\begin{aligned}&\widetilde{E}_N^{\mathrm{cJ}}(\phi;(\omega_1,\omega_2),\mu;\xi^*) \\ &\quad\propto \frac{1}{(1+s^2)^{N\mu/2}} \widetilde{E}_N^{\mathrm{Cy}}(s;(N+\omega_1+\mu/2,\omega_2),\mu;\xi)\Big|_{s=\cot\phi/2}.\end{aligned} \tag{1.19}$$

The second Jacobi ensemble integral in (1.6) shares the feature of the Cauchy ensemble integral (1.11) of being related to an average in the CUE. This is done using the integral identity [24]

$$\begin{aligned}(1.20)\quad \int_0^1 dt_1\, t_1^{\epsilon-1} \cdots \int_0^1 dt_N\, t_N^{\epsilon-1}\, f(t_1,\dots,t_N) &= \Big(\frac{\pi}{\sin\pi\epsilon}\Big)^N \\ &\times \int_{-1/2}^{1/2} dx_1\, e^{2\pi i\epsilon x_1} \cdots \int_{-1/2}^{1/2} dx_N\, e^{2\pi i\epsilon x_N}\, f(-e^{2\pi i x_1},\dots,-e^{2\pi i x_N})\end{aligned}$$

valid for f a Laurent polynomial, and making use of Carlson's theorem. One finds

(1.21)
$$\Big\langle \prod_{l=1}^N (1-tx_l)^\mu \Big\rangle_{\mathrm{JUE}} = \frac{M_N(0,0)}{M_N(a',b')} \Big\langle \prod_{l=1}^N z_l^{(a'-b')/2} |1+z_l|^{a'+b'} (1+tz_l)^\mu \Big\rangle_{\mathrm{CUE}}$$

where $a' = N+a+b$, $b' = -(N+a)$ and

$$\begin{aligned}(1.22)\quad M_N(a',b') := \int_{-1/2}^{1/2} dx_1 \cdots \int_{-1/2}^{1/2} dx_N \prod_{l=1}^N z_l^{(a'-b')/2} |1+z_l|^{a'+b'} \\ \times \prod_{1\le j<k\le N} |z_j - z_k|^2, \quad z_l := e^{2\pi i x_l}\end{aligned}$$

(note that the left hand side of (1.21) corresponds to the second integral in (1.6) after interchanging μ and b). The normalisation (1.22) results from the Jacobi ensemble normalisation

(1.23)
$$J_N(a,b) := \int_0^1 dx_1\, x_1^a (1-x_1)^b \cdots \int_0^1 dx_N\, x_N^a (1-x_N)^b \prod_{1\le j<k\le N} (x_k - x_j)^2.$$

Both (1.22) and (1.23) have gamma function evaluations,

$$(1.24)\qquad M_N(a,b) = \prod_{j=0}^{N-1} \frac{\Gamma(a+b+1+j)\Gamma(2+j)}{\Gamma(a+1+j)\Gamma(b+1+j)}$$

$$(1.25)\qquad J_N(a,b) = \prod_{j=0}^{N-1} \frac{\Gamma(a+1+j)\Gamma(b+1+j)\Gamma(2+j)}{\Gamma(a+b+1+N+j)},$$

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with the latter following from the Selberg integral [62], and the former by applying (1.20) to the evaluation (1.25).

The Jacobi average (1.10) and the circular Jacobi average (1.18) (which according to (1.19) is equivalent to the Cauchy average (1.11)) admit various scaled $N \to \infty$ limits. There are three distinct possibilities in that the limiting averages can correspond either to the soft edge, hard edge or a spectrum singularity in the bulk. The averages at the soft and hard edge have been studied in our previous papers [29], [30], giving rise to $\mathrm{P_{II}}$ and $\mathrm{P_{III}}$ transcendents respectively, and will not be discussed here. Instead, attention will be focussed on the scaling to a spectrum singularity in the bulk. This results by replacing $X \mapsto X/N$ in (1.18), making a suitable choice of the constant C, and taking the $N \to \infty$ limit. The problem with C can be avoided by taking the logarithmic derivative with respect to X, leading us to consider the scaled quantity

$$(1.26)\quad u(X;(\omega_1,\omega_2),\mu;\xi) := \lim_{N\to\infty} X\frac{d}{dX} \times \log\Big\langle \prod_{l=1}^{N}\big(1-\xi^*\chi^{(l)}_{(\pi-X/N,\pi)}\big)e^{\omega_2\theta_l}|1+z_l|^{2\omega_1}|e^{i(\pi-X/N)}-z_l|^{2\mu}\Big\rangle_{\mathrm{CUE}}$$

(d/dX is multiplied by X for later convenience).

1.3. Summary of the characterisation of the averages as Painlevé σ-functions

Previous studies [65], [35], [72], [12] have characterised $\widetilde{E}^{\mathrm{J}}_N(s;a,b,\mu = 0;\xi)$ in terms of the solution of nonlinear equations related to $\mathrm{P_{VI}}$. The nonlinear equations obtained have been third order [65], [35] and second order second degree [35], [72], [12]. By the work of Cosgrove and Scoufis [16], these equations are equivalent, and are in fact examples of the so called Jimbo-Miwa-Okamoto σ-form of the $\mathrm{P_{VI}}$ differential equation,

$$(1.27)\quad h'\Big(t(1-t)h''\Big)^2 + \Big(h'[2h-(2t-1)h'] + b_1b_2b_3b_4\Big)^2 = \prod_{k=1}^{4}(h'+b_k^2).$$

This fact is most explicit in the work of Borodin and Deift [12] who have shown that

$$(1.28)\quad \sigma(t) := -t(t-1)\frac{d}{dt}\log\widetilde{E}^{\mathrm{J}}_N(1-t;b,a,0;\xi) + b_1b_2t + \frac{1}{2}(-b_1b_2+b_3b_4)$$

with

$$b_1 = b_2 = N + \frac{a+b}{2}, \quad b_3 = \frac{a+b}{2}, \quad b_4 = \frac{a-b}{2} \tag{1.29}$$

satisfies (1.27). The quantity (1.11) in the case $\mu = 0$ has been similarly characterised. In particular, we know from [71] that

$$\sigma(s) := (1+s^2)\frac{d}{ds}\log \widetilde{E}_N^{\mathrm{Cy}}(s;(a+N,0),0;\xi) \tag{1.30}$$

satisfies the equation

$$\begin{aligned}(1+s^2)^2(\sigma'')^2 + 4(1+s^2)(\sigma')^3 - 8s\sigma(\sigma')^2 + 4\sigma^2(\sigma'-a^2)& \\ + 8a^2 s\sigma\sigma' + 4\big[N(N+2a) - a^2s^2\big](\sigma')^2 = 0&.\end{aligned} \tag{1.31}$$

To relate (1.31) to (1.27), change variables $t \mapsto (is+1)/2$, $h(t) \mapsto \frac{i}{2}h(s)$ in the latter so it reads

$$h'\Big((1+s^2)h''\Big)^2 + 4\Big(h'(h-sh') - ib_1b_2b_3b_4\Big)^2 + 4\prod_{k=1}^{4}(h'+b_k^2) = 0. \tag{1.32}$$

With

$$h = \sigma - a^2 s, \quad \mathbf{b} = (-a, 0, N+a, a), \tag{1.33}$$

(1.32) reduces to (1.31).

In this paper we generalise the $\mathrm{P_{VI}}$ σ-function characterisations (1.28) and (1.30). Consider first the Jacobi case. In Proposition 13 we show that

$$\widehat{U}_N^{\mathrm{J}}(t;a,b,\mu;\xi) := e_2'[\hat{\mathbf{b}}]t - \frac{1}{2}e_2[\hat{\mathbf{b}}] + t(t-1)\frac{d}{dt}\log \widetilde{E}_N^{\mathrm{J}}(t;a,b,\mu;\xi) \tag{1.34}$$

with

$$\hat{\mathbf{b}} = \Big(\frac{1}{2}(a+b)+N, \frac{1}{2}(b-a), -\frac{1}{2}(a+b), -\frac{1}{2}(a+b) - N - \mu\Big) \tag{1.35}$$

satisfies (1.27) (the quantities $e_2'[\hat{\mathbf{b}}]$ and $e_2[\hat{\mathbf{b}}]$ are defined in Proposition 1). This characterisation can be made unique in the cases $\xi = 0, 1$, when we have the boundary conditions

$$\widehat{U}_N^{\mathrm{J}}(t;a,b,\mu;0) \underset{|t|\to\infty}{\sim} (e_2'[\hat{\mathbf{b}}] + N\mu)t + O(1) \tag{1.36}$$

$$\begin{aligned}\widehat{U}_N^{\mathrm{J}}(t;a,b,\mu;1) \underset{t\to 0}{\sim} -\frac{1}{2}e_2[\hat{\mathbf{b}}] - N(a+\mu+N)& \\ + \Big(e_2'[\hat{\mathbf{b}}] + N(N+a+\mu) + bN\frac{a+N}{a+\mu+2N}\Big)t&\end{aligned} \tag{1.37}$$

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((3.12) and (3.11)). Also, when $\mu = 0$ with ξ general we have

$$\begin{aligned}\widehat{U}_N^{\mathrm{J}}(t; a, b, 0; \xi) - e_2'[\hat{\mathbf{b}}]t + \frac{1}{2}e_2[\hat{\mathbf{b}}] &\underset{t\to 1^-}{\sim} \xi(t-1)\rho^{\mathrm{J}}(t) \\ \rho^{\mathrm{J}}(t) \underset{t\to 1^-}{\sim} (1-t)^b \frac{\Gamma(a+b+N+1)\Gamma(b+N+1)}{\Gamma(N)\Gamma(a+N)\Gamma(b+1)\Gamma(b+2)} &\end{aligned} \tag{1.38}$$

where $\rho^{\mathrm{J}}(t)$ denotes the eigenvalue density in the JUE. Regarding the Cauchy case, in Proposition 15 we show

$$U_N^{\mathrm{Cy}}(t; (\eta, 0), \mu; \xi) = (t^2+1)\frac{d}{dt}\log\Big((t^2+1)^{e_2'[\mathbf{b}]/2}\widetilde{E}_N^{\mathrm{Cy}}(t; (\eta, 0), \mu; \xi)\Big), \tag{1.39}$$

with

$$\mathbf{b} = (N-\eta, 0, \eta, -\mu+\eta-N) \tag{1.40}$$

satisfies (1.32). Proposition 15 in fact contains the characterisation of the more general Cauchy average (1.12). It follows from this and (1.19) that

$$\begin{aligned}\widetilde{E}_N^{\mathrm{cJ}}(\phi; (\omega_1, \omega_2), \mu; \xi^*) &= \frac{M_N(\omega_1 - i\omega_2 + \mu/2, \omega_1 + i\omega_2 + \mu/2)}{M_N(\omega_1, \omega_1)} \\ &\times \exp\Big\{-\frac{1}{2}\int_0^\phi \Big(U_N^{\mathrm{Cy}}\big(\cot\frac{\theta}{2}; (\omega, \omega_2), \mu; \xi\big) + i(e_2'[\mathbf{b}] - e_2[\mathbf{b}]) \\ &\qquad - (e_2'[\mathbf{b}] + N\mu)\cot\frac{\theta}{2}\Big)\Big|_{\omega = N+\omega_1+\mu/2} d\theta\Big\}\end{aligned} \tag{1.41}$$

(eq. (3.25)) where $U_N^{\mathrm{Cy}}(t; (\eta_1, \eta_2), \mu; \xi)$ satisfies (1.32) with

$$\mathbf{b} = (N-\eta_1, i\eta_2, \eta_1, -\mu+\eta_1-N).$$

The scaled limit $\phi \mapsto X/N$, $N \to \infty$ of the logarithmic derivative of (1.41) is essentially (1.26). This gives rise to the differential equation [55]

$$(th_V'')^2 - (h_V - th_V' + 2(h_V')^2)^2 + 4\prod_{k=1}^{4}(h_V' + v_k) = 0 \tag{1.42}$$

(cf. (1.27)) where

$$v_1 + v_2 + v_3 + v_4 = 0 \tag{1.43}$$

satisfied by a particular auxiliary Hamiltonian h_V in the τ-function of $\mathrm{P_V}$. We show (Proposition 19) that the scaled average (1.26) is such that

$$h(t) = u(it; (\omega_1, \omega_2), \mu; \xi) + \frac{i\omega_2}{2} t + 2\omega_1 \mu + \frac{\omega_2^2}{2}$$

satisfies (1.42) with

$$\tilde{v}_1 = \mu,\ \tilde{v}_2 = -\mu,\ \tilde{v}_3 = \omega,\ \tilde{v}_4 = -\bar{\omega},\ \tilde{v}_j := v_j + \frac{i\omega_2}{2},\ \omega := \omega_1 + i\omega_2. \tag{1.44}$$

§2. Overview of the Okamoto τ-function theory of $\mathrm{P_{VI}}$

2.1. The Jimbo-Miwa-Okamoto σ-form of $\mathrm{P_{VI}}$

The sixth Painlevé equation $\mathrm{P_{VI}}$ reads

$$\begin{aligned} q'' = \frac{1}{2}\Big(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t}\Big)(q')^2 - \Big(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\Big)q' \\ + \frac{q(q-1)(q-t)}{t^2(t-1)^2}\Big(\alpha + \frac{\beta t}{q^2} + \frac{\gamma(t-1)}{(q-1)^2} + \frac{\delta t(t-1)}{(q-t)^2}\Big). \end{aligned} \tag{2.1}$$

It has been known since the work of Malmquist in the early 1920's [43] that (2.1) can be obtained by eliminating p from a Hamiltonian system

$$q' = \frac{\partial H}{\partial p}, \quad p' = -\frac{\partial H}{\partial q}. \tag{2.2}$$

In the notation of [41] the required Hamiltonian can be written

$$\begin{aligned} t(t-1)H = q(q-1)(q-t)p^2 - \big[\alpha_4(q-1)(q-t) + \alpha_3 q(q-t) \\ + (\alpha_0 - 1)q(q-1)\big]p + \alpha_2(\alpha_1 + \alpha_2)(q-t), \end{aligned} \tag{2.3}$$

where the parameters $\alpha_0, \ldots, \alpha_4$ in (2.3) are inter-related by

$$\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1 \tag{2.4}$$

and are related to the parameters $\alpha, \ldots, \delta$ in (2.1) by

$$\alpha = \frac{1}{2}\alpha_1^2, \quad \beta = -\frac{1}{2}\alpha_4^2, \quad \gamma = \frac{1}{2}\alpha_3^2, \quad \delta = \frac{1}{2}(1 - \alpha_0^2). \tag{2.5}$$

One sees that the Hamiltonian can be written as an explicit rational function of the $\mathrm{P_{VI}}$ transcendent and its derivative. This follows from the fact that with the substitution (2.3), the first of the Hamilton equations is linear in p, so p can be written as a rational function of q, q' and t. The sought form of H then follows by substituting this expression for p in (2.3).

After the addition of a certain linear function in t, the Hamiltonian (2.3) has the crucial feature for random matrix applications of satisfying the second order, second degree differential equation (1.27).

Proposition 1. ([38], [54]) *Rewrite the parameters* $\alpha_0, \ldots, \alpha_4$ *of* (2.3) *in favour of the parameters*

$$\begin{aligned} b_1 &= \frac{1}{2}(\alpha_3+\alpha_4), & b_2 &= \frac{1}{2}(\alpha_4-\alpha_3), \\ b_3 &= \frac{1}{2}(\alpha_0+\alpha_1-1), & b_4 &= \frac{1}{2}(\alpha_0-\alpha_1-1), \end{aligned} \tag{2.6}$$

and introduce the auxiliary Hamiltonian h *by*

$$\begin{aligned} h &= t(t-1)H + e_2'[\mathbf{b}]t - \frac{1}{2}e_2[\mathbf{b}] \\ &= t(t-1)H + (b_1b_3+b_1b_4+b_3b_4)t - \frac{1}{2}\sum_{1\le j<k\le 4} b_jb_k, \end{aligned} \tag{2.7}$$

where $e_j'[\mathbf{b}]$ *denotes the* j*th degree elementary symmetric function in* b_1, b_3 *and* b_4 *while* $e_j[\mathbf{b}]$ *denotes the* j*th degree elementary symmetric function in* $b_1, \ldots, b_4$. *The auxiliary Hamiltonian satisfies the Jimbo-Miwa-Okamoto* σ*-form of* $\mathrm{P_{VI}}$, (1.27).

Proof. Following [54], we note from (2.3), (2.2) and (2.6) that

$$h' = -q(q-1)p^2 + \{b_1(2q-1)-b_2\}p - b_1^2, \tag{2.8}$$

or equivalently

$$q(q-1)(h'+b_1^2) = -\Big(q(q-1)p\Big)^2 + \Big(b_1(2q-1)-b_2\Big)q(q-1)p. \tag{2.9}$$

We see from (2.8) and (2.9) that a differential equation for h will result if we can express q and $q(q-1)p$ in terms of h and its derivatives. For this purpose we note from (2.7) and (2.8) that

$$h - th' = q(-h' + e_2'[\mathbf{b}]) - (b_3+b_4)q(q-1)p - \frac{1}{2}e_2[\mathbf{b}], \tag{2.10}$$

while differentiation of this formula and use of the Hamilton equations shows

$$t(t-1)h'' = 2q(e_1'[\mathbf{b}]h' - e_3'[\mathbf{b}]) - 2q(q-1)p(h'-b_3b_4) - e_1[\mathbf{b}]h' + e_3[\mathbf{b}]. \tag{2.11}$$

The equations (2.10) and (2.11) are linear in q and $q(q-1)p$. Solving for these quantities and substituting in (2.9) gives (1.27). ∎

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The τ-function is defined in terms of the Hamiltonian by

$$H = \frac{d}{dt} \log \tau(t). \tag{2.12}$$

In terms of h, it follows from (2.7) that

$$h = t(t-1)\frac{d}{dt}\log\Big((t-1)^{e_2'[\mathbf{b}]-\frac{1}{2}e_2[\mathbf{b}]}t^{\frac{1}{2}e_2[\mathbf{b}]}\tau(t)\Big). \tag{2.13}$$

2.2. Bäcklund transformations and Toda lattice equation
Bäcklund transformations

$$T(\mathbf{b}; q, p, t, H) = (\bar{\mathbf{b}}; \bar{q}, \bar{p}, \bar{t}, \bar{H})$$

are birational canonical transformations of the symplectic form. Thus the Hamilton equations are satisfied in the variables $(\bar{\mathbf{b}}; \bar{q}, \bar{p}, \bar{t}, \bar{H})$. Because there are particular T possessing the property

$$TH = H\big|_{\mathbf{b}\mapsto T\mathbf{b}}, \tag{2.14}$$

Bäcklund transformations allow an infinite family of solutions of the $\mathrm{P_{VI}}$ system to be generated from one seed solution.

Okamoto [54] identified the affine Weyl group $W_a(D_4^{(1)})$ as being realised by a set of t-invariant Bäcklund transformations of the $\mathrm{P_{VI}}$ system (if Bäcklund transformations altering t are permitted, a realisation of the affine F_4 reflection group is obtained [54]). The group $W_a(D_4^{(1)})$ is generated by the operators $s_0, \dots, s_4$ obeying the algebraic relations

$$(s_i s_j)^{m_{ij}} = 1, \quad 0 \le i, j \le 4 \tag{2.15}$$

where

$$[m_{ij}] = \begin{bmatrix} 1 & 2 & 3 & 2 & 2 \\ 2 & 1 & 3 & 2 & 2 \\ 3 & 3 & 1 & 3 & 3 \\ 2 & 2 & 3 & 1 & 2 \\ 2 & 2 & 3 & 2 & 1 \end{bmatrix}. \tag{2.16}$$

The entries m_{ij} are related to the Dynkin diagram for the affine root system $D_4^{(1)}$,

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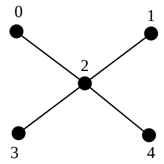

Figure 1: $D_4^{(1)}$ Dynkin diagram.

Thus for $i \neq j$, $(0 \leq i < j \leq 4)$, $m_{ij} = 2$ if the nodes i and j are not connected, while $m_{ij} = 3$ if the nodes are connected. An equivalent way to specify the algebra (2.15) is via the relations

$$\begin{aligned} &s_i^2 = 1 \quad (0 \leq i, j \leq 4), \\ &s_i s_j = s_j s_i \quad (i, j \neq 2), \\ &(s_i s_2)^3 = (s_2 s_i)^3 = 1 \quad (i \neq 2). \end{aligned} \tag{2.17}$$

The operators $s_0, \ldots, s_4$ are associated with affine vectors $\alpha_0, \ldots, \alpha_4$ ($\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1$, with the α's regarded as coordinates) in a four dimensional vector space such that s_i corresponds to a reflection in the subspace perpendicular to α_i, and thus $s_i \alpha_i = -\alpha_i$. The action of s_i on the other affine vectors is given by

$$s_i(\alpha_j) = \alpha_j - \alpha_i a_{ij} \tag{2.18}$$

where $A = (a_{ij})$ is the Cartan matrix

$$A = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ -1 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix} \tag{2.19}$$

(the off diagonal elements in (2.19) are obtained from those in (2.16) by replacing all 2's in the latter by 0's, and all 3's by -1's). As first identified by Okamoto [54], the operators s_i are t-invariant Bäcklund transformations for the P_{VI} system with action on the parameters specified by (2.18).

It remains to specify the action of the s_i on p and q. For this purpose the most systematic way to proceed is to make use of recent work of Noumi and Yamada [50] (see also their subsequent work [51]), who give a symmetric formulation of the Bäcklund transformations for Painlevé type systems. In

	α_0	α_1	α_2	α_3	α_4	p	q
s_0	$-\alpha_0$	α_1	$\alpha_2+\alpha_0$	α_3	α_4	$p-\frac{\alpha_0}{q-t}$	q
s_1	α_0	$-\alpha_1$	$\alpha_2+\alpha_1$	α_3	α_4	p	q
s_2	$\alpha_0+\alpha_2$	$\alpha_1+\alpha_2$	$-\alpha_2$	$\alpha_3+\alpha_2$	$\alpha_4+\alpha_2$	p	$q+\frac{\alpha_2}{p}$
s_3	α_0	α_1	$\alpha_2+\alpha_3$	$-\alpha_3$	α_4	$p-\frac{\alpha_3}{q-1}$	q
s_4	α_0	α_1	$\alpha_2+\alpha_4$	α_3	$-\alpha_4$	$p-\frac{\alpha_4}{q}$	q
r_1	α_1	α_0	α_2	α_4	α_3	$-\frac{p(q-t)^2+\alpha_2(q-t)}{t(t-1)}$	$\left(\frac{q-1}{q-t}\right)t$
r_3	α_3	α_4	α_2	α_0	α_1	$-\frac{q}{t}[qp+\alpha_2]$	$\frac{t}{q}$

Table 1: Bäcklund transformations for the $\mathrm{P_{VI}}$ Hamiltonian (2.3).

the general formalism of [50], the action (2.18) has as its counterpart the action

$$s_i(f_j) = f_j + \frac{\alpha_i}{f_i} u_{ij} \tag{2.20}$$

where $U = [u_{ij}]$ is the orientation matrix defined by

$$U = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{2.21}$$

The key points are that (2.17) is realised by (2.20) with the f_j specified in terms of p, q, t by

$$f_0 = q-t, \quad f_1 = q-\infty, \quad f_2 = -p, \quad f_3 = q-1, \quad f_4 = q, \tag{2.22}$$

(the quantities subtracted from q are the location of the fixed singularities of $\mathrm{P_{VI}}$), and that the transformations (2.18) and (2.20) together are then Bäcklund transformations for $\mathrm{P_{VI}}$. Following [41], the action of the s_i on the α_j, p and q is summarised in Table 1.

The symmetric formalism of [50], [52] naturally extends the Bäcklund transformations from a realisation of $W_a(D_4^{(1)})$ to a realisation of $W_a(D_4^{(1)} \rtimes \Omega)$, where Ω denotes particular diagram automorphisms of $D_4^{(1)}$. The latter are operators r_1 and r_3 (another natural diagram automorphism is $r_4\boldsymbol{\alpha} = (\alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0)$ and is related to r_1 and r_3 by $r_4 = r_1 r_3$) with the actions

$$r_1\boldsymbol{\alpha} = (\alpha_1, \alpha_0, \alpha_2, \alpha_4, \alpha_3), \quad r_3\boldsymbol{\alpha} = (\alpha_3, \alpha_4, \alpha_2, \alpha_0, \alpha_1)$$

of interchanging outer pairs of vertices in the diagram of Figure 1. These operators obey the algebraic relations

$$
\begin{aligned}
&r_1^2 = r_3^2 = 1, \quad r_1 s_2 = s_2 r_1, \quad r_3 s_2 = s_2 r_3 \quad r_i r_j = r_j r_i, \; i \neq j, \\
&s_1 = r_1 s_0 r_1, \quad s_4 = r_1 s_3 r_1, \quad s_3 = r_3 s_0 r_3, \quad s_4 = r_3 s_1 r_3.
\end{aligned}
$$

Their action on the p and q is given in Table 1. Making use of Table 1 the action of the fundamental operators on the Hamiltonian

$$t(t-1)H =: K \tag{2.23}$$

as specified by (2.3) is found to be

$$
\begin{aligned}
s_0 K &= K - \alpha_0 \frac{t(t-1)}{q-t} + \alpha_0(\alpha_3 - 1)t + \alpha_0(\alpha_4 - 1)(t-1) \\
s_1 K &= K \\
s_2 K &= K + \alpha_2(1 + \alpha_1 - \alpha_0)t - \alpha_2(\alpha_1 + \alpha_2 + \alpha_3) \\
s_3 K &= K - \alpha_3(1 - \alpha_0)t \\
s_4 K &= K - \alpha_4(1 - \alpha_0)(t-1) \\
r_1 K &= K - q(q-1)p - \alpha_2 q + \alpha_2(\alpha_1 - \alpha_0)t + \alpha_2(\alpha_0 + \alpha_2 + \alpha_4) \\
r_3 K &= K + (1-t)qp + \alpha_2(\alpha_0 + \alpha_2 + \alpha_4)(1-t)
\end{aligned}
$$

(the first five of these equations can be found in [68]).

Consider the composite operator

$$T_3 := r_1 s_0 s_1 s_2 s_3 s_4 s_2, \tag{2.24}$$

which from Table 1 has the action on the α parameters

$$T_3 \boldsymbol{\alpha} = (\alpha_0 + 1, \alpha_1 + 1, \alpha_2 - 1, \alpha_3, \alpha_4) \tag{2.25}$$

or equivalently, using (2.6), the action on the b parameters

$$T_3 \mathbf{b} = (b_1, b_2, b_3 + 1, b_4) \tag{2.26}$$

(in [54] this was referred to as the parallel transformation ℓ_3). With K specified by (2.23) we see from (2.24) and Table 1 that

$$T_3 K = K\big|_{\boldsymbol{\alpha} \mapsto T_3 \boldsymbol{\alpha}} = K - q(q-1)p - (\alpha_1 + \alpha_2)(q-t), \tag{2.27}$$

which was derived in [54] in a different way. There are only three other fundamental shift operators that share the property (2.27). They have the actions

$$\begin{aligned}
T_{10}K &:= r_1 s_1 s_2 s_3 s_4 s_2 s_1 K = K\big|_{\boldsymbol{\alpha}\mapsto(\alpha_0+1,\alpha_1-1,\alpha_2,\alpha_3,\alpha_4)} \\
&\qquad\qquad\qquad\qquad\quad = K - q(q-1)p - \alpha_2(q-t), \\
T_{30}K &:= r_3 s_3 s_2 s_1 s_4 s_2 s_3 K = K\big|_{\boldsymbol{\alpha}\mapsto(\alpha_0+1,\alpha_1,\alpha_2,\alpha_3-1,\alpha_4)} = K - (t-1)qp, \\
T_{40}K &:= r_4 s_4 s_2 s_1 s_3 s_2 s_4 K = K\big|_{\boldsymbol{\alpha}\mapsto(\alpha_0+1,\alpha_1,\alpha_2,\alpha_3,\alpha_4-1)} = K - t(q-1)p,
\end{aligned}$$

although we will not develop the theory of these cases in this work.

The result (2.27) motivates introducing the sequence of Hamiltonians

$$T_3^n H = H\big|_{\boldsymbol{\alpha}\mapsto(\alpha_0+n,\alpha_1+n,\alpha_2-n,\alpha_3,\alpha_4)},$$

and the corresponding sequence of τ-functions specified by

$$T_3^n H = \frac{d}{dt}\log\tau_3[n], \qquad \tau_3[n] = \tau_3[n](t) = \tau_3(t; b_1, b_2, b_3+n, b_4). \tag{2.28}$$

Okamoto [54] proved that $\tau_3[n]$ satisfies the Toda lattice equation. We will give the derivation of this result using the strategy of Kajiwara et al. [41].

PROPOSITION 2. *The τ-function sequence* (2.28) *satisfies the Toda lattice equation*

$$\delta^2 \log\bar{\tau}_3[n] = \frac{\bar{\tau}_3[n-1]\bar{\tau}_3[n+1]}{\bar{\tau}_3^2[n]}, \qquad \delta = t(t-1)\frac{d}{dt} \tag{2.29}$$

where

$$\bar{\tau}_3[n] = \big(t(t-1)\big)^{(n+b_1+b_3)(n+b_3+b_4)/2}\tau_3[n]. \tag{2.30}$$

Proof. From the definitions

$$\begin{aligned}
\delta\log\frac{\tau_3[n-1]\tau_3[n+1]}{\tau_3^2[n]} &= \Big(T_3K[n] - K\Big) - \Big(K[n] - T_3^{-1}K\Big) \\
&= -q(q-1)p - (\alpha_1+\alpha_2)(q-t) \\
&\quad + T_3^{-1}\Big(q(q-1)p + (\alpha_1+\alpha_2)(q-t)\Big)\Big|_{\boldsymbol{\alpha}\mapsto(\alpha_0+n,\alpha_1+n,\alpha_2-n,\alpha_3,\alpha_4)},
\end{aligned} \tag{2.31}$$

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where the second equality follows from (2.27). Now T_3 is specified by (2.24), and each of the elementary operators is an involution so

$$T_3^{-1} = s_2 s_4 s_3 s_2 s_1 s_0 r_1 = s_2 s_3 s_4 s_2 s_0 s_1 r_1 \tag{2.32}$$

where the second equality follows from the commutation relations $s_3 s_4 = s_4 s_3$, $s_0 s_1 = s_1 s_0$ implied by (2.17). Using (2.32) and Table 1 we compute that

$$\begin{aligned} &-T_3^{-1}\Big(q(q-1)p + (\alpha_1+\alpha_2)(q-t)\Big) \\ &\quad = q(q-1)p + \alpha_2(q-1) + (1-\alpha_0)q + \frac{\alpha_2(1-\alpha_0)}{p} + (\alpha_1+\alpha_2)t \\ &\qquad - (\alpha_1+\alpha_2+\alpha_3) + (\alpha_2+\alpha_3+\alpha_4)(1-\alpha_0) \\ &\qquad\quad \times \frac{1}{p}\frac{q(q-1)p + \alpha_2 q + \alpha_2(q-1) + \alpha_2^2/p}{q(q-1)p - \alpha_3 q - \alpha_4(q-1) - \alpha_2(\alpha_2+\alpha_3+\alpha_4)/p}. \end{aligned} \tag{2.33}$$

Substituting this in (2.31), we can verify that the resulting expression can be written as

$$\delta \log\Big[q(q-1)p^2 - [\alpha_3 q + \alpha_4(q-1)]p - \alpha_2(\alpha_2+\alpha_3+\alpha_4)\Big]. \tag{2.34}$$

To do this we first compute the derivative in terms of p and q using the Hamilton equations, and then check that the expression so obtained is indeed equal to the rational function in p and q obtained by substituting (2.33) in (2.31) using computer algebra. But according to (2.3), (2.34) is equal to

$$\delta \log\Big[\frac{d}{dt}K + \alpha_2(1-\alpha_0)\Big]. \tag{2.35}$$

Substituting (2.12) in (2.35) and equating the resulting expression with the left hand side of (2.31), we see that

$$A\frac{\tau_3[n-1]\tau_3[n+1]}{\tau_3^2[n]} = \frac{d}{dt}\delta \log \tau_3[n] + (\alpha_2 - n)(1-\alpha_0-n),$$

where $A \neq 0$ is arbitrary. Recalling (2.6) to replace the α's by the b's and choosing $A = 1$ gives (2.29). ∎

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2.3. Classical solutions

An identity of Sylvester (see [46]) gives that if

$$\bar{\tau}_3[0] = 1, \tag{2.36}$$

then the general solution of (2.29) is given by

$$\bar{\tau}_3[n] = \det\left[\delta^{j+k}\bar{\tau}_3[1]\right]_{j,k=0,1,\dots,n-1}. \tag{2.37}$$

As noted by Okamoto [54] and Watanabe [68], the solution (2.36) is permitted by restricting the parameters so that

$$b_1 + b_3 = 0 \quad \text{or equivalently} \quad \alpha_2 = 0 \tag{2.38}$$

(this corresponds to a chamber wall in the affine $D_4^{(1)}$ root system). Thus with $\alpha_2 = 0$ we see that (2.3) permits the solution

$$p = 0, \quad H = 0 \tag{2.39}$$

and thus $\tau_3[0] = \bar{\tau}_3[0] = 1$. Furthermore, with this initial condition $\tau_3[1]$ is given by a solution of the Gauss hypergeometric equation.

PROPOSITION 3. ([54]) *Let the parameters* $\boldsymbol{\alpha}$ *in* (2.3) *be initially restricted by* (2.38), *then apply the operator* T_3 *so that*

$$T_3 H[0] = H[1] = \frac{d}{dt}\log\tau_3[1](t). \tag{2.40}$$

The function $\tau_3[1](t)$ *satisfies the Gauss hypergeometric equation*

$$t(1-t)\tau_3''[1](t) + \Big(c - (a+b+1)t\Big)\tau_3'[1](t) - ab\tau_3[1](t) = 0 \tag{2.41}$$

where

$$a = -\alpha_1 = b_1 + b_4, \ \ b = \alpha_0 = 1 + b_3 + b_4, \ \ c = \alpha_0 + \alpha_4 = 1 + b_2 + b_4. \tag{2.42}$$

Proof. It follows from (2.27), (2.23) and (2.38), (2.39) that

$$T_3 K[0] = -\alpha_1(q - t) \tag{2.43}$$

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where q is the solution of the Hamilton equation

$$\delta(q-t) = \frac{\partial K}{\partial p}\bigg|_{\substack{\alpha_2=0\\ p=0}} - t(t-1) \tag{2.44}$$
$$= \alpha_1(q-t)^2 + \big[(\alpha_1+\alpha_4)t + (\alpha_1+\alpha_3)(t-1)\big](q-t) - \alpha_0 t(t-1).$$

Rewriting the left hand side of (2.43) in terms of $\tau_3'[1](t)/\tau_3[1](t)$ according to (2.40), then substituting the resulting expression in (2.44) and simplifying gives (2.41). $\square$

Our interest is in particular integral solutions of (2.41), which being a second order linear equation has in general two linearly independent solutions. Consider first the solution analytic at the origin — the Gauss hypergeometric function ${}_2F_1$ — written as its Euler integral representation

$$ {}_2F_1(a,b;c;t) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 u^{b-1}(1-u)^{c-b-1}(1-ut)^{-a}\,du. \tag{2.45}$$

We then have

$$\tau_3[1](t) = \tau_3(t; b_1, b_2, -b_1+1, b_4) \tag{2.46}$$
$$= {}_2F_1(b_1+b_4, 1-b_1+b_4, 1+b_2+b_4; t).$$

Integral solutions of (2.41) which in general are not analytic at the origin are given by

$$f(a,b,c;t) \propto \int_p^q u^{a-c}(1-u)^{c-b-1}(t-u)^{-a}\,du \tag{2.47}$$

where p and q are any of $0, 1, t, \nu\,\infty$ $(\nu = 1)$ such that the integrand vanishes [36]. Forming from this a linear combination with $(p,q) = (0,1)$ and $(p,q) = (t,1)$ we deduce from this that $\tau_3[1](t) = f^{\mathrm{J}}(a,b,c;t)$ satisfies (2.41), where

$$f^{\mathrm{J}}(a,b,c;t) \propto \Big(\int_0^1 - \xi \int_t^1\Big) u^{a-c}(1-u)^{c-b-1}(t-u)^{-a}\,du. \tag{2.48}$$

Another case of interest is the linear combination of (2.47) with $(p,q) = (-\infty,\infty)$ and $(p,q) = (t,\infty)$. After suitably deforming these contours in the complex plane we deduce that $\tau_3[1](t) = f^{\mathrm{Cy}}(a,b,c;t)$ satisfies (2.41), where

$$f^{\mathrm{Cy}}\Big(a,b,c;\frac{1+it}{2}\Big) \propto \Big(\int_{-\infty}^{\infty} - \xi \int_t^{\infty}\Big)(1+iu)^{a-c}(1-iu)^{c-b-1}(t-u)^{-a}\,du. \tag{2.49}$$

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Let $F(a,b,c;t)$ be any particular solution of (2.41). Then recalling (2.38), (2.42) and (2.30) we see from (2.37) that

$$\bar{\tau}_3[n] = \det\big[\delta^{j+k} t^{b/2}(t-1)^{b/2} F(a,b,c;t)\big]_{j,k=0,\dots,n-1}.$$

A useful check on further working is to note that if $\{\bar{\tau}_3[n](t)\}$ satisfies the Toda lattice equation (2.29) with $\bar{\tau}_3[0]=1$, then $\{t^{\gamma n/2}(t-1)^{-\gamma n/2}\bar{\tau}[n](t)\}$, γ arbitrary, is also a solution which is given by the determinant formula (2.37). Thus

$$\bar{\tau}_3[n] = t^{-\gamma n/2}(t-1)^{\gamma n/2} \det\big[\delta^{j+k} t^{A}(t-1)^{B} F(a,b,c;t)\big]_{j,k=0,\dots,n-1} \tag{2.50}$$

where

$$A+B=b, \quad A-B=\gamma \tag{2.51}$$

and the right hand side must be independent of γ. Our task is to substitute the particular solutions (2.45), (2.48) and (2.49) in the $n\times n$ determinant (2.50) and show that these can be reduced to three distinct n-dimensional multiple integrals.

Firstly we consider (2.50) by taking the solution (2.45).

Proposition 4. *Let $F(a,b,c;t)$ be given by* (2.45), *and let δ be given as in* (2.29). *Then assuming the constraint* (2.51),

$$\bar{\tau}_3[n] \propto t^{bn/2}(t-1)^{bn/2+n(n-1)/2} \det[{}_2F_1(a-j,b+k;c;t)]_{j,k=0,\dots,n-1}. \tag{2.52}$$

Proof. A fundamental differential-difference relation for the Gauss hypergeometric function is

$$t\frac{d}{dt}{}_2F_1(a,b;c;t) = b\Big({}_2F_1(a,b+1;c;t) - {}_2F_1(a,b;c;t)\Big).$$

It follows from this that

$$\begin{aligned}\delta\Big(t^{A}(t-1)^{B}{}_2F_1(a,b;c;t)\Big) \\ &= \big(b-A+t(A+B-b)\big)t^{A}(t-1)^{B}{}_2F_1(a,b;c;t) \\ &\quad + bt^{A}(t-1)^{B+1}{}_2F_1(a,b+1;c;t).\end{aligned}$$

In the special case of the constraint (2.51), this reads

$$\begin{aligned}\delta\Big(t^{A}(t-1)^{B}{}_2F_1(a,b;c;t)\Big) \\ &= Bt^{A}(t-1)^{B}{}_2F_1(a,b;c;t) + bt^{A}(t-1)^{B+1}{}_2F_1(a,b+1;c;t).\end{aligned} \tag{2.53}$$

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Another fundamental differential-difference relation for the Gauss hypergeometric function is

$$\begin{aligned} t(1-t)\frac{d}{dt}{}_2F_1(a,b;c;t) \\ = (c-a){}_2F_1(a-1,b;c;t) + (a-c+bt){}_2F_1(a,b;c;t). \end{aligned}$$

This implies

$$\begin{aligned} &\delta\Big(t^A(t-1)^B{}_2F_1(a,b;c;t)\Big) \\ &\quad = \big(c-a-A+t(A+B-b)\big)t^A(t-1)^B{}_2F_1(a,b;c;t) \\ &\qquad + (a-c)t^A(t-1)^B{}_2F_1(a-1,b;c;t), \end{aligned}$$

which in the case of the constraint (2.51) reads

$$\begin{aligned} &\delta\Big(t^A(t-1)^B{}_2F_1(a,b;c;t)\Big) \\ &\quad = (c-a-A)t^A(t-1)^B{}_2F_1(a,b;c;t) \\ &\qquad + (a-c)t^A(t-1)^B{}_2F_1(a-1,b;c;t). \end{aligned} \tag{2.54}$$

The identity (2.53) can be used to eliminate the operation δ^k from the right hand side of (2.50). Thus substituting (2.53) in column k, and subtracting B times column $k-1$ ($k=n-1,n-2,\dots,1$ in that order) shows

$$\begin{aligned} &\det\Big[\delta^{j+k}t^A(t-1)^B{}_2F_1(a,b;c;t)\Big]_{j,k=0,\dots,n-1} \\ &\quad = b^{n-1}\det\Big[\delta^j t^A(t-1)^B{}_2F_1(a,b;c;t) \ \cdots \\ &\qquad\qquad \delta^{j+k-1}t^A(t-1)^{B+1}{}_2F_1(a,b+1;c;t)\Big]_{\substack{j=0,\dots,n-1 \\ k=1,\dots,n-1}}. \end{aligned}$$

Repeating this procedure on column k ($k=n-1,n-2,\dots,k'$), for each of $k'=2,3,\dots,n-2$ in that order shows

$$\begin{aligned} &\det\Big[\delta^{j+k}t^A(t-1)^B{}_2F_1(a,b;c;t)\Big]_{j,k=0,\dots,n-1} \\ &\quad = \prod_{l=1}^{n-1}(b)_l \det\Big[\delta^j t^A(t-1)^{B+k}{}_2F_1(a,b+k;c;t)\Big]_{j,k=0,\dots,n-1}. \end{aligned}$$

To eliminate δ^j from this expression we substitute (2.54) in row j and subtract $(c-a-A)$ times row $j-1$ ($j=n-1,n-2,\dots,1$ in that order). This shows

$$\det\Big[\delta^j t^A(t-1)^{B+k}{}_2F_1(a,b+k;c;t)\Big]_{j,k=0,\dots,n-1} = (a-c)^{n-1} \times \prod_{l=1}^{n-1}(b)_l \det\begin{bmatrix} t^A(t-1)^{B+k}{}_2F_1(a,b+k;c;t) \\ \delta^{j-1}t^A(t-1)^{B+k}{}_2F_1(a-1,b+k;c;t)\end{bmatrix}_{\substack{j=1,\dots,n-1\\ k=0,\dots,n-1}}.$$

Repeating this procedure on row j, $j=n-1,n-2,\dots,j'$ for each of $j'=3,4,\dots,n-2$ in that order gives

$$\det\Big[\delta^{j+k}t^A(t-1)^B{}_2F_1(a,b;c;t)\Big]_{j,k=0,\dots,n-1} = (-1)^{n(n-1)/2} \times \prod_{j=1}^{n-1}(b)_j(c-a)_j \det\Big[t^A(t-1)^{B+k}{}_2F_1(a-j,b+k;c;t)\Big]_{j,k=0,\dots,n-1}.$$

Removing the factor $t^A(t-1)^{B+k}$ from each column and using (2.51) gives (2.52). □

Proposition 5. *It follows from* (2.52) *that*

$$\tau_3[n](t) := \tau_3(t;b_1,b_2,b_3+n,b_4)\big|_{b_1+b_3=0} \propto \int_0^1 du_1\cdots\int_0^1 du_n \times \prod_{i=1}^n u_i^{b_3+b_4}(1-u_i)^{b_2-b_3-n}(1-tu_i)^{b_3-b_4}\prod_{1\le j<k\le n}(u_k-u_j)^2. \tag{2.55}$$

Proof. From (2.30) we have

$$\tau_3[n](t) = (t(t-1))^{-n(n-1+b)/2}\bar\tau_3[n](t)$$

where use has been made of the equation $b_1+b_3=0$ from (2.38) and $b=1-b_1+b_4$ from (2.42). Substituting (2.52) then shows

$$\tau_3[n](t) \propto t^{-n(n-1)/2}\det[{}_2F_1(a-j,b+k;c;t)]_{j,k=0,\dots,n-1}. \tag{2.56}$$

Substituting the integral representation (2.45) and recalling (2.42) shows

$$\begin{aligned}(2.57)\quad \tau_3[n](t) &\propto t^{-n(n-1)/2} \\ &\times \det\Big[\int_0^1 du\, u^{b_3+b_4+k}(1-u)^{b_2-b_3-k-1}(1-tu)^{j-b_1-b_4}\Big]_{j,k=0,\dots,n-1} \\ &= t^{-n(n-1)/2}\int_0^1 du_1\cdots\int_0^1 du_n \prod_{i=1}^n u_i^{b_3+b_4}(1-u_i)^{b_2-b_3-n}(1-tu_i)^{-b_1-b_4} \\ &\qquad \times \det\Big[u_{j+1}^k(1-u_{j+1})^{n-1-k}(1-tu_{j+1})^j\Big]_{j,k=0,\dots,n-1}.\end{aligned}$$

The integrand can be symmetrised without changing the value of the integral provided we divide by $n!$. The function of the u_i's outside the determinant is symmetric in the u_i's, so symmetrising the integrand is equivalent to symmetrising the determinant. We have

$$\begin{aligned}(2.58)\quad &\operatorname{Sym}\det\Big[u_{j+1}^k(1-u_{j+1})^{n-1-k}(1-tu_{j+1})^j\Big]_{j,k=0,\dots,n-1} \\ &= \operatorname{Sym}\prod_{j=0}^{n-1}(1-tu_{j+1})^j \det\Big[u_{j+1}^k(1-u_{j+1})^{n-1-k}\Big]_{j,k=0,\dots,n-1} \\ &= \Big(\operatorname{Asym}\prod_{j=0}^{n-1}(1-tu_{j+1})^j\Big)\det\Big[u_{j+1}^k(1-u_{j+1})^{n-1-k}\Big]_{j,k=0,\dots,n-1} \\ &= \det\Big[(1-tu_{j+1})^k\Big]_{j,k=0,\dots,n-1}\det\Big[u_{j+1}^k(1-u_{j+1})^{n-1-k}\Big]_{j,k=0,\dots,n-1} \\ &= \prod_{j=0}^{n-1}(1-u_{j+1})^{n-1}\det\Big[(1-tu_{j+1})^k\Big]_{j,k=0,\dots,n-1} \\ &\qquad \times \det\Big[\Big(\frac{u_{j+1}}{1-u_{j+1}}\Big)^k\Big]_{j,k=0,\dots,n-1}.\end{aligned}$$

But the Vandermonde determinant identity gives that for any $a_1, a_2, \dots, a_n$,

$$\det\big[a_{j+1}^k\big]_{j,k=0,\dots,n-1} = \prod_{1\le j<k\le n}(a_k - a_j).$$

This allows the determinants in (2.58) to be written as products. Substituting the results for the determinant in (2.57) gives (2.55). □

Next we will show that by choosing F in (2.50) to equal the solution (2.48), a generalised multiple integral representation for $\bar{\tau}_3[n]$ can be obtained.

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PROPOSITION 6. *Let* $F(a, b, c; t)$ *in* (2.50) *be given by*

$$F(a,b,c;t) = \Big(\int_0^1 - \xi \int_t^1 \Big) u^{a-c}(1-u)^{c-b-1}(t-u)^{-a}\, du. \tag{2.59}$$

Then

$$\bar{\tau}_3[n](t) \propto t^{bn/2}(t-1)^{bn/2+n(n-1)/2} \det[F(a-j, b+k, c; t)]_{j,k=0,\dots,n-1}. \tag{2.60}$$

Proof. The proof of Proposition 4 shows that the sufficient conditions for reducing (2.50) to (2.60) is that F satisfies the differential-difference relations

$$t\frac{d}{dt}F(a,b,c;t) = C_0 F(a,b+1,c;t) - bF(a,b,c;t) \tag{2.61}$$

$$t(1-t)\frac{d}{dt}F(a,b,c;t) = C_1 F(a-1,b,c;t) + (C_2 + bt)F(a,b,c;t), \tag{2.62}$$

independent of the explicit form of C_0, C_1, C_2. To derive the form (2.61) we first change variables $u \mapsto tu$ in (2.59) so it reads

$$F(a,b,c;t) = t^{-c+1}\Big(\int_0^{1/t} - \xi \int_1^{1/t}\Big) u^{a-c}(1-tu)^{c-b-1}(1-u)^{-a}\, du.$$

Differentiating this shows

$$\begin{aligned} t\frac{d}{dt}F(a,b,c;t) &= (-c+1)F(a,b,c;t) + t^{-c+1}(c-b-1) \\ &\quad \times \Big(\int_0^{1/t} - \xi \int_1^{1/t}\Big) u^{a-c}\Big(\frac{-tu}{1-tu}\Big)(1-tu)^{c-b-1}(1-u)^{-a}\, du. \end{aligned}$$

Writing

$$-\frac{tu}{1-tu} = 1 - \frac{1}{1-tu} \tag{2.63}$$

we see from this that

$$t\frac{d}{dt}F(a,b,c;t) = -bF(a,b,c;t) - (c-b-1)F(a,b+1,c;t), \tag{2.64}$$

thus establishing (2.61).

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To establish the structure (2.62), we first multiply both sides of (2.64) by $(1-t)$ to get

$$t(1-t)\frac{d}{dt}F(a,b,c;t) = btF(a,b,c;t) - bF(a,b,c;t) \\ - (c-b-1)F(a,b+1,c;t) + (c-b-1)tF(a,b+1,c;t).$$

But

$$tF(a,b+1,c;t) = t^{-c+1}\Big(\int_0^{1/t} - \xi\int_1^{1/t}\Big)u^{a-c-1} \\ \times\Big(\frac{tu}{1-tu}\Big)(1-tu)^{c-b-1}(1-u)^{-a}\,du. \tag{2.65}$$

Using (2.63) and the equally simple manipulation

$$(1-u)^{-a+1} = (1-u)^{-a} - u(1-u)^{-a}$$

shows that the right hand side of (2.65) is equal to

$$-\Big(F(a,b,c;t) + F(a-1,b,c;t)\Big) + \Big(F(a,b+1,c;t) + F(a-1,b+1,c;t)\Big)$$

and thus

$$t(1-t)\frac{d}{dt}F(a,b,c;t) = btF(a,b,c;t) - (c-1)F(a,b,c;t) \\ - (c-b-1)\Big(F(a-1,b,c;t) - F(a-1,b+1,c;t)\Big).$$

But

$$\begin{aligned} &- (c-b-1)\Big(F(a-1,b,c;t) - F(a-1,b+1,c;t)\Big) \\ &= (c-b-1)t^{-c+1}\Big(\int_0^{1/t} - \xi\int_1^{1/t}\Big)u^{a-c-1}(1-tu)^{c-b-1}(1-u)^{-a+1} \\ &\qquad \times \frac{tu}{1-tu}\,du \\ &= -t^{-c+1}\Big(\int_0^{1/t} - \xi\int_1^{1/t}\Big)u^{a-c}\frac{d}{du}(1-tu)^{c-b-1}(1-u)^{-a+1} \\ &= t^{-c+1}(a-c)\Big(\int_0^{1/t} - \xi\int_1^{1/t}\Big)u^{a-c-1}(1-tu)^{c-b-1}(1-u)^{-a+1}\,du \\ &\qquad + t^{-c+1}(a-1)\Big(\int_0^{1/t} - \xi\int_1^{1/t}\Big)u^{a-c}(1-tu)^{c-b-1}(1-u)^{-a}\,du \\ &= (a-c)F(a-1,b,c;t) + (a-1)F(a,b,c;t) \end{aligned}$$

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so we have

$$t(1-t)\frac{d}{dt}F(a,b,c;t) = btF(a,b,c;t) + (a-c)F(a,b,c;t) + (a-c)F(a-1,b,c;t),$$

in agreement with (2.62). ∎

Following the steps from which (2.55) was deduced from (2.52) allows us to deduce from (2.60) the following multiple integral representation.

PROPOSITION 7. *We can rewrite* (2.60) *and so deduce*

$$\begin{aligned}(2.66)\quad \tau_3[n](t) &= \tau_3(t;b_1,b_2,b_3+n,b_4)\big|_{b_1+b_3=0} \\ &\propto \Big(\int_0^1 - \xi \int_t^1\Big) du_1 \cdots \Big(\int_0^1 - \xi \int_t^1\Big) du_n \\ &\times \prod_{i=1}^n u_i^{-b_2-(b_3+n)}(1-u_i)^{b_2-(b_3+n)}(t-u_i)^{-(b_1+b_4)} \prod_{1\le j<k\le n}(u_k-u_j)^2.\end{aligned}$$

Finally we show that by choosing F in (2.50) to be given by the linear combination (2.49), $\tau_3[n]$ can be written as an n-dimensional integral having a form different from the previous cases.

PROPOSITION 8. *Let* $F(a,b,c;t)$ *in* (2.50) *be given by*

$$(2.67)\qquad F(a,b,c;t) = \Big(\int_{-\infty}^{\infty} - \xi \int_t^{\infty}\Big) u^{a-c}(1-u)^{c-b-1}(t-u)^{-a}\, du,$$

where it is required the integrand be integrable in the neighbourhood of $u = t, \pm\infty$ *but not necessarily* $u = 0, 1$ (*the path of integration can be deformed around these points*). *Then*

$$(2.68)\quad \bar\tau_3[n](t) \propto t^{bn/2}(t-1)^{bn/2+n(n-1)/2} \det[F(a-j,b+k,c;t)]_{j,k=0,\dots,n-1}.$$

Following the steps from which (2.55) was deduced from (2.52) allows us to deduce from (2.68) the following multiple integral representation.

Proposition 9. *We can rewrite* (2.68) *and so deduce*

$$\begin{aligned}(2.69)\quad \tau_3[n](t) &= \tau_3(t;b_1,b_2,b_3+n,b_4)\big|_{b_1+b_3=0}\\ &\propto \Big(\int_{-\infty}^{\infty} -\xi\int_t^{\infty}\Big)du_1\cdots\Big(\int_{-\infty}^{\infty} -\xi\int_t^{\infty}\Big)du_n\\ &\times\prod_{i=1}^n u_i^{-b_2-(b_3+n)}(1-u_i)^{b_2-(b_3+n)}(t-u_i)^{-(b_1+b_4)}\prod_{1\le j<k\le n}(u_k-u_j)^2.\end{aligned}$$

The only necessary detail of the contours (intervals) of integration in (2.69) is that the integrand vanishes at the endpoints. Deforming the contours in this manner we see from (2.69) that

$$\begin{aligned}(2.70)\quad &\tau_3\Big(\frac{1+it}{2};b_1,b_2,b_3+n,b_4\Big)\Big|_{b_1+b_3=0}\\ &\propto \Big(\int_{-\infty}^{\infty} -\xi\int_t^{\infty}\Big)du_1\cdots\Big(\int_{-\infty}^{\infty} -\xi\int_t^{\infty}\Big)du_n\\ &\times\prod_{i=1}^n (1+iu_i)^{-b_2-(b_3+n)}(1-iu_i)^{b_2-(b_3+n)}(t-u_i)^{b_3-b_4}\prod_{1\le j<k\le n}(u_k-u_j)^2.\end{aligned}$$

2.4. Schlesinger transformations

In this part we develop difference equations arising from the sequence generated by the action of T_3, which are also known as Schlesinger transformations because the formal monodromy exponents are shifted by integers. In doing so we will demonstrate that this recurrence in the canonical variables can be expressed in a form which is precisely that of the discrete fifth Painlevé equation dP_{V}. We thus establish directly that the discrete dP_{V} is the contiguity relation of the continuous P_{VI} equation in contrast to other treatments [61], [60], [33], [45]. The T_3, T_3^{-1} operators are equivalent to the $R_{(9)}$, $R_{(10)}$ operators respectively in [45].

Proposition 10. *The sequence* $\{q[n],p[n]\}_{n=0}^{\infty}$ *generated by the shift operator* T_3 *with parameters* $\boldsymbol{\alpha}=(\alpha_0+n,\alpha_1+n,\alpha_2-n,\alpha_3,\alpha_4)$ *defines an auxiliary sequence, equivalent to the sequence* $\{g[n],f[n]\}_{n=0}^{\infty}$ *satisfying the discrete fifth Painlevé equation* dP_{V}

$$(2.71)\qquad g[n+1]g[n]=\frac{t}{t-1}\frac{(f[n]+1-\alpha_2)(f[n]+1-\alpha_2-\alpha_4)}{f[n](f[n]+\alpha_3)}$$

$$(2.72)\qquad f[n]+f[n-1]=-\alpha_3+\frac{\alpha_1}{g[n]-1}+\frac{\alpha_0 t}{t(g[n]-1)-g[n]},$$

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where

$$g := \frac{q}{q-1}, \qquad f := q(q-1)p + (1-\alpha_2-\alpha_4)(q-1) - \alpha_3 q - \alpha_0 \frac{q(q-1)}{q-t}. \tag{2.73}$$

Proof. To begin with we construct the forward and backward shifts in the canonical variables under the action of T_3 using Table 1 ($q := q[n]$, $p := p[n]$)

$$\begin{aligned} q[n+1] &:= \widehat{q} \\ &= \frac{t}{q-t}[(q-1)(q-t)p + (\alpha_1+\alpha_2)(q-t) - \alpha_0(t-1)] \\ &\times [q(q-1)(q-t)p + (\alpha_1+\alpha_2)q(q-t) - \alpha_0(t-1)q + \alpha_4(q-t)]\frac{1}{X_u} \end{aligned} \tag{2.74}$$

$$\begin{aligned} p[n+1] := \widehat{p} = -\frac{(q-t)}{t(t-1)} &\frac{[(q-t)p+\alpha_1+\alpha_2]X_u}{[q(q-t)p + (\alpha_1+\alpha_2)(q-t) - \alpha_0 t]} \\ &\times \frac{1}{[(q-1)(q-t)p + (\alpha_1+\alpha_2)(q-t) - \alpha_0(t-1)]} \end{aligned} \tag{2.75}$$

$$q[n-1] := \breve{q} = t\frac{[(q-1)p+\alpha_2][q(q-1)p+\alpha_2 q+\alpha_4]}{X_d} \tag{2.76}$$

$$\begin{aligned} p[n-1] := \breve{p} = \frac{X_d}{t(t-1)}\Bigg\{ &-\frac{[(q-t)p+\alpha_2]}{[qp+\alpha_2][(q-1)p+\alpha_2]} \\ &+ \frac{\alpha_0-1}{q(q-1)p^2 - (\alpha_4(q-1)+\alpha_3 q)p - \alpha_2(\alpha_2+\alpha_3+\alpha_4)}\Bigg\}, \end{aligned} \tag{2.77}$$

where

$$\begin{aligned} X_u = q(q-1)(q-t)^2p^2 &- \big[\alpha_4(q-1)(q-t) + \alpha_3 q(q-t) \\ &+ (\alpha_0 - 1 - \alpha_1)q(q-1)\big](q-t)p + (\alpha_1+\alpha_2)^2(q-t)^2 \\ &+ (\alpha_1+\alpha_2)\big[(\alpha_1+\alpha_2+\alpha_4)t + (\alpha_1+\alpha_2+\alpha_3)(t-1)\big](q-t) \\ &- \alpha_0(\alpha_1+1)t(t-1) \end{aligned} \tag{2.78}$$

$$\begin{aligned} X_d = q(q-1)(q-t)p^2 &- \big[\alpha_4(q-1)(q-t) + \alpha_3 q(q-t) \\ &+ (\alpha_0 - 1 + \alpha_1)q(q-1)\big]p + \alpha_2^2(q-t) + \alpha_2(\alpha_2+\alpha_4)t \\ &+ \alpha_2(\alpha_2+\alpha_3)(t-1), \end{aligned} \tag{2.79}$$

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(note that the factors X_d, X_u are distinguished by being quadratic in p). From these variables certain products can be constructed that have simple factorisable forms

$$\begin{aligned}(2.80)\quad \widehat{q}\,\widehat{p} &= -\frac{[(q-t)p+\alpha_1+\alpha_2]}{t-1} \\ &\times \frac{[q(q-1)(q-t)p+(\alpha_1+\alpha_2)q(q-t)-\alpha_0(t-1)q+\alpha_4(q-t)]}{[q(q-t)p+(\alpha_1+\alpha_2)(q-t)-\alpha_0 t]}\end{aligned}$$

$$\begin{aligned}(2.81)\quad (\widehat{q}-1)\widehat{p} &= -\frac{[(q-t)p+\alpha_1+\alpha_2]}{t} \\ &\times \frac{[q(q-1)(q-t)p+(\alpha_1+\alpha_2)(q-1)(q-t)-\alpha_0 t(q-1)-\alpha_3(q-t)]}{[(q-1)(q-t)p+(\alpha_1+\alpha_2)(q-t)-\alpha_0(t-1)]}\end{aligned}$$

$$\begin{aligned}(2.82)\quad (\widehat{q}-t)\widehat{p} = &-\frac{[(q-t)p+\alpha_1+\alpha_2]}{[q(q-t)p+(\alpha_1+\alpha_2)(q-t)-\alpha_0 t]} \\ &\times \frac{1}{[(q-1)(q-t)p+(\alpha_1+\alpha_2)(q-t)-\alpha_0(t-1)]} \\ &\times \Big\{q(q-1)(q-t)^2p^2 \\ &\quad -[\alpha_4(q-1)(q-t)+\alpha_3 q(q-t)+2\alpha_0 q(q-1)](q-t)p \\ &\quad +\alpha_0[\alpha_4(q-1)(q-t)+\alpha_3 q(q-t)+\alpha_0 q(q-1)] \\ &\quad -(1-\alpha_2)(\alpha_0+\alpha_1+\alpha_2)(q-t)^2\Big\}\end{aligned}$$

$$(2.83)\quad \frac{\smile{q}-1}{\smile{q}-t} = \frac{1}{t}\frac{q^2(q-1)p^2+[2\alpha_2 q(q-1)-\alpha_3 q]p+\alpha_2^2 q-\alpha_2(\alpha_2+\alpha_3)}{q(q-1)p^2-[\alpha_4(q-1)+\alpha_3 q]p-\alpha_2(\alpha_2+\alpha_3+\alpha_4)}.$$

From the ratio of (2.80), (2.81) one notices that it can be simply expressed in terms of the single quantity f defined by (2.73) which is the first of the coupled recurrences for $\mathrm{dP_V}$ (2.71). To find the other member of the pair we evaluate $\smile{f} := f[n-1]$ using the above formulas and find

$$(2.84)\qquad \smile{f} = -(q-1)(qp+\alpha_2),$$

so that when this is added to f we arrive at (2.72). ∎

Although of theoretical interest, Proposition 10 does not immediately lead to a difference equation for the Hamiltonian itself. However such an equation can be derived by using the workings from the derivation of Proposition 10.

PROPOSITION 11. *The sequence of Hamiltonians* $\{K[n]\}_{n=0}^{\infty}$ *generated by the shift operator* T_3 *with parameters* $\boldsymbol{\alpha} = (\alpha_0+n, \alpha_1+n, \alpha_2-n, \alpha_3, \alpha_4)$ *satisfies the third order difference equation*

(2.85)
$$
\begin{aligned}
&\Big[(1-\alpha_0)\overset{\frown}{K} + \alpha_0 K\Big]\Big[(1+\alpha_1)K - \alpha_1\overset{\frown}{K} + (\alpha_1+\alpha_2)(1+\alpha_1-\alpha_0)t \\
&\quad - (\alpha_1+\alpha_2)(\alpha_1+\alpha_2+\alpha_3)\Big]\Big[\overset{\frown}{K} - \overset{\smile}{K} + 1 - \alpha_0 - \alpha_4 \\
&\quad - (1+\alpha_1-\alpha_0+2\alpha_2)t\Big]\Big[\overset{\frown}{\overset{\frown}{K}} - K - \alpha_0 - \alpha_4 + (2\alpha_0+\alpha_3+\alpha_4)t\Big] \\
&\quad + t(t-1)\Big[(1-\alpha_0-\alpha_1)K - \alpha_1(1-\alpha_0)(\overset{\frown}{K} - \overset{\smile}{K}) \\
&\quad + (1-\alpha_0)(\alpha_1+\alpha_2)(1-\alpha_0+\alpha_1)t - (1-\alpha_0)(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+\alpha_3)\Big] \\
&\quad \times \Big[(1+\alpha_0+\alpha_1)\overset{\frown}{K} - \alpha_0(1+\alpha_1)(\overset{\frown}{\overset{\frown}{K}} - K) + \alpha_0(\alpha_1+\alpha_2)(1-\alpha_0+\alpha_1)t \\
&\quad - \alpha_0(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+\alpha_3)\Big] = 0
\end{aligned}
$$

where $K := t(t-1)H[n]$, $\overset{\smile}{K} := t(t-1)H[n-1]$, $\overset{\frown}{K} := t(t-1)H[n+1]$, $\overset{\frown}{\overset{\frown}{K}} := t(t-1)H[n+2]$.

Proof. We focus our attention on the quantity

$$Z := -q(q-1)p - (\alpha_1+\alpha_2)(q-t), \tag{2.86}$$

which from (2.27) is $Z = K[n+1] - K[n]$. First we consider $\overset{\smile}{Z} := Z[n-1]$. From the definition Z is related to f so employing (2.84) and (2.83) for the downshifted variables we have

$$
\begin{aligned}
\overset{\smile}{Z} &= (q-1)(qp+\alpha_2) + (1-\alpha_0)q - \alpha_3 + (\alpha_1+\alpha_2)(t-1) \\
&\quad + (1-\alpha_0)\frac{(2\alpha_2+\alpha_3+\alpha_4)[q(q-1)p+\alpha_2 q] - \alpha_2(\alpha_2+\alpha_3)}{q(q-1)p^2 - [\alpha_4(q-1)+\alpha_3 q]p - \alpha_2(\alpha_2+\alpha_3+\alpha_4)}.
\end{aligned} \tag{2.87}
$$

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We transform this expression by replacing the canonical variables q, $q-1$, p where possible by K, Z and retaining $q-t$. The resulting equation is then one which is a linear equation for $q-t$ in terms of K, Z, $\breve{Z}$,

$$(2.88)\quad q-t = \Big[K+(1-\alpha_0)Z\Big] \times \Big[Z+\breve{Z}+\alpha_2+\alpha_3-(\alpha_1+\alpha_2)(t-1)-(1+\alpha_2-\alpha_0)t\Big] \Big/ \Big[(1-\alpha_0-\alpha_1)K-\alpha_1(1-\alpha_0)(Z+\breve{Z}) + (1-\alpha_0)[-\alpha_1(\alpha_2+\alpha_3)+(\alpha_1^2+\alpha_1\alpha_2+\alpha_2^2+\alpha_2\alpha_3)(t-1) + (\alpha_1+\alpha_1\alpha_2-\alpha_0\alpha_1+\alpha_2^2+\alpha_2\alpha_4)t]\Big].$$

We use a similar strategy and evaluate the upshifted $\widehat{Z} := Z[n+1]$ using (2.81), (2.80), (2.82), (2.75). We find after considerable simplification

$$(2.89)\quad \widehat{Z} = q(q-1)p+(\alpha_1+\alpha_2)(q-t)-(2\alpha_0+\alpha_3+\alpha_4)t - (1+\alpha_0+\alpha_1)\frac{t(t-1)}{q-t}+(1+\alpha_1)\frac{t(t-1)}{(q-t)X_u} \times \Big\{(1+\alpha_0+\alpha_1)q(q-1)(q-t)p+(\alpha_1+\alpha_2)(1+\alpha_0+\alpha_1)q^2 + \Big[-(1+\alpha_0+\alpha_1)(\alpha_0+\alpha_1+\alpha_2)t+\alpha_0(\alpha_0+\alpha_4) - (\alpha_1+\alpha_2)(\alpha_1+\alpha_2+\alpha_3)\Big]q+(\alpha_0+\alpha_1+\alpha_2)(1-\alpha_2-\alpha_4)t\Big\},$$

and after the variable replacements described above one has a linear equation for

$$(2.90)\quad \frac{t(t-1)}{q-t} = -\Big[K-\alpha_1Z+(\alpha_1+\alpha_2)[(1-\alpha_0+\alpha_1)t-\alpha_1-\alpha_2-\alpha_3]\Big] \times \Big[Z+\widehat{Z}-\alpha_0-\alpha_4+(2\alpha_0+\alpha_3+\alpha_4)t\Big] \Big/ \Big\{(1+\alpha_0+\alpha_1)K+(1+\alpha_1-\alpha_0\alpha_1)Z-\alpha_0(1+\alpha_1)\widehat{Z} + \alpha_0(\alpha_1+\alpha_2)(1-\alpha_0+\alpha_1)t-\alpha_0(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+\alpha_3)\Big\}$$

By eliminating $q-t$ between (2.88) and (2.90) and further simplification one arrives at the stated result (2.85). ∎

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We remark that the right-hand side of (2.86) can be written in terms of f and g, and then coupled with the $\mathrm{dP_V}$ equations (2.71), (2.72) to provide an alternative scheme to calculate $K[n]$. A similar recurrence appears in the recent work of Borodin [10].

§3. Application to the finite JUE and CyUE

We are now in a position to identify the multiple integral representations for the τ-functions of the classical solutions to the $\mathrm{P_{VI}}$ system, (2.55), (2.66) and (2.70), with the spectral averages defined by (1.6), (1.10) and (1.12) respectively.

3.1. The JUE

The τ-function solution (2.46) is relevant to the average (1.6), for with $N = 1$ we have

$$\begin{aligned}(3.1)\quad \widetilde{E}_1^J(t;a,b,\mu) &\propto t^{a+\mu+1}{}_2F_1(-b,a+1;\mu+a+2;t)\\ &= t^{a+\mu+1}\tau_3\Big(t;-\frac{1}{2}(a+b),\mu+1+\frac{1}{2}(a+b),1+\frac{1}{2}(a+b),\frac{1}{2}(a-b)\Big).\end{aligned}$$

The multiple integral representation (2.55) of $\tau_3[n](t)$ is of the type occurring in the definition (1.6) of $\widetilde{E}_N(t;a,b;\mu)$. This allows the latter to be identified with a τ-function for the $\mathrm{P_{VI}}$ system, and its logarithmic derivative identified with an auxiliary Hamiltonian (2.7) and so characterised as a solution of the Jimbo-Miwa-Okamoto σ-form of the $\mathrm{P_{VI}}$ equation (1.27).

PROPOSITION 12. *Let $\tau_3[n](t) = \tau_3(t;b_1,b_2,b_3+n,b_4)\big|_{b_1+b_3=0}$ refer to the τ-function sequence (2.28) with $\tau_3[0] = 1$ and $\tau_3[1](t)$ given by (2.46). Then we have*

$$(3.2)\qquad \widetilde{E}_N^{\mathrm{J}}(t;a,b,\mu) = Ct^{(b_1+b_3)(b_2+b_4)}\tau_3(t;\mathbf{b}),$$
$$\mathbf{b} = \Big(-\frac{1}{2}(a+b),\mu+N+\frac{1}{2}(a+b),N+\frac{1}{2}(a+b),\frac{1}{2}(a-b)\Big)$$

and consequently

$$\begin{aligned}(3.3)\quad t(t-1)\frac{d}{dt}\log\Big((t-1)^{e_2'[\mathbf{b}]-\frac{1}{2}e_2[\mathbf{b}]}t^{\frac{1}{2}e_2[\mathbf{b}]}t^{-(b_1+b_3)(b_2+b_4)}\widetilde{E}_N^{\mathrm{J}}(t;a,b,\mu)\Big)&\\ = U_N^{\mathrm{J}}(t;a,b,\mu)&\end{aligned}$$

where $U_N^{\mathrm{J}}(t;a,b,\mu)$ satisfies the Jimbo-Miwa-Okamoto σ-form of the $\mathrm{P_{VI}}$ equation (1.27) with $\mathbf{b}$ specified as in (3.2), and $e_2'[\mathbf{b}]$, $e_2[\mathbf{b}]$ defined as in

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Proposition 1*. The latter is to be solved subject to the boundary condition*

$$(3.4)\quad \begin{aligned} U_N^{\mathrm{J}}(t;a,b,\mu) &\underset{t\to 0}{\sim} -\frac{1}{2}e_2[\mathbf{b}] + \Big(e_2'[\mathbf{b}] + bN\frac{a+N}{a+\mu+2N}\Big)t \\ &\underset{t\to 0}{\sim} -\frac{1}{2}N^2 - \frac{1}{2}(\mu+a-b)N + \frac{1}{4}b(a+b) - \frac{1}{4}\mu(a-b) \\ &\qquad + \left(-bN - \frac{1}{4}(a+b)^2 + bN\frac{a+N}{a+\mu+2N}\right)t \end{aligned}$$

with U_N^{J} given by a power series in t about $t=0$.

Proof. The first equation follows immediately upon comparing (2.55) with (1.6), and rewriting the exponent in the first factor in (1.6) in terms of the b's. The equation (3.3) then follows from (2.13). For the boundary condition, we see from the second integral in (1.6) and the definition of the b's that

$$(3.5)\quad \begin{aligned} t^{-N(a+\mu+N)}\widetilde{E}_N(t;a,b,\mu) &= t^{-(b_1+b_3)(b_2+b_4)}\widetilde{E}_N^{\mathrm{J}}(t;a,b,\mu) \\ &\underset{t\to 0}{\sim} \frac{J_N(a,\mu)}{C}\left(1 - b\frac{J_N(a,\mu)\big[\sum_{j=1}^N x_j\big]}{J_N(a,\mu)}t\right) \end{aligned}$$

where $J_N(a,\mu)\big[\sum_{j=1}^N x_j\big]$ is the integral (1.23) with an extra factor of $\sum_{j=1}^N x_j$ in the integrand. The ratio of integrals in (3.5) can be evaluated using a generalisation of the Selberg integral due to Aomoto [3] to give (3.4). Alternatively, the $O(1)$ term in (3.4) can be substituted in (1.27) to deduce the $O(t)$ term. ∎

Let us now reconcile Proposition 12 with the result that (1.28) satisfies (1.27) with parameters (1.29). For this purpose we note from (3.2) that with $\mu=0$, $b_2=b_3$. Then the exponents in (3.3) simplify, and we see that

$$(3.6)\quad U_N^{\mathrm{J}}(t;a,b,0) = -tb_2^2 + \frac{1}{2}(b_1b_4 + b_2^2) + t(t-1)\frac{d}{dt}\log \widetilde{E}_N^{\mathrm{J}}(t;a,b,0).$$

Interchanging a and b as required by (1.28) changes the sign of b_4, so with $\mathbf{b}$ given as in (3.2) with $\mu=0$, we see from (3.6) that

$$-U_N^{\mathrm{J}}(1-t;b,a,0) = -tb_2^2 - \frac{1}{2}(b_1b_4 - b_2^2) + t(t-1)\frac{d}{dt}\log \widetilde{E}_N^{\mathrm{J}}(1-t;a,b,0).$$

But in general, if $h(t)$ satisfies (1.27) with parameters $\mathbf{b}$, then $-h(1-t)$ also satisfies (1.27) but with an odd number of $b_1, \dots, b_4$ reversed in sign. Choosing to reverse the sign of b_4, it follows that

$$-tb_2^2 + \frac{1}{2}(b_1 b_4 + b_2^2) + t(t-1)\frac{d}{dt}\log \widetilde{E}_N^{\mathrm{J}}(1-t; a, b, 0)$$

satisfies (1.27) with $\mathbf{b}$ given by (a suitable permutation of) (1.29), which is in agreement with (1.28).

A corollary of Proposition 12, which follows from (1.21) and the second equality in (1.6), is the formula

$$\begin{aligned}(3.7)\quad U_N^{\mathrm{J}}(t; a, \mu, b) &= t(t-1)\frac{d}{dt} \\ &\times \log\Big((t-1)^{e_2'[\mathbf{b}^*] - \frac{1}{2}e_2[\mathbf{b}^*]} t^{\frac{1}{2}e_2[\mathbf{b}^*]} \Big\langle \prod_{l=1}^{N} z_l^{N+a+b/2} |1+z_l|^b (1+tz_l)^\mu \Big\rangle_{\mathrm{CUE}_N} \Big), \\ \mathbf{b}^* &= \Big(-\frac{1}{2}(a+\mu), b+N+\frac{1}{2}(a+\mu), N+\frac{1}{2}(a+\mu), \frac{1}{2}(a-\mu) \Big).\end{aligned}$$

Comparison of the τ-function solution not analytic at the origin (2.48) with (1.10) shows, for $N=1$

$$\begin{aligned}(3.8)\quad \widetilde{E}_1^{\mathrm{J}}(t; a, b, \mu; \xi) &\propto f^{\mathrm{J}}(-\mu, -\mu-a-1-b, -\mu+a; t) \\ &= \tau_3\Big(t; 1+\frac{1}{2}(a+b), \frac{1}{2}(b-a), -\frac{1}{2}(a+b), -\mu-1-\frac{1}{2}(a+b) \Big).\end{aligned}$$

Furthermore comparing the n-dimensional integral (2.66) with (1.10) allows us to characterise the latter in terms of a solution of (1.27).

Proposition 13. *Let* $\tau_3[n](t) = \tau_3(t; b_1, b_2, b_3+n, b_4)\big|_{b_1+b_3=0}$ *refer to the* τ*-function sequence* (2.28) *with* $\tau_3[0]=1$ *and* $\tau_3[1](t)$ *given by* (2.59)*. Then we have*

$$\begin{aligned}(3.9)\quad \widetilde{E}_N^{\mathrm{J}}(t; a, b, \mu; \xi) &= C\tau_3(t; \hat{\mathbf{b}}), \\ \hat{\mathbf{b}} &= \Big(\frac{1}{2}(a+b)+N, \frac{1}{2}(b-a), -\frac{1}{2}(a+b), -\frac{1}{2}(a+b)-N-\mu \Big)\end{aligned}$$

and consequently

$$(3.10)\quad e_2'[\hat{\mathbf{b}}]t - \frac{1}{2}e_2[\hat{\mathbf{b}}] + t(t-1)\frac{d}{dt}\log \widetilde{E}_N^{\mathrm{J}}(t; a, b, \mu; \xi) = \widehat{U}_N^{\mathrm{J}}(t; a, b, \mu; \xi)$$

where $\widehat{U}_N^{\mathrm{J}}(t;a,b,\mu;\xi)$ *satisfies the Jimbo-Miwa-Okamoto* σ*-form of the* $\mathrm{P_{VI}}$ *equation* (1.27) *with* $\mathbf{b}$ *specified as in* (3.9)*, and* $e_2'[\hat{\mathbf{b}}]$*,* $e_2[\hat{\mathbf{b}}]$ *defined as in Proposition* 1*. In the case* $\xi = 1$ *we have the boundary condition*

$$(3.11)\quad \widehat{U}_N^{\mathrm{J}}(t;a,b,\mu;1) \underset{t\to 0}{\sim} -\frac{1}{2}e_2[\hat{\mathbf{b}}] - N(a+\mu+N) + \Big(e_2'[\hat{\mathbf{b}}] + (N+\mu+a)N + bN\frac{a+N}{a+\mu+2N}\Big)t$$
$$\underset{t\to 0}{\sim} -\frac{1}{2}N^2 - \frac{1}{2}(\mu+a-b)N + \frac{1}{4}b(a+b) - \frac{1}{4}\mu(a-b) + \Big(-bN - \frac{1}{4}(a+b)^2 + bN\frac{a+N}{a+\mu+2N}\Big)t$$

with $\widehat{U}_N^{\mathrm{J}}$ *a power series in* t *about* $t = 0$*, while in the case* $\xi = 0$

$$(3.12)\quad \widehat{U}_N^{\mathrm{J}}(t;a,b,\mu;0) \underset{|t|\to\infty}{\sim} \Big(e_2'[\hat{\mathbf{b}}] + N\mu\Big)t + O(1) \underset{|t|\to\infty}{\sim} -\Big(N + \frac{1}{2}(a+b)\Big)^2 t + O(1)$$

with $\widehat{U}_N^{\mathrm{J}}$*, apart from the leading term, a power series in* $1/t$*.*

Proof. The only remaining task is the specification of the boundary conditions (3.11) and (3.12). The first of these follows from the fact that $\widetilde{E}_N^{\mathrm{J}}(t;a,b,\mu;1) = \widetilde{E}_N^{\mathrm{J}}(t;a,b,\mu)$ and then substituting (3.5) in (3.10), while the second follows from the fact that $\widetilde{E}_N(t;a,b,\mu;0) = F_N^{\mathrm{J}}(t;a,b,\mu)$ and noting from (1.7) that $F_N^{\mathrm{J}}(t;a,b,\mu) \sim t^{N\mu}$ as $t \to \infty$. □

Comparing the definition (1.10) with (1.6) shows

$$(3.13)\qquad \widetilde{E}_N^{\mathrm{J}}(t;a,b,\mu;1) = \widetilde{E}_N^{\mathrm{J}}(t;a,b,\mu)$$

so the σ-function $\widehat{U}$ in (3.10) must be related to the σ-function U in (3.3). To explore this point, writing $\mathbf{b}$ in (3.2) as $\mathbf{b} = (b_1, b_2, b_3, b_4)$ we see from (3.9) that

$$(3.14)\qquad \hat{\mathbf{b}} = (b_3, -b_4, b_1, -b_2).$$

The differential equation (1.27) is unchanged by the replacement of $\mathbf{b}$ by (3.14), so $\widehat{U}$ and U in fact satisfy the same equation. In fact $\widehat{U}$ and U are in

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this case the same function, as the identity (3.13) together with the readily verified formula

$$e_2'[\mathbf{b}]t - \frac{1}{2}e_2[\mathbf{b}] - (t-1)(b_1+b_3)(b_2+b_4) = e_2'[\hat{\mathbf{b}}]t - \frac{1}{2}e_2[\hat{\mathbf{b}}]$$

show that the left hand sides of (3.3) and (3.10) agree.

Another special case of (1.10) of interest is the coefficient of ξ^N,

$$\begin{aligned}[\xi^N]\widetilde{E}_N^{\mathrm{J}}(s;a,b,\mu;\xi) = \frac{(-1)^N}{C}\int_s^1 dx_1\, x_1^a(1-x_1)^b(s-x_1)^\mu \cdots \\ \times \int_s^1 dx_N\, x_N^a(1-x_N)^b(s-x_N)^\mu \prod_{1\le j<k\le N}(x_k-x_j)^2.\end{aligned}$$

This exhibits the functional property

$$[\xi^N]\widetilde{E}_N^{\mathrm{J}}\Big(\frac{1}{s};a,b,\mu;\xi\Big) = (-1)^N s^{-N(a+b+\mu+N)}[\xi^N]\widetilde{E}_N^{\mathrm{J}}(s;a,b,\mu;\xi).$$

The τ-function evaluation (3.9) then implies

$$\tau_3\Big(\frac{1}{t};\hat{\mathbf{b}}\Big) = (-1)^N t^{(\hat{b}_3+\hat{b}_4)(\hat{b}_1+\hat{b}_3)}\tau_3(t;\bar{\mathbf{b}}) \tag{3.15}$$

where

$$\begin{aligned}\bar{\mathbf{b}} = \Big(\frac{1}{2}(\hat{b}_1-\hat{b}_2+\hat{b}_3-\hat{b}_4), \frac{1}{2}(-\hat{b}_1+\hat{b}_2+\hat{b}_3-\hat{b}_4), \\ \frac{1}{2}(\hat{b}_1+\hat{b}_2+\hat{b}_3+\hat{b}_4), \frac{1}{2}(-\hat{b}_1-\hat{b}_2+\hat{b}_3+\hat{b}_4)\Big)\end{aligned} \tag{3.16}$$

($\bar{\mathbf{b}}$ is obtained from $\hat{\mathbf{b}}$ by simply interchanging b and μ in the latter; recall (3.14) and (3.2)). This result can be understood within the context of the general theory of the $\mathrm{P_{VI}}$ equation. Thus the mapping

$$(q,p,H,t;\hat{\mathbf{b}}) \longmapsto \Big(\frac{1}{q}, (b_1+b_3)q - q^2p, -\frac{H}{t^2}+\Phi(t), \frac{1}{t};\bar{\mathbf{b}}\Big),$$
$$\Phi(t) = -\frac{1}{t}(\hat{b}_3+\hat{b}_1)(\hat{b}_3+\hat{b}_4)$$

has been identified in [54] as a canonical transformation of the $\mathrm{P_{VI}}$ system (the value of $\Phi(t)$ was not given explicitly in [54]). Recalling (2.12), up to a proportionality constant this immediately implies (3.15).

A consequence of the difference equation (2.85) is that a difference equation for $U_N^{\mathrm{J}}(t;a,b,\mu;\xi)$ with respect to μ can be found.

Proposition 14. *The Jimbo-Miwa-Okamoto σ-function $U_N^{\mathrm{J}}(t;a,b,\mu;\xi)$ satisfies a third order difference equation in μ, for all N,a,b,t*

(3.17)
$$\begin{aligned}
&\Big[(N+a+\mu+1)U-(N+a+\mu)\widehat{U}+1/4(a-b)^2t-1/4(a^2+b^2)\\
&\quad-1/4(a+b)N\Big]\Big[(N+b+\mu+1)U-(N+b+\mu)\widehat{U}+1/4(a-b)^2t\\
&\quad+1/2ab+1/4(a+b)N\Big]\Big[\widehat{U}-\check{U}+(2N+2\mu+a+b)(t-1/2)\Big]\\
&\quad\times\Big[\widehat{\widehat{U}}-U+(2N+2\mu+2+a+b)(t-1/2)\Big]\\
&=t(t-1)\Big[(N+a+\mu)(N+b+\mu)(\widehat{U}-\check{U})-(2N+2\mu+a+b)U\\
&\quad-1/4(a-b)^2(2N+2\mu+a+b)t\\
&\quad+1/4(b-a)\big(2Nb+(b-a)\mu+b(a+b)\big)\Big]\\
&\quad\times\Big[(N+a+\mu+1)(N+b+\mu+1)(\widehat{\widehat{U}}-U)-(2N+2\mu+2+a+b)\widehat{U}\\
&\quad-1/4(a-b)^2(2N+2\mu+2+a+b)t\\
&\quad+1/4(b-a)(2Nb+(b-a)(\mu+1)+b(a+b))\Big],
\end{aligned}$$

where $U:=U_N^{\mathrm{J}}(t;a,b,\mu;\xi)$, $\check{U}:=U_N^{\mathrm{J}}(t;a,b,\mu-1;\xi)$, $\widehat{U}:=U_N^{\mathrm{J}}(t;a,b,\mu+1;\xi)$ etc.

Proof. This follows from (2.85) using the parameters in (3.2) with $b_2\leftrightarrow b_3$ and the invariance of h or U under this interchange, and the relation

(3.18)
$$\begin{aligned}U_N^{\mathrm{J}}(t;a,b,\mu;\xi)&=K[\mu]-\big[b(N+\mu)+1/4(a+b)^2\big]t\\
&\quad-1/2N^2-1/2N(\mu+a-b)-1/4\mu(a-b)+1/4b(a+b),\end{aligned}$$

and the shifted variants. □

One can easily verify that (3.17) is invariant under the transformations $a\leftrightarrow b$, $t\mapsto 1-t$ and $U_N^{\mathrm{J}}\mapsto -U_N^{\mathrm{J}}$.

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3.2. The CyUE and cJUE

Comparison of the τ-function solution (2.49) with the average (1.12) shows, for $N=1$

$$\begin{aligned}(3.19)\quad \widetilde{E}_1^{\mathrm{Cy}}(t;(\eta_1,\eta_2),\mu;\xi) &\propto f^{\mathrm{Cy}}\Big(-\mu, 2\eta_1-\mu-1, \eta_1+i\eta_2-\mu; \frac{1+it}{2}\Big)\\ &= \tau_3\Big(\frac{1+it}{2}; -\eta_1+1, i\eta_2, \eta_1, \eta_1-\mu-1\Big),\end{aligned}$$

where in obtaining the second line use has been made of (2.42) with $b_3 = -b_1$. For $n \ge 1$ comparison of (2.70) with (1.12) allows us to deduce the analogue of Proposition 12 for $\widetilde{E}_N^{\mathrm{Cy}}$.

PROPOSITION 15. *Let $\tau_3[n](t) = \tau_3(t; b_1, b_2, b_3+n, b_4)\big|_{b_1+b_3=0}$ refer to the τ-function sequence* (2.28) *with $\tau_3[0]=1$ and $\tau_3[1](t)$ given by* (2.67)*. Then we have*

$$\begin{aligned}(3.20)\qquad \widetilde{E}_N^{\mathrm{Cy}}(t;(\eta_1,\eta_2),\mu;\xi) &= C\tau_3\Big(\frac{it+1}{2}; \mathbf{b}\Big),\\ \mathbf{b} &= (N-\eta_1, i\eta_2, \eta_1, -\mu+\eta_1-N)\end{aligned}$$

and consequently

$$\begin{aligned}(3.21)\quad (t^2+1)\frac{d}{dt}\log\Big((it-1)^{e_2'[\mathbf{b}]-\frac{1}{2}e_2[\mathbf{b}]}(it+1)^{\frac{1}{2}e_2[\mathbf{b}]}\widetilde{E}_N^{\mathrm{Cy}}(t;(\eta_1,\eta_2),\mu;\xi)\Big)\\ = U_N^{\mathrm{Cy}}(t;(\eta_1,\eta_2),\mu;\xi)\end{aligned}$$

where $U_N^{\mathrm{Cy}}(t;(\eta_1,\eta_2),\mu;\xi)$ satisfies the transformed Jimbo-Miwa-Okamoto σ-form of the $\mathrm{P_{VI}}$ equation (1.32) *with $\mathbf{b}$ specified as in* (3.20)*, and $e_2'[\mathbf{b}]$, $e_2[\mathbf{b}]$ defined as in Proposition* 1*. In the case $\xi=1$ we have the boundary condition*

$$\begin{aligned}(3.22)\qquad U_N^{\mathrm{Cy}}(t;(\eta_1,\eta_2),\mu;1) &\underset{t\to-\infty}{\sim} \Big(e_2'[\mathbf{b}] + N(N+\mu-2\eta_1)\Big)t\\ &\underset{t\to-\infty}{\sim} -\eta_1^2 t - \frac{\eta_2(N-\eta_1)(N+\mu-\eta_1)}{\eta_1} + O(1/t)\end{aligned}$$

while in the case $\xi=0$

$$\begin{aligned}(3.23)\qquad U_N^{\mathrm{Cy}}(t;(\eta_1,\eta_2),\mu;0) &\underset{t\to-\infty}{\sim} \Big(e_2'[\mathbf{b}] + N\mu\Big)t + O(1)\\ &\underset{t\to-\infty}{\sim} -(N-\eta_1)^2 t + O(1)\end{aligned}$$

We see from (3.21) that with $\eta_2 = 0$ we have $b_2 = 0$. This in turn implies that $e_2[\mathbf{b}] = e_2'[\mathbf{b}]$ and so (3.21) reduces to

(3.24)
$$(t^2+1)\frac{d}{dt}\log\Big((t^2+1)^{e_2'[\mathbf{b}]/2}\widetilde{E}_N^{\rm Cy}(t;(\eta_1,0),\mu;\xi)\Big) = U_N^{\rm Cy}(t;(\eta_1,0),\mu;\xi).$$

That this with $\mu = 0$, $\eta_1 = N+a$, $\mathbf{b} = (-a,0,N+a,a)$ satisfies (1.32) is in precise agreement with the fact noted earlier that (1.32) with the substitution (1.33) gives the equation (1.31) satisfied by (1.30).

According to (1.19), we can write down from Proposition 15 an analogous result for $\widetilde{E}_N^{\rm cJ}(\phi;(\omega_1,\omega_2),\mu;\xi)$. Using the fact that for $\phi\to 0$ this quantity tends to a constant, and evaluating the latter in terms of $M_N(a,b)$ as specified by (1.22), we have

$$(3.25)\quad \widetilde{E}_N^{\rm cJ}(\phi;(\omega_1,\omega_2),\mu;\xi^*) = \frac{M_N(\omega_1 - i\omega_2+\mu/2,\omega_1+i\omega_2+\mu/2)}{M_N(\omega_1,\omega_1)} \\ \times\exp\Big\{-\frac{1}{2}\int_0^\phi\Big(U_N^{\rm Cy}\big(\cot\frac{\theta}{2};(N+\omega_1+\mu/2,\omega_2),\mu;\xi\big) \\ +\omega_2(N+\omega_1-{}^1\!/\!_2\mu)+(\omega_1+{}^1\!/\!_2\mu)^2\cot\frac{\theta}{2}\Big)d\theta\Big\}.$$

In the case $\omega_2 = 0$ the average $\widetilde{E}_N^{\rm cJ}$ satisfies a functional relation which according to (3.25) must be related to a corresponding functional relation of $U_N^{\rm Cy}$. Thus, in the case $\omega_2 = 0$ the integrand of the multi-dimensional integral specifying $\widetilde{E}_N^{\rm cJ}$ is periodic, which in turn allows the change of variable $\theta_l\mapsto\theta_l-\phi$. From the latter it follows

$$(3.26)\qquad \widetilde{E}_N^{\rm cJ}(\phi;(\omega_1,0),2\mu;\xi) = \widetilde{E}_N^{\rm cJ}(\phi;(\mu,0),2\omega_1;\xi).$$

To exhibit this symmetry in (3.25), we note from (3.20) that in the special case $\omega_2 = 0$ we must substitute in (3.25)

$$(3.27)\qquad \mathbf{b} = (-(\omega_1+\mu/2),0,N+\omega_1+\mu/2,\omega_1-\mu/2), \\ e_2'[\mathbf{b}] = e_2[\mathbf{b}],\quad e_2'[\mathbf{b}]+N\mu = -(\omega_1+\mu/2)^2.$$

Replacing μ by 2μ as required by the left hand side of (3.26) we see that all quantities in (3.27) are then symmetric in μ and ω_1 as required by the right hand side of (3.26), except for the component $b_4 = \omega_1-\mu/2$, which is antisymmetric under these operations. However we see the equation (1.27) in the case $b_2 = 0$ is unchanged by the mapping $b_4 \mapsto -b_4$ and so indeed (3.25) is consistent with (3.26).

With ξ^* specified as in (1.13), the average defining $\widetilde{E}_N^{\mathrm{cJ}}(\phi; (\omega_1, \omega_2), 2\mu; \xi^*)$ is proportional to the CUE average

$$\tag{3.28} \begin{aligned} &\Big\langle \prod_{l=1}^{N} (1 - \xi^* \chi^{(l)}_{(\pi-\phi,\pi)}) e^{\omega_2 \theta_l} |1 + z_l|^{2\omega_1} \Big(\frac{1}{t z_l}\Big)^{\mu} (1 + t z_l)^{2\mu} \Big\rangle_{\mathrm{CUE}} \Big|_{t=e^{i\phi}} \\ &\qquad =: A_N(t; \omega_1, \omega_2, \mu; \xi^*)\big|_{t=e^{i\phi}}. \end{aligned}$$

It then follows from (3.25) that

$$\tag{3.29} \begin{aligned} -it\frac{d}{dt} \log A_N(t; \omega_1, \omega_2, \mu; \xi^*) &= \frac{1}{2}\Big(U_N^{\mathrm{Cy}}\big(i\frac{t+1}{t-1}; (N + \omega_1 + \mu, \omega_2), 2\mu; \xi\big) \\ &\qquad + i(e_2'[\mathbf{b}] - e_2[\mathbf{b}]) - i\frac{t+1}{t-1}(e_2'[\mathbf{b}] + 2N\mu)\Big) \end{aligned}$$

where U_N^{Cy} satisfies (1.32) with

$$\tag{3.30} \mathbf{b} = (-(\omega_1 + \mu), i\omega_2, N + \omega_1 + \mu, \omega_1 - \mu) =: \mathbf{b}^{\mathrm{Cy}}.$$

But, in the case $\xi^* = 0$, the same average (3.28) results from (3.7) upon introducing a factor of $t^{-\mu N/2}$ into the average and making the replacements

$$b \longmapsto 2\omega_1, \quad \mu \longmapsto 2\mu, \quad a \longmapsto -(N + \omega_1 + i\omega_2 + \mu).$$

The quantity U_N^{J} in (3.7) then satisfies (1.27) with

$$\tag{3.31} \begin{aligned} \mathbf{b} = \Big(&\frac{1}{2}(N + \omega - \mu), \bar{\omega} + \frac{1}{2}(N + \omega + \mu), \\ &\frac{1}{2}(N - \omega + \mu), -\mu - \frac{1}{2}(N + \omega + \mu)\Big), \quad \omega = \omega_1 + i\omega_2. \end{aligned}$$

It follows that

$$\tag{3.32} \begin{aligned} &t(t-1)\frac{d}{dt} \log A_N(t; \omega_1, \omega_2, \mu; 0) \\ &\qquad = U_N^{\mathrm{J}}(t; -(N + \omega_1 + i\omega_2 + \mu), 2\mu, 2\omega_1) - C_1 t + C_2 \end{aligned}$$

where

$$\tag{3.33} C_1 = e_2'[\mathbf{b}] + \mu N, \quad C_2 = \frac{1}{2}e_2[\mathbf{b}] + \mu N.$$

Comparing (3.29) and (3.32) shows

$$
\begin{aligned}
(3.34)\quad i\frac{(t-1)}{2}\Big(U_N^{\mathrm{Cy}}\big(i\frac{t+1}{t-1};(N+\omega_1+\mu,\omega_2),2\mu;0\big)\\
+i(e_2'[\mathbf{b}]-e_2[\mathbf{b}])-i\frac{t+1}{t-1}(e_2'[\mathbf{b}]+2N\mu)\Big)\\
=U_N^{\mathrm{J}}(t;-(N+\omega_1+i\omega_2+\mu),2\mu,2\omega_1)-C_1t+C_2.
\end{aligned}
$$

In fact (3.34) is a special case of the following transformation property of (1.27).

PROPOSITION 16. *Let h satisfy* (1.27), *and put*

$$
(3.35)\qquad h(t)=i(t-1)\frac{1}{2}f\Big(i\frac{t+1}{t-1}\Big)-\frac{1}{4}(b_1^2+b_2^2+b_3^2+b_4^2)t.
$$

Then $f(s)$ satisfies (1.32) *with $\mathbf{b}\mapsto\bar{\mathbf{b}}$, where*

$$
(3.36)\qquad
\begin{aligned}
\bar b_1&=\frac{1}{2}(b_1-b_2+b_3+b_4), & \bar b_2&=\frac{1}{2}(b_1-b_2-b_3-b_4)\\
\bar b_3&=\frac{1}{2}(b_1+b_2+b_3-b_4), & \bar b_4&=\frac{1}{2}(b_1+b_2-b_3+b_4)
\end{aligned}
$$

(*or any permutation of these values*).

Proof. Substituting (3.35) in (1.27), changing variables to

$$
s=i\frac{t+1}{t-1}
$$

and equating like terms with (1.32) modified so that $h\mapsto f$ and $b\mapsto\bar b$, shows the statement of the proposition is correct provided

(3.37)

$$
\begin{aligned}
\bar b_1\bar b_2\bar b_3\bar b_4&=\frac{1}{16}(e_1[\mathbf{b}^2])^2-\frac{1}{2}b_1b_2b_3b_4-\frac{1}{4}e_2[\mathbf{b}^2]\\
-e_2[\bar{\mathbf{b}}^2]&=-\frac{3}{8}(e_1[\mathbf{b}^2])^2-3b_1b_2b_3b_4+\frac{1}{2}e_2[\mathbf{b}^2]\\
-e_3[\bar{\mathbf{b}}^2]&=-\frac{1}{2}e_1[\mathbf{b}^2]b_1b_2b_3b_4-\frac{1}{16}(e_1[\mathbf{b}^2])^2+\frac{1}{4}e_1[\mathbf{b}^2]e_2[\mathbf{b}^2]-e_3[\mathbf{b}^2].
\end{aligned}
$$

Direct substitution of (3.36) into these equations verifies their validity. ∎

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We can check immediately that $\mathbf{b}^{\mathrm{Cy}}$ in (3.30) is related to $\mathbf{b}$ as in (3.31) according to (3.36), and furthermore that

$$\frac{(t-1)}{2}(e_2'[\mathbf{b}^{\mathrm{Cy}}]-e_2[\mathbf{b}^{\mathrm{Cy}}])+\frac{1}{2}(1+t)(e_2'[\mathbf{b}^{\mathrm{Cy}}]+2N\mu) \\ +C_1t-C_2+\frac{1}{4}(b_1^2+b_2^2+b_3^2+b_4^2)t=0,$$

which together show (3.34) follows from Proposition 16.

An analogous result of the difference equation in μ for the Jacobi case (3.17) holds also for the Cauchy Jimbo-Miwa-Okamoto σ-function $U_N^{\mathrm{Cy}}(t;(\eta_1,\eta_2),\mu;\xi)$.

PROPOSITION 17. *The Jimbo-Miwa-Okamoto σ-function $U_N^{\mathrm{Cy}}(t;(\eta_1,\eta_2),\mu;\xi)$ satisfies a third order difference equation in μ, for all N,η_1,η_2,t*

$$\begin{aligned}(3.38)\quad &\Big[(N+\mu+1-2\eta_1)U-(N+\mu-2\eta_1)\widehat{U}+\eta_1^2t+\eta_2(N-\eta_1)\Big]\\ &\times\Big[(N+\mu+1)U-(N+\mu)\widehat{U}+\eta_1^2t-\eta_2(N-\eta_1)\Big]\\ &\times\Big[\widehat{U}-\breve{U}+2(N+\mu-\eta_1)t\Big]\Big[\widehat{\widehat{U}}-U+2(N+\mu+1-\eta_1)t\Big]\\ &=(1+t^2)\Big[(N+\mu)(N+\mu-2\eta_1)(\widehat{U}-\breve{U})-2(N+\mu-\eta_1)U\\ &\qquad-2\eta_1^2(N+\mu-\eta_1)t-2\eta_1\eta_2(N-\eta_1)\Big]\\ &\times\Big[(N+\mu+1)(N+\mu+1-2\eta_1)(\widehat{\widehat{U}}-U)-2(N+\mu+1-\eta_1)\widehat{U}\\ &\qquad-2\eta_1^2(N+\mu+1-\eta_1)t-2\eta_1\eta_2(N-\eta_1)\Big],\end{aligned}$$

where $U:=U_N^{\mathrm{Cy}}(t;(\eta_1,\eta_2),\mu;\xi)$, $\breve{U}:=U_N^{\mathrm{Cy}}(t;(\eta_1,\eta_2),\mu-1;\xi)$, $\widehat{U}:=U_N^{\mathrm{Cy}}(t;(\eta_1,\eta_2),\mu+1;\xi)$ etc.

Proof. This follows from (2.85) using the parameters in (3.20) with $b_3\leftrightarrow b_4$ and the invariance of h or U under this interchange, and the relation

$$(3.39)\qquad U_N^{\mathrm{Cy}}(t;(\eta_1,\eta_2),\mu;\xi)=K[\mu]-\big[(N-\eta_1)^2+\mu N\big]t \\ +{}^1\!/_2\big[(N-\eta_1)^2+\mu N+i\eta_2(\mu-\eta_1)\big],$$

and the shifted variants. Finally one has to transform the variables $t\mapsto(1+it)/2$ and $U_N^{\mathrm{Cy}}\mapsto iU_N^{\mathrm{Cy}}/2$. ∎

3.3. Duality relations

In our previous studies of the average (1.4) in the Gaussian and Laguerre cases we have exhibited a duality in μ and N which in fact extends to random matrix ensembles in which the exponent 2 in the product of differences (1.1) is replaced by a continuous parameter β. The same holds true of the average (1.21). Let us define by $\mathrm{C}\beta\mathrm{E}_N$ the eigenvalue PDF proportional to

$$\prod_{1\le j<k\le N} |z_k - z_j|^\beta, \quad z_j = e^{i\theta_j}, \quad -\pi < \theta_j \le \pi. \tag{3.40}$$

When $\beta = 2$ this is the CUE, while the cases $\beta = 1$ and $\beta = 4$ are known in the random matrix literature as the COE (circular orthogonal ensemble) and CSE (circular symplectic ensemble) respectively. Similarly, let us define by $\mathrm{J}\beta\mathrm{E}_n$ the eigenvalue PDF proportional to

$$\prod_{l=1}^n x_l^{\lambda_1}(1-x_l)^{\lambda_2} \prod_{1\le j<k\le n} |x_k - x_j|^\beta, \quad 0 \le x_k \le 1.$$

In a previous study the two particle distribution functions for the ensembles $\mathrm{C}\beta\mathrm{E}_N$ in the cases β even have been expressed as a β-dimensional integral [22], with the latter having the form of an average in the ensemble $\mathrm{J}(4/\beta)\mathrm{E}_\beta$. Generalising the derivation of this identity (see Section 3.4) gives

$$\begin{aligned}\Big\langle \prod_{l=1}^N z_l^{(\eta_1-\eta_2)/2}|1+z_l|^{\eta_1+\eta_2}(1+tz_l)^m \Big\rangle_{\mathrm{C}\beta\mathrm{E}_N} \\ \propto \Big\langle \prod_{l=1}^m (1-(1-t)x_l)^N \Big\rangle_{\mathrm{J}(4/\beta)\mathrm{E}_m}\Big|_{\substack{\lambda_1=2(\eta_2-m+1)/\beta-1 \\ \lambda_2=2(\eta_1+1)/\beta-1}}.\end{aligned} \tag{3.41}$$

In the case $\beta = 2$ it then follows from (1.21) that for $\mu \in \mathbb{Z}_{\ge 0}$

$$\begin{aligned}\Big\langle \prod_{l=1}^N z_l^{(\eta_1-\eta_2)/2}|1+z_l|^{\eta_1+\eta_2}(1+tz_l)^\mu \Big\rangle_{\mathrm{CUE}_N} \\ \propto \Big\langle \prod_{l=1}^\mu z_l^{(\eta_1+2\eta_2)/2}|1+z_l|^{\eta_1}(1+(1-t)z_l)^N \Big\rangle_{\mathrm{CUE}_\mu}.\end{aligned} \tag{3.42}$$

As we have characterised the averages in (3.42) in terms of solutions of the σ form of the P_{VI} equation (1.27), it must be that both sides of (3.42) satisfy (1.27) with the same parameters $\mathbf{b}$.

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To check this, we note that according to (3.7)

$$(3.43)\quad U_N^{\mathrm{J}}(t; -N-\eta_2, \mu, \eta_1+\eta_2) = t(t-1)\frac{d}{dt}$$
$$\times \log\Big((t-1)^{e_2'[\mathbf{b}]-\frac{1}{2}e_2[\mathbf{b}]} t^{\frac{1}{2}e_2[\mathbf{b}]} \Big\langle \prod_{l=1}^N z_l^{(\eta_1-\eta_2)/2} |1+z_l|^{\eta_1+\eta_2} (1+tz_l)^\mu \Big\rangle_{\mathrm{CUE_N}} \Big)$$

satisfies (1.27) with

$$\mathbf{b} = \Big(\frac{N+\eta_2-\mu}{2}, \eta_1 + \frac{N+\eta_2+\mu}{2}, \frac{N-\eta_2+\mu}{2}, -\frac{N+\eta_2+\mu}{2} \Big)$$
$$=: (b_1, b_2, b_3, b_4).$$

Now the right hand side of (3.42) is obtained from the left hand side of (3.42) by the mappings

$$t \longmapsto 1-t, \quad \mu \longleftrightarrow N, \quad \eta_1 \longmapsto \eta_1+\eta_2, \quad \eta_2 \longmapsto -\eta_2$$

which when applied to (3.43) tells us that

$$(3.44)\quad U_N^{\mathrm{J}}(1-t; -\mu+\eta_2, N, \eta_1) = -t(1-t)\frac{d}{dt}$$
$$\times \log\Big((t-1)^{\frac{1}{2}e_2[\mathbf{b}]} t^{e_2'[\mathbf{b}]-\frac{1}{2}e_2[\mathbf{b}]} \Big\langle \prod_{l=1}^N z_l^{(\eta_1-\eta_2)/2} |1+z_l|^{\eta_1+\eta_2} (1+tz_l)^\mu \Big\rangle_{\mathrm{CUE_N}} \Big)$$

satisfies (1.27) with $\mathbf{b}$ replaced by

$$\tilde{\mathbf{b}} := (-b_1, b_2, -b_4, -b_3).$$

But we have already commented (recall the paragraph including (3.6)) that $-h(1-t)$ satisfies (1.27) with the sign of an odd number of the b's reversed. Furthermore (1.27) is symmetric in the b's so we deduce $t(t-1)$ times the logarithmic derivative of the averages in (3.43) and (3.44) satisfy the same equation provided

$$te_2'[\mathbf{b}] - \frac{1}{2}e_2[\mathbf{b}] = te_2'[\tilde{\mathbf{b}}] - e_2'[\tilde{\mathbf{b}}] + \frac{1}{2}e_2[\tilde{\mathbf{b}}],$$

which is readily verified.

Another manifestation of the duality relation is that for $\xi = 0$ and μ a positive integer, $\widetilde{E}_N^{\mathrm{J}}(t; a, b, \mu; \xi)$ as specified by (1.10) can be written as a $\mu \times \mu$ determinant.

Proposition 18. *Let* $\tau_3[\mu](t)$, $\mu \in \mathbb{Z}_{\geq 0}$, *denote the* τ*-function sequence* (2.56) *with*

$$\tau_3[1](t) \propto {}_2F_1(-N, N+a+b+1; 1+a; t).$$

Then

$$\begin{aligned}&\widetilde{E}_N^{\mathrm{J}}(t; a, b, \mu; 0) \propto \tau_3[\mu](t)\\&\quad \propto t^{-\mu(\mu-1)/2} \det\Big[{}_2F_1(-N-j, N+a+b+1+k; 1+a; t)\Big]_{j,k=0,\dots,\mu-1}.\end{aligned}$$

Proof. When $\mu = 1$ we have

$$\begin{aligned}\widetilde{E}_N^{\mathrm{J}}(t; a, b, 1; 0) &= \Big\langle \prod_{l=1}^{N} (t - x_l) \Big\rangle_{\mathrm{JUE}} \propto P_N^{(a,b)}(1-2t)\\&\propto {}_2F_1(-N, N+a+b+1; 1+a; t)\end{aligned} \tag{3.45}$$

where the first proportionality follows from a standard result in [63] ($P_N^{(a,b)}$ denotes the Jacobi polynomial of degree N). Recalling Proposition 3, we therefore have

$$\begin{gathered}\widetilde{E}_N^{\mathrm{J}}(t; a, b, 1; 0) \propto \tau_3[1](t; \mathbf{b}),\\ \mathbf{b} = \Big(-N - \frac{1}{2}(a+b), -\frac{1}{2}(b-a), N + \frac{1}{2}(a+b) + 1, \frac{1}{2}(a+b)\Big).\end{gathered}$$

Thus

$$\tau_3[\mu](t) = \tau_3(t; \mathbf{b}), \quad \mathbf{b} = (-\hat{b}_1, -\hat{b}_2, -\hat{b}_4, -\hat{b}_3) \tag{3.46}$$

where $\hat{\mathbf{b}} = (\hat{b}_1, \hat{b}_2, \hat{b}_3, \hat{b}_3)$ is specified by (3.9). The operation of reversing the signs of all the $\hat{b}$'s and interchanging $\hat{b}_3$ and $\hat{b}_4$ leaves $e_2'[\hat{\mathbf{b}}]$, $e_2[\hat{\mathbf{b}}]$ and the differential equation (1.27) unchanged. It follows from (2.7) that $\tau_3(t; \mathbf{b})$ itself is unchanged, so in fact

$$\tau_3[\mu](t) = \tau_3(t; \hat{\mathbf{b}}),$$

which is just the equation (3.9). That the case $\xi = 0$ of (3.9) is singled out by (3.46) follows from the latter being a polynomial in t. ∎

3.4. Relationship to generalised hypergeometric functions

In this section we will show that $\widetilde{E}_N^{\mathrm{J}}(s;a,b,\mu)$ can be identified as an integral representation of a generalised hypergeometric function [42] based on Jack polynomials evaluated at a special point. To present this theory requires some notation. Let $\kappa := (\kappa_1,\dots,\kappa_N)$ denote a partition so that $\kappa_i \ge \kappa_j$ $(i<j)$ and $\kappa_i \in \mathbb{Z}_{\ge 0}$. Let m_κ denote the monomial symmetric function corresponding to the partition κ (e.g. if $\kappa=(2,1)$ then $m_\kappa = z_1^2 z_2 + z_1 z_2^2$), and for partitions $|\kappa| = |\mu|$ define the dominance partial ordering by the statement that $\kappa > \mu$ if $\kappa \ne \mu$ and $\sum_{j=1}^p \kappa_j \ge \sum_{j=1}^p \mu_j$ for each $p = 1,\dots,N$. Introduce the Jack polynomial $P_\kappa^{(1/\alpha)}(z_1,\dots,z_N) =: P_\kappa^{(1/\alpha)}(z)$ as the unique homogeneous polynomial of degree $|\kappa|$ with the structure

$$P_\kappa^{(1/\alpha)}(z) = m_\kappa + \sum_{\mu<\kappa} a_{\kappa\mu} m_\mu$$

(the $a_{\kappa\mu}$ are some coefficients in $\mathbb{Q}(\alpha)$) and which satisfy the orthogonality

$$\langle P_\kappa^{(1/\alpha)}, P_\rho^{(1/\alpha)} \rangle^{(\alpha)} \propto \delta_{\kappa,\rho}$$

where

$$\langle f,g\rangle^{(\alpha)} := \int_{-1/2}^{1/2} dx_1 \cdots \int_{-1/2}^{1/2} dx_N \, \overline{f(z_1,\dots,z_N)} g(z_1,\dots,z_N) \times \prod_{1\le j<k\le N} |z_k - z_j|^{2\alpha}, \quad z_j := e^{2\pi i x_j}.$$

We remark that when $\alpha = 1$ the Jack polynomial coincides with the Schur polynomial. Introduce the generalised factorial function

$$[u]_\kappa^{(\alpha)} = \prod_{j=1}^N \frac{\Gamma(u-(j-1)/\alpha + \kappa_j)}{\Gamma(u-(j-1)/\alpha)}.$$

Let

$$d'_\kappa = \prod_{(i,j)\in\kappa} \Big(\alpha(a(i,j)+1) + l(i,j)\Big),$$

where the notation $(i,j) \in \kappa$ refers to the diagram of κ, in which each part κ_i becomes the nodes (i,j), $1 \le j \le \kappa_i$ on a square lattice labelled as is conventional for a matrix. The quantity $a(i,j)$ is the so called arm length (the number of nodes in row i to the right of column j), while $l(i,j)$

is the leg length (number of nodes in column j below row i). Define the renormalised Jack polynomial

$$C_\kappa^{(\alpha)}(z) := \frac{\alpha^{|\kappa|}|\kappa|!}{d'_\kappa} P_\kappa^{(\alpha)}(z). \tag{3.47}$$

Then the generalised hypergeometric function ${}_pF_q^{(\alpha)}$ based on the Jack polynomial (3.47) is specified by the series

$${}_pF_q^{(\alpha)}(a_1,\dots,a_p;b_1,\dots,b_q;z) := \sum_\kappa \frac{1}{|\kappa|!} \frac{[a_1]_\kappa^{(\alpha)}\cdots[a_p]_\kappa^{(\alpha)}}{[b_1]_\kappa^{(\alpha)}\cdots[b_q]_\kappa^{(\alpha)}} C_\kappa^{(\alpha)}(z) \tag{3.48}$$

(when $N=1$ this reduces to the classical definition of ${}_pF_q$).

The relevance of the generalised hypergeometric functions to the present study is that it was shown in [21] that

(3.49)
$$\begin{aligned}
&\frac{1}{C}\int_0^1 dx_1\, x_1^{\lambda_1}(1-x_1)^{\lambda_2}(1-tx_1)^{-r}\cdots\int_0^1 dx_N\, x_N^{\lambda_1}(1-x_N)^{\lambda_2}(1-tx_N)^{-r}\\
&\qquad\times\prod_{1\le j<k\le N}|x_j-x_k|^{2/\alpha} := \Big\langle \prod_{l=1}^N (1-tx_l)^{-r}\Big\rangle_{\mathrm{J}(2/\alpha)\mathrm{E}_N}\\
&\quad= {}_2F_1^{(\alpha)}\Big(r,\frac{1}{\alpha}(N-1)+\lambda_1+1;\frac{2}{\alpha}(N-1)+\lambda_1+\lambda_2+2;\\
&\qquad\qquad\qquad t_1,\dots,t_N\Big)\Big|_{t_1=\cdots=t_N=t}
\end{aligned}$$

In the case $\alpha=1$, $\lambda_1=a$, $\lambda_2=\mu$, $r=-b$, the integrand in (3.49) coincides with the integral in the second equality of (1.6). Hence, by Proposition 12 we have that

$$\tau_3(t;\mathbf{b}) = {}_2F_1^{(1)}(-b,N+a;2N+a+\mu;t_1,\dots,t_N)\big|_{t_1=\cdots=t_N=t} \tag{3.50}$$

where $\mathbf{b}$ is specified by (3.2) and τ_3 is normalised so that $\tau_3(0;\mathbf{b})=1$. This, after inserting the known value of $P_\kappa^{(1)}$ evaluated with all arguments equal in (3.48), can seen to be in agreement with a conjecture of Noumi et al. [49] subsequently proved by Taneda [64] using a different argument to that here.

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For future reference we note that for $-r =: m$, $m \le N$, a non-negative integer, a result of Kaneko [42] gives that the average in (3.49) has the alternative generalised hypergeometric function evaluation

$$
\begin{aligned}
(3.51)\quad & \Big\langle \prod_{l=1}^{N} (1 - t x_l)^m \Big\rangle_{\mathrm{J}(2/\alpha)\mathrm{E}_N} \\
&= {}_2F_1^{(1/\alpha)}(-N, -(N-1) - \alpha(\lambda_1 + 1); -2(N-1) - \alpha(\lambda_1 + \lambda_2 + 2); \\
&\qquad\qquad t_1, \dots, t_m)\big|_{t_1 = \dots = t_m = t} \\
&\propto {}_2F_1^{(1/\alpha)}(-N, -(N-1) - \alpha(\lambda_1 + 1); \alpha(\lambda_2 + m); \\
&\qquad\qquad 1 - t_1, \dots, 1 - t_m)\big|_{t_1 = \dots = t_m = t}
\end{aligned}
$$

where the final formula follows from an identity in [22]. Note that the role of m and N is interchanged relative to (3.49). In fact it is the equality of (3.49) and (3.51) which implies the duality relation (3.41). Also from [42] we have that

$$
\begin{aligned}
(3.52)\quad & \\
& \Big\langle \prod_{l=1}^{N} (t - x_l)^m \Big\rangle_{\mathrm{J}(2/\alpha)\mathrm{E}_N} \propto {}_2F_1^{(1/\alpha)}(-N, \alpha(\lambda_1 + \lambda_2 + m + 1) + N - 1; \\
&\qquad\qquad \alpha(\lambda_1 + m); t_1, \dots, t_m)\big|_{t_1 = \dots = t_m = t}.
\end{aligned}
$$

In the case $\alpha = 1$, and with $\lambda_1 = a$, $\lambda_2 = b$, $m = \mu$ this coincides with the definition (1.10) of $\widetilde{E}_N^{\mathrm{J}}(t; a, b, \mu; 0)$. Hence from Proposition 13 we have a formula similar to (3.50) valid for $\mu, N \in \mathbb{Z}_{\ge 0}$

$$
(3.53)\quad \tau_3(t; \hat{\mathbf{b}}) \propto {}_2F_1^{(1)}(-N, N + a + b + \mu; a + \mu; t_1, \dots, t_\mu)\big|_{t_1 = \dots = t_\mu = t}.
$$

Results relating generalised hypergeometric functions to τ-functions can also be found in [1].

§4. $\mathrm{P_V}$ scaling limit

4.1. Scaling limit of the cJUE at the spectrum singularity

We now turn our attention to the scaled limit (1.26). To anticipate that it is well defined, one notes that the integrand defining the average has an interpretation in classical statistical mechanics as a log-gas. Thus it can be

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written in the form of a Boltzmann factor $e^{-\beta U}$ with inverse temperature $\beta = 2$ and potential energy

$$(4.1)\qquad U = -\sum_{j=1}^{N}\Big(\frac{\omega_2}{2}\theta_l + \omega_1 \log|1+z_l| + \mu \log|e^{i(\pi - X/N)} - z_l|\Big) - \sum_{1\le j<k\le N} \log|z_k - z_j|.$$

The classical particle system corresponding to (4.1) has N identical mobile charges interacting via a repulsive logarithmic potential, and subject to the discontinuous external potential $-(\omega_2/2)\theta_l$ $(-\pi < \theta < \pi)$. The mobile charges also interact with impurity charges at $\theta = \pi$ (of strength ω_1) and at $\theta = \pi - X/N$ (of strength μ). In random matrix language these impurities correspond to spectrum singularities of degenerate eigenvalues. We anticipate (1.26) to be well defined because changing variables $\theta_l \mapsto \theta_l/N$ gives a system which has spacing between particles of order unity, and X is measured in units of this spacing.

In the notation of (3.28), the average in (1.26) is denoted $A_N(t;\omega_1,\omega_2,\mu;\xi^*)$. It follows from (3.29), (3.34) and Proposition 16 that

$$t(t-1)\frac{d}{dt}\log A_N(t;\omega_1,\omega_2,\mu;\xi^*) = \sigma(t) - C_1 t + C_2$$

where $\sigma(t)$ satisfies (1.27) with $\mathbf{b}$ given by (3.31) and C_1, C_2 given by (3.33). Since with $t = e^{iX/N}$,

$$t(t-1)\frac{d}{dt} \sim X\frac{d}{dX}$$

it follows from the definition of u in (1.26) that

$$\lim_{N\to\infty}\Big((\sigma(t) - C_1 t + C_2)\big|_{t=e^{iX/N}}\Big) = u(X;(\omega_1,\omega_2),\mu;\xi).$$

This limit can be taken directly in the differential equation (1.27).

Proposition 19. *Let C_1, C_2 be defined as in* (3.33), *let* $\mathbf{b}$ *be given by* (3.31), *and replace h in* (1.27) *by $u + C_1 t - C_2$. Then with the change of variable $t = e^{iX/N}$, as $N \to \infty$ the leading terms of* (1.27) *are of $O(N^2)$, and give that*

$$(4.2)\qquad h(t) = u(it;(\omega_1,\omega_2),\mu;\xi) + \frac{i\omega_2}{2}t + 2\omega_1\mu + \frac{\omega_2^2}{2}$$

satisfies (1.42) *with*

$$\tilde{v}_1 = \mu,\ \tilde{v}_2 = -\mu,\ \tilde{v}_3 = \omega,\ \tilde{v}_4 = -\bar{\omega},\ \tilde{v}_j := v_j + \frac{i\omega_2}{2},\ \omega := \omega_1 + i\omega_2. \tag{4.3}$$

Proof. Direct substitution with the change of variables $t = e^{iX/N}$ and expanding in N for N large shows that the leading order term is proportional to N^2. Equating the coefficient of N^2 to zero gives

$$-\Big(X\frac{d^2u}{dX^2}\Big)^2 - 4X\Big(\frac{du}{dX}\Big)^3 + 4u\Big(\frac{du}{dX}\Big)^2 + [4(\mu+\omega_1)^2 - 4\omega_2 X - X^2]\Big(\frac{du}{dX}\Big)^2$$
$$+ 4\big(\omega_2 + \frac{X}{2}\big)u\frac{du}{dX} + 8\mu\big(\omega_1\omega_2 + \mu\omega_2 + \frac{\omega_1}{2}X\big)\frac{du}{dX} - u^2 - 4\omega_1\mu u$$
$$+ (2\mu\omega_2)^2 = 0.$$

Changing variables $X \mapsto it$ and comparing the resulting equation with (1.42) after the replacement $h_V \mapsto h + a_1 t + a_2$ shows that the equations for $u(it)$ and h coincide provided

$$a_1 = \frac{1}{2}i\omega_2, \quad a_2 = \frac{1}{2}\omega_2^2 + 2\omega_1\mu$$

and the equations

$$\begin{aligned} e_1[\tilde{\mathbf{v}}] &= 2i\omega_2, & e_2[\tilde{\mathbf{v}}] &= -(\mu^2 + \omega_1^2 + \omega_2^2) \\ e_3[\tilde{\mathbf{v}}] &= -2i\mu^2\omega_2, & e_4[\tilde{\mathbf{v}}] &= \mu^2(\omega_1^2 + \omega_2^2) \end{aligned} \tag{4.4}$$

hold. Direct substitution verifies (4.4) is satisfied by (4.3). ∎

The boundary condition which (4.2) is to satisfy can be predicted in the cases $\xi = 0$ and $\xi = 1$ from the log-gas interpretation of the average in (1.26). Consider first the case $\xi = 0$. Then the potential energy function (4.1) is being averaged over the whole circle. For large X (after the scaled limit has been taken), to leading order the impurity charge originally at $z = e^{i(\pi - X/N)}$ on the circle is expected to decouple from the impurity charge at $z = e^{i\pi}$, meaning that the average in (1.26) will be independent of X. However, for this to happen the potential energy must be modified to include the term $-2\omega_1\mu \log|1 + e^{-iX/N}|$ corresponding to the interaction between the two impurity charges (see [18] for similar applications of this argument). This leads to the prediction

$$u(X; (\omega_1, \omega_2), \mu; 0) \underset{X\to\infty}{\sim} -2\omega_1\mu \tag{4.5}$$
$$\text{and thus} \quad h(t) \underset{t\to -i\infty}{\sim} \frac{i\omega_2}{2}t + \frac{\omega_2^2}{2}.$$

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In the case $\xi = 1$, the average (1.26) corresponds to excluding the particles from the interval $(\pi - X/N, \pi)$. To leading order one would expect the large X behaviour to be independent of the parameters ω_1, ω_2, μ and thus to be given by the $\omega_1 = \omega_2 = \mu = 0$ results [69], [17] which implies

$$u(X; (\omega_1, \omega_2), \mu; 1) \underset{X\to\infty}{\sim} -\frac{X^2}{16} + O(1) \tag{4.6}$$
$$\text{and thus} \quad h(t) \underset{t\to -i\infty}{\sim} \frac{t^2}{16} + \frac{i\omega_2}{2} t + O(1).$$

4.2. Multi-dimensional integral solutions

As commented in the Introduction, the differential equation (1.42) is the analogue of (1.27) for the PV system. Specifically, one considers [55] a Hamiltonian H_V associated with the Painlevé V equation,

$$tH_V = q(q-1)^2p^2 - \Big\{(v_2 - v_1)(q-1)^2 - 2(v_1 + v_2)q(q-1) + tq\Big\}p \\ + (v_3 - v_1)(v_4 - v_1)(q-1)$$

where the parameters $v_1, \dots, v_4$ are constrained by

$$v_1 + v_2 + v_3 + v_4 = 0.$$

It is well known that eliminating p in the Hamilton equations (2.2) gives the general $\mathrm{P_V}$ equation with $\delta = -1/2$ in the usual notation. Our interest is in the further fact that the auxiliary Hamiltonian

$$h_V = tH_V + (v_3 - v_1)(v_4 - v_1) - v_1 t - 2v_1^2$$

satisfies (1.42), which is the analogue of the fact that (2.7) satisfies (1.27). Using the Hamiltonian formalism, we have previously constructed determinant/multi-dimensional integral solutions of (1.42) [30]. To make use of these results, we first note that with

$$h_V = \sigma - v_1 t - 2v_1^2 \tag{4.7}$$

it follows from (1.42) that the quantity σ satisfies

$$(t\sigma'')^2 - [\sigma - t\sigma' + 2(\sigma')^2 + (\nu_0 + \nu_1 + \nu_2 + \nu_3)\sigma']^2 \\ + 4(\nu_0 + \sigma')(\nu_1 + \sigma')(\nu_2 + \sigma')(\nu_3 + \sigma') = 0 \tag{4.8}$$

with

$$\nu_0 = 0, \quad \nu_1 = v_2 - v_1, \quad \nu_2 = v_3 - v_1, \quad \nu_3 = v_4 - v_1. \tag{4.9}$$

Now, a family of multi-dimensional integral solutions of (4.8) found in [30, Prop. 3.1 together with (1.45)] states that

$$t\frac{d}{dt}\log\Big(t^{an+n^2}\big\langle e^{-t\sum_{j=1}^n x_j}\big\rangle_{\mathrm{JUE}_n}\Big) = U_n^{\mathrm{L}}(t;a,b) \tag{4.10}$$

where $U_n^{\mathrm{L}}(t;a,b)$ satisfies (4.8) with

$$\nu_0 = 0, \quad \nu_1 = -b, \quad \nu_2 = n+a, \quad \nu_3 = n. \tag{4.11}$$

The relevance of (4.10) for the present purposes is that in the special case $m \in \mathbb{Z}^+$ it follows from (3.42) that

$$\begin{aligned} \lim_{N\to\infty} X\frac{d}{dX}\log\Big\langle\prod_{l=1}^N z_l^{(\eta_1-\eta_2)/2}|1+z_l|^{\eta_1+\eta_2}(1+e^{-X/N}z_l)^m\Big\rangle_{\mathrm{CUE}_N} \\ = X\frac{d}{dX}\log\big\langle e^{-X\sum_{j=1}^m x_j}\big\rangle_{\mathrm{JUE}_m}\Big|_{\substack{\lambda_1=\eta_2-m \\ \lambda_2=\eta_1}}. \end{aligned} \tag{4.12}$$

We recognise the left hand side of (4.12) as a rewrite of the left hand side of (1.26) in the case $\xi = \xi^*$, where $\xi^* = 0$ for m even and $\xi^* = 2$ for m odd. Consequently

$$\begin{aligned} X\frac{d}{dX}\log\big\langle e^{-X\sum_{j=1}^m x_j}\big\rangle_{\mathrm{JUE}_m}\Big|_{\substack{\lambda_1=a \\ \lambda_2=b}} \\ = u\Big(iX;\big(\frac{a+b+m}{2}, i\frac{b-a}{2}\big), \frac{m}{2};\xi^*\Big) - \frac{m}{2}X. \end{aligned} \tag{4.13}$$

It thus follows from (4.12) and Proposition 19 that

$$\begin{aligned} t\frac{d}{dt}\log\big\langle e^{-t\sum_{j=1}^n x_j}\big\rangle_{\mathrm{JUE}_n} \\ = h(t) + \frac{1}{4}(b-a-2n)t - \frac{n}{2}(a+b+n) + \frac{1}{8}(b-a)^2 \end{aligned} \tag{4.14}$$

where $h(t)$ satisfies (4.8) with $v_1 = n/2 + (b-a)/4$ and $\nu_0, \dots, \nu_3$ as in (4.11). Use of (4.7) to substitute for $h(t)$ in (4.14) reclaims the fact that (4.10) satisfies (4.8) with parameters (4.11).

4.3. Generalised hypergeometric function expressions

The generalised hypergeometric function ${}_2F_1^{(\alpha)}$ has the confluence property [73]

$$\lim_{L\to\infty} {}_2F_1^{(\alpha)}(L,b;c;x/L) = {}_1F_1^{(\alpha)}(b;c;x), \tag{4.15}$$

which follows easily from the series definition (3.48). Applying (4.15) to (3.49) with $t \mapsto -t/r$, $r \to \infty$ gives [20]

$$\begin{aligned}\Big\langle \prod_{l=1}^n e^{-tx_l} \Big\rangle_{\mathrm{J}(2/\alpha)\mathrm{E}_n} = {}_1F_1^{(\alpha)}\Big(&\frac{1}{\alpha}(n-1)+\lambda_1+1;\\ &\frac{2}{\alpha}(n-1)+\lambda_1+\lambda_2+2; -t_1,\dots,-t_n\Big)\Big|_{t_1=\cdots=t_n=t}.\end{aligned} \tag{4.16}$$

In the case $\alpha = 1$, $\lambda_1 = a$, $\lambda_2 = b$ we can substitute (4.16) in (4.10) to deduce

$$\begin{aligned} t\frac{d}{dt}\log t^{an+n^2}\, {}_1F_1^{(1)}(n+a, 2n+a+b; -t_1,\dots,-t_n)\big|_{t_1=\cdots=t_n=t} \\ = U_n^{\mathrm{L}}(t;a,b)\end{aligned} \tag{4.17}$$

where $U_n^{\mathrm{L}}(t;a,b)$ satisfies (4.8) with

$$\nu_0 = 0, \quad \nu_1 = -b, \quad \nu_2 = n+a, \quad \nu_3 = n.$$

Furthermore, applying (1.20) to the left hand side of (4.16) we deduce [20]

$$\begin{aligned}\Big\langle \prod_{l=1}^n z_l^{(a'-b')/2} |1+z_l|^{a'+b'} e^{-tz_l} \Big\rangle_{\mathrm{C}(2/\alpha)\mathrm{E}_n} \\ \propto {}_1F_1^{(\alpha)}\Big(-b'; \frac{1}{\alpha}(n-1)+a'+1; t_1,\dots,t_n\Big)\Big|_{t_1=\cdots=t_n=t}\end{aligned} \tag{4.18}$$

§5. Applications

5.1. Eigenvalue distributions in ensembles with unitary symmetry

Let a sequence of N eigenvalues $x_1,\dots,x_N$ have joint distribution $p_N(x_1,\dots,x_N)$ and let the support of these eigenvalues be an interval I. Let $I_0 \subset I$, and consider the multi-dimensional integral

$$E_N(I_0;\xi) := \Big(\int_I - \xi\int_{I_0}\Big)dx_1 \cdots \Big(\int_I - \xi\int_{I_0}\Big)dx_N\, p_N(x_1,\dots,x_N). \tag{5.1}$$

We see immediately that the probability $E_{N,n}(I_0)$ of there being exactly n eigenvalues in the interval I_0 is related to (5.1) by

$$E_{N,n}(I_0) = \frac{(-1)^n}{n!}\frac{\partial^n}{\partial\xi^n} E_N(I_0;\xi)\Big|_{\xi=1}.$$

Equivalently

$$\sum_{n=0}^{\infty}(1-\xi)^n E_{N,n}(I_0) = E_N(I_0;\xi). \tag{5.2}$$

In particular, in an obvious notation it follows from (1.10) and (1.18) that

$$E^{\rm J}_{N,n}((s,1);a,b) = \frac{(-1)^n}{n!}\frac{\partial^n}{\partial\xi^n}\widetilde{E}^{\rm J}_N(s;a,b,0;\xi)\Big|_{\xi=1} \tag{5.3}$$

$$E^{\rm cJ}_{N,n}((\pi-\phi,\pi);(\omega_1,\omega_2)) = \frac{(-1)^n}{n!}\frac{\partial^n}{\partial\xi^n}\widetilde{E}^{\rm cJ}_N(\phi;(\omega_1,\omega_2),0;\xi)\Big|_{\xi=1}. \tag{5.4}$$

According to Proposition 13, and the fact that with $s=1$, $\widetilde{E}^{\rm J}_N$ in (5.3) equals unity, we have

$$\begin{aligned} E^{\rm J}_{N,n}((s,1);a,b) = {} & \frac{(-1)^n}{n!}\frac{\partial^n}{\partial\xi^n} \\ & \times \exp\Big\{-\int_s^1\Big(\widehat{U}^{\rm J}_N(t;a,b,0;\xi) - e_2'[\hat{\mathbf{b}}]t + \frac{1}{2}e_2[\hat{\mathbf{b}}]\Big)\frac{dt}{t(t-1)}\Big\}\Big|_{\xi=1} \end{aligned} \tag{5.5}$$

where $\widehat{U}^{\rm J}_N$ is specified as in Proposition 13 with boundary condition (1.38). The result (1.41) with $\mu=0$ gives an analogous evaluation of (5.4).

Suppose now that $I_0=(s,t)$. Then the quantity

$$\begin{aligned} E_N(I_0;s;\xi) := (N+1)\Big(\int_I - \xi\int_{I_0}\Big)dx_1\cdots\Big(\int_I - \xi\int_{I_0}\Big)dx_N & \\ \times\, p_{N+1}(s,x_1,\dots,x_N) & \end{aligned} \tag{5.6}$$

is related to the probability density $p_{N,n}(I_0,s)$ of there being an eigenvalue at s as well as n eigenvalues in the interval $I_0=(s,t)$. Thus we have

$$p_{N,n}(I_0,s) = \frac{(-1)^n}{n!}\frac{\partial^n}{\partial\xi^n}E_N(I_0;s;\xi)\Big|_{\xi=1}.$$

Now for the JUE with $I_0 = (s, 1)$ the quantity (5.6) is proportional to $s^a(1-s)^b$ times $\widetilde{E}_N^{\rm J}(s; a, b, 2; \xi)$ as specified by (1.10). Writing the proportionality in terms of the integral (1.23), it follows from Proposition 13 that

$$
\text{(5.7)}\quad p^{\rm J}_{N,n}((s,1);a,b) = (N+1)\frac{J_N(a,b+2)}{J_{N+1}(a,b)} s^a(1-s)^b \frac{(-1)^n}{n!}\frac{\partial^n}{\partial \xi^n} \times \exp\left\{ -\int_s^1 \left(\widehat{U}^{\rm J}_N(t;a,b,2;\xi) - e_2'[\hat{\mathbf{b}}]t + \frac{1}{2}e_2[\hat{\mathbf{b}}]\right)\frac{dt}{t(t-1)}\right\}\Bigg|_{\xi=1}
$$

where $\widehat{U}^{\rm J}_N$ satisfies the differential equation specified as in Proposition 13. The boundary condition which $\widehat{U}^{\rm J}_N$ must satisfy is given by the following result.

Proposition 20. *We have*

$$
\text{(5.8)}\quad \widehat{U}^{\rm J}_N(t;a,b,2;\xi) \underset{t\to 1^-}{\sim} e_2'[\hat{\mathbf{b}}] - \frac{1}{2}e_2[\hat{\mathbf{b}}] + O(1-t) + d_0(1-t)^{b+3}[1 + O(1-t)] + \cdots
$$

where

$$
\text{(5.9)}\quad d_0 = -2\xi \frac{1}{(b+1)(b+2)\Gamma(b+3)\Gamma(b+4)} \frac{\Gamma(a+b+N+3)}{\Gamma(a+N)} \frac{\Gamma(b+N+3)}{\Gamma(N)}
$$

and the terms $O(1-t)$ are analytic in $1-t$.

Proof. The constant term follows immediately from the requirement that the integrand in (5.7) be integrable at $t = 1$. To deduce the explicit form of the leading non analytic term we note that the analogue of (5.2) applied to (5.7) gives

$$
\text{(5.10)}\quad p^{\rm J}_{N-1,0}((s,1);a,b) + (1-\xi)p^{\rm J}_{N-1,1}((s,1);a,b) \underset{s\to 1^-}{\sim} N\frac{J_{N-1}(a,b+2)}{J_N(a,b)} s^a(1-s)^b \times \exp\left\{ -\int_s^1 \left(\widehat{U}^{\rm J}_N(t;a,b,2;\xi) - e_2'[\hat{\mathbf{b}}]t + \frac{1}{2}e_2[\hat{\mathbf{b}}]\right)\Big|_{N\mapsto N-1}\frac{dt}{t(t-1)}\right\}.
$$

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But it follows from the definitions that

$$p^{\mathrm{J}}_{N-1,0}((s,1);a,b) \underset{s\to 1^-}{\sim} \rho^{\mathrm{J}}_{N,1}(s) - \int_s^1 \rho^{\mathrm{J}}_{N,2}(s,t)\,dt + \cdots \tag{5.11}$$

$$p^{\mathrm{J}}_{N-1,1}((s,1);a,b) \underset{s\to 1^-}{\sim} \int_s^1 \rho^{\mathrm{J}}_{N,2}(s,t)\,dt + \cdots \tag{5.12}$$

where $\rho^{\mathrm{J}}_{N,n}$ denotes the n-point distribution function in the JUE with N eigenvalues. Hence, making use also of the ansatz (5.8), (5.10) reduces to

$$\begin{aligned}\rho^{\mathrm{J}}_{N,1}(s) - \xi \int_s^1 \rho^{\mathrm{J}}_{N,2}(s,t)\,dt + \cdots &\underset{s\to 1^-}{\sim} N\frac{J_{N-1}(a,b+2)}{J_N(a,b)} s^a(1-s)^b \\ &\times \Big\{1 + O(1-s) + \cdots + \frac{d_0|_{N\mapsto N-1}}{b+3}(1-s)^{b+3}[1+O(1-s)] + \cdots\Big\}.\end{aligned} \tag{5.13}$$

For the first term on the left hand side of (5.13) we note from the definition of $\rho^{\mathrm{J}}_{N,1}(s)$ as a multiple integral that

$$\rho^{\mathrm{J}}_{N,1}(s) \underset{s\to 1^-}{\sim} N\frac{J_{N-1}(a,b+2)}{J_N(a,b)} s^a(1-s)^b,$$

which is in agreement with the first term on the right hand side of (5.13). For the term proportional to ξ we note from the definition of $\rho^{\mathrm{J}}_{N,2}(s,t)$ as a multiple integral that

$$\rho^{\mathrm{J}}_{N,2}(s,t) \underset{s,t\to 1^-}{\sim} N(N-1)\frac{J_{N-2}(a,b+4)}{J_N(a,b)}(1-s)^b(1-t)^b(s-t)^2,$$

which gives

$$\begin{aligned}&\int_s^1 \rho^{\mathrm{J}}_{N,2}(s,t)\,dt \\ &\quad\underset{s\to 1^-}{\sim} N(N-1)\frac{J_{N-2}(a,b+4)}{J_N(a,b)}(1-s)^{2b+3}\frac{2}{(b+1)(b+2)(b+3)}.\end{aligned}$$

Substituting in (5.13) and making use of (1.25) shows this is consistent with the value of d_0 in (5.9). ∎

The probability $p_{N,n}$ for the JUE with $I_0 = (0,s)$ can be similarly characterised. In fact by changing variables $x_l \mapsto 1 - x_l$ in (1.10) we see that

$$p^{\mathrm{J}}_{N,n}((0,s);a,b) = p^{\mathrm{J}}_{N,n}((s,1);b,a). \tag{5.14}$$

Again with $I_0 = (s,t)$ we introduce the quantity

$$E_N(I_0;(s,t);\xi) := (N+2)(N+1)\Big(\int_I - \xi\int_{I_0}\Big)dx_1\cdots \times\Big(\int_I - \xi\int_{I_0}\Big)dx_N\, p_{N+2}(s,t,x_1,\dots,x_N). \tag{5.15}$$

We have

$$p_{N,n}(I_0,(s,t)) = \frac{1}{\rho_{N+2}(t)}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial\xi^n}E_N(I_0;(s,t);\xi)\Big|_{\xi=1}$$

where $p_{N,n}(I_0,(s,t))$ denotes the probability density of there being an eigenvalue at s and n eigenvalues in $I_0 = (s,t)$, given there is an eigenvalue at t ($\rho_{N+2}(t)$ denotes the eigenvalue density at t). For the CUE the eigenvalue density is a constant. Choosing $(s,t) = (\pi-\phi,\pi)$, (5.15) is proportional to $\sin^2\phi/2$ times $\widetilde{E}_N^{\mathrm{cJ}}(\phi;(1,0),2;\xi)$ as specified by (1.18). Using (1.41) and (1.24) we deduce

$$\Big(\frac{2\pi}{N}\Big)p^{\mathrm{CUE}}_{N-2,n}\Big(\frac{2\pi X}{N}\Big) = \frac{1}{3}(N^2-1)\sin^2\frac{\pi X}{N}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial\xi^n} \times\exp\Big\{-\frac{1}{2}\int_0^{2\pi X/N}\Big(U^{\mathrm{Cy}}_{N-2}\big(\cot\frac{\theta}{2};(N,0),2;\xi\big)+4\cot\frac{\theta}{2}\Big)d\theta\Big\}\Big|_{\xi=1}. \tag{5.16}$$

The boundary condition can be deduced by adapting a strategy analogous to that used in the proof of Proposition 20. We find

$$\frac{1}{2}\Big(U^{\mathrm{Cy}}_{N-2}\big(\cot\frac{\theta}{2};(N,0),2;\xi\big)+4\cot\frac{\theta}{2}\Big) \sim \frac{2}{15}(N^2-4)\tan\frac{\theta}{2} + O(\tan^3\frac{\theta}{2}) + \frac{(\xi-1)}{540\pi}\frac{(N+2)!}{(N-3)!}\Big(\tan^4\frac{\theta}{2}+O\big(\tan^6\frac{\theta}{2}\big)\Big) \tag{5.17}$$

where are term $O(\tan^3\frac{\theta}{2})$ are odd in θ while all terms $O(\tan^6\frac{\theta}{2})$ are even in θ.

The eigenvalue density is another statistical property of the eigenvalues accessible from the average (1.10) (for the JUE) and (1.18) (for the cJUE). We must choose $\mu = 2$ and $\xi = 0$, and multiply (1.10) by $x^a(1-x)^b$ and (1.18) by $e^{\omega_2\phi}|1+e^{i\phi}|^{2\omega_1}$. We then obtain expressions proportional to the definition of the density in an ensemble of $N+1$ eigenvalues, ρ_{N+1} say. Making use of Proposition 13 it therefore follows that

$$\rho^{\mathrm{J}}_{N+1}(s) = (N+1)\frac{J_N(a+2,b)}{J_{N+1}(a,b)}s^a(1-s)^b \times \exp\Big\{\int_0^s \Big(\widehat{U}^{\mathrm{J}}_N(t;a,b,2;0) - e_2'[\hat{\mathbf{b}}]t + \frac{1}{2}e_2[\hat{\mathbf{b}}]\Big)\frac{dt}{t(t-1)}\Big\} \tag{5.18}$$

where $\widehat{U}^{\mathrm{J}}_N$ is specified as in Proposition 13 with boundary condition (1.36). An analogous formula can be written down for $\rho^{\mathrm{cJ}}_{N+1}(\phi)$ using (1.41).

The term in (5.18) involving the exponential function is proportional to $\widetilde{E}^{\mathrm{J}}_N(s;a,b,2;0)$. According to Proposition 18 and (3.45)

$$\widetilde{E}^{\mathrm{J}}_N(s;a,b,2;0) \propto s^{-1}\begin{vmatrix} P_N^{(a,b)}(1-2s) & P_N^{(a,b+1)}(1-2s) \\ P_{N+1}^{(a,b-1)}(1-2s) & P_{N+1}^{(a,b)}(1-2s)\end{vmatrix}. \tag{5.19}$$

On the other hand, from the fact that $\big\{P_n^{(a,b)}(1-2s)\big\}_{n=0,1,\dots}$ are orthogonal with respect to the weight function $s^a(1-s)^b$, $(0<s<1)$, well known direct integration methods give that

$$\widetilde{E}^{\mathrm{J}}_N(s;a,b,2;0) \propto \begin{vmatrix} P_{N+1}^{(a,b)}(1-2s) & \frac{d}{ds}P_{N+1}^{(a,b)}(1-2s) \\ P_{N}^{(a,b)}(1-2s) & \frac{d}{ds}P_{N}^{(a,b)}(1-2s)\end{vmatrix}. \tag{5.20}$$

Using Jacobi polynomial identities, (5.20) can be reduced to (5.19).

The above results apply to the finite N ensembles. Also of interest are scaled $N\to\infty$ limits of these results. As mentioned in the introduction, we are restricting attention of such limits to the spectrum singularity. One example of the latter type is

$$p_n^{\mathrm{bulk}}(X) := \lim_{N\to\infty}\frac{2\pi}{N}p^{\mathrm{CUE}}_{N-2,n}\Big(\frac{2\pi X}{N}\Big).$$

For its evaluation we read off from (5.16) that

$$p_0^{\mathrm{bulk}}(X) \underset{X\to 0}{\sim} \frac{\pi^2}{3}X^2,$$

and note that the scaled limit of (5.15) for the CUE is proportional to the average in (1.26) with

$$\omega_2 = 0, \quad \omega_1 = \mu = 1. \tag{5.21}$$

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Thus

$$p_n^{\text{bulk}}(X) = \frac{\pi^2}{3} X^2 \frac{(-1)^n}{n!} \frac{\partial^n}{\partial \xi^n} \exp\left\{ \int_0^{2\pi X} u(t;(1,0),1;\xi) \frac{dt}{t} \right\}\Big|_{\xi=1} \tag{5.22}$$

where according to Proposition 19 the quantity $h(t) = u(it;(1,0),1;\xi) + 2$ satisfies (1.42). Substituting the parameter values (5.21) in (4.3) it follows from (4.2) and (1.42) that u itself satisfies

$$(su'')^2 + (u - su')\{u - su' + 4 - 4(u')^2\} - 16(u')^2 = 0 \tag{5.23}$$

with boundary condition

$$u(t;(1,0),1;\xi) \underset{t\to 0}{\sim} -\frac{1}{15}t^2 + O(t^4) - \frac{(\xi-1)}{8640\pi}\left(t^5 + O(t^7)\right) \tag{5.24}$$

where all terms $O(t^4)$ are even, while all terms $O(t^7)$ are odd. In the case $n = 0$, $p_n^{\text{bulk}}(X)$ is the distribution of consecutive neighbour spacings in the infinite CUE, scaled so that the mean eigenvalue spacing is unity. The formula (5.22) gives

$$p_0^{\text{bulk}}(X) = \frac{\pi^2}{3} X^2 \exp\left(\int_0^{2\pi X} \frac{h(-it;\mathbf{v}) - 2}{t}\, dt \right), \quad \mathbf{v} = (1,-1,1,-1). \tag{5.25}$$

Previous studies have given two alternative Painlevé σ-function evaluations of this same quantity. The first, due to Jimbo et al. [39], can be written

$$p_0^{\text{bulk}}(X) = \frac{d^2}{dX^2} \exp\left(\int_0^{2\pi X} \frac{h(-it;\mathbf{v})}{t}\, dt \right), \quad \mathbf{v} = (0,0,0,0) \tag{5.26}$$

while the second, due to the present authors [28], reads

$$p_0^{\text{bulk}}(X) = -\frac{d}{dX} \exp\left(\int_0^{2\pi X} \frac{h(-it;\mathbf{v})}{t}\, dt \right), \quad \mathbf{v} = (0,0,1,-1) \text{ or } (1,-1,0,0) \tag{5.27}$$

(the boundary conditions in (5.26) and (5.27) are not required for the present purpose). In each of (5.25)–(5.27) the function $h(t;\mathbf{v})$ satisfies (1.42) with $\mathbf{v}$ as specified.

The equality between (5.26) and (5.27) implies the identity

$$h(s;(1,-1,0,0)) = -1 + h(s;(0,0,0,0)) + s\frac{h'(s;(0,0,0,0))}{h(s;(0,0,0,0))}. \tag{5.28}$$

A direct proof of this identity was given in [71] using the formulas of [16] for h_V and h'_V in (1.42) in terms of a particular $\mathrm{P_V}$ transcendent and its derivative. Analogous to (5.28), the equality between (5.27) and (5.25) implies the identity

$$h(s;(1,-1,1,-1)) = -1 + h(s;(0,0,1,-1)) + s\frac{h'(s;(0,0,1,-1))}{h(s;(0,0,1,-1))}. \tag{5.29}$$

In fact both (5.28) and (5.29) are particular examples of an identity between functions satisfying the σ-form of the $\mathrm{P_V}$ equation proved in our paper [28]. With $\sigma = \sigma(t;\mathbf{v})$ specified as a solution of (4.8) with $\mathbf{v}$ given by

$$\begin{aligned}\mathbf{v} = \Big(-\frac{1}{4}(2N+a-\mu), -\frac{1}{4}(2N+a+3\mu),\\ \frac{1}{4}(2N+3a+\mu), \frac{1}{4}(2N-a+\mu)\Big)\end{aligned} \tag{5.30}$$

the identity states

$$\sigma(t;\mathbf{v})\Big|_{\mu=2} = -(a+1+2N) + t + \Big(\sigma(t;\mathbf{v}) + t\frac{\sigma'(t;\mathbf{v})}{\sigma(t;\mathbf{v})}\Big)\Big|_{\substack{\mu=0\\ N\mapsto N+1}}. \tag{5.31}$$

Now, from (4.7),

$$\begin{aligned} h(t;(1,-1,1,-1)) &= \sigma(t;(1,-1,1,-1)) - t - 2,\\ h(t;(1,-1,0,0)) &= \sigma(t;(1,-1,0,0)) - t - 2,\\ h(t;(0,0,1,-1)) &= \sigma(t;(0,0,1,-1)),\\ h(t;(0,0,0,0)) &= \sigma(t;(0,0,0,0)). \end{aligned} \tag{5.32}$$

To deduce (5.28) from (5.31) we set $\mu = 2$, $N = -1$, $a = 0$ in (5.30), giving $\mathbf{v} = (1,-1,0,0)$ on the left hand side, then set $\mu = 0$, $N = 0$, $a = 0$ giving $\mathbf{v} = (0,0,0,0)$ on the right hand side of (5.31). The identity (5.28) then follows by making use of the second and fourth formulas in (5.32). Similarly, (5.29) follows from (5.31) by choosing $\mu = 2$, $N = -2$, $a = 2$, then $\mu = 0$, $N = -1$, $a = 2$, and making use of the first and third identities in (5.32).

There is an analogue of (5.31) for functions satisfying the σ-form (1.27) of the $\mathrm{P_{VI}}$ equation. This can be obtained by substituting in the formula

$$p^{\mathrm{J}}_{N,n}((s,1);a,b) = \frac{d}{ds}E^{\mathrm{J}}_{N+1,n}((s,1);a,b) \tag{5.33}$$

the exact evaluations (5.7) and (5.5), and taking the logarithmic derivative. We find

(5.34)

$$\begin{aligned}
\widehat{U}^{\mathrm{J}}_N(s;a,b,2;\xi) &= -\Big(a+b+2+e_2'[\hat{\mathbf{b}}]\big|_{\substack{\mu=0\\ N\mapsto N+1}} - e_2'[\hat{\mathbf{b}}]\big|_{\mu=2}\Big)s \\
&\quad + \Big(a+1+\frac{1}{2}e_2[\hat{\mathbf{b}}]\big|_{\substack{\mu=0\\ N\mapsto N+1}} - \frac{1}{2}e_2[\hat{\mathbf{b}}]\big|_{\mu=2}\Big) + \widehat{U}^{\mathrm{J}}_{N+1}(s;a,b,0;\xi) \\
&\quad + s(s-1)\frac{d}{ds}\log\Big(\widehat{U}^{\mathrm{J}}_{N+1}(s;a,b,0;\xi) - e_2'[\hat{\mathbf{b}}]\big|_{\substack{\mu=0\\ N\mapsto N+1}} s + \frac{1}{2}e_2[\hat{\mathbf{b}}]\big|_{\substack{\mu=0\\ N\mapsto N+1}}\Big) \\
&= 1/2 - s + \widehat{U}^{\mathrm{J}}_{N+1}(s;a,b,0;\xi) + s(s-1)\frac{d}{ds}\log\Big(\widehat{U}^{\mathrm{J}}_{N+1}(s;a,b,0;\xi) \\
&\quad + \big(N+1+\frac{1}{2}(a+b)\big)^2(s-1/2) + \frac{1}{8}(a^2-b^2)\Big).
\end{aligned}$$

An alternative derivation of (5.34) can be given. Following the method first introduced in [71], and also used in [30] to derive (5.31), we use (2.26) and the action

$$s_1\mathbf{b} = (b_1,b_2,b_4,b_3)$$

to note that

$$s_1T_3^{-1}s_1T_3^{-1}(b_1,b_2,b_3+1,b_4-1) = (b_1,b_2,b_3,b_4-2).$$

This means that if we temporarily reorder the components in (1.35),

$$\mathbf{b} \longmapsto \Big(\frac{1}{2}(b-a), -\frac{1}{2}(a+b), \frac{1}{2}(a+b)+N, -\frac{1}{2}(a+b)-N-\mu\Big)$$

(as (1.27) is symmetric in the $\mathbf{b}$'s, this does not affect the $\widehat{U}^{\mathrm{J}}$, which are defined as solutions of (1.27)), then

$$\widehat{U}^{\mathrm{J}}_N(s;a,b,2;\xi) = s_1T_3^{-1}s_1T_3^{-1}\widehat{U}^{\mathrm{J}}_{N+1}(s;a,b,0;\xi) \tag{5.35}$$

Using (2.3), (2.7), (2.8), Table 1, (2.24) and (2.32), we can use computer algebra to verify that the right hand side of (5.35) agrees with the right hand side of (5.34), thus providing an independent derivation of this result.

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5.2. Relationship to the hard edge gap probability in the scaled LOE and LSE

The Laguerre orthogonal ensemble (LOE) refers to the PDF

$$\frac{1}{C}\prod_{l=1}^{N} x_l^a e^{-x_l/2} \prod_{1\le j<k\le N} |x_k - x_j|, \quad x_l \ge 0. \tag{5.36}$$

For $a = n-N-1$ ($n \ge N$), this is realised as the joint eigenvalue distribution of random matrices $A^T A$, where A is an $n \times N$ matrix with independent, identically distributed standard Gaussian random variables. Scaling by $x_l \mapsto X_l/4N$, and taking the limit $N \to \infty$ gives well defined distributions in the vicinity of $x = 0$ (the hard edge) [21]. One particular quantity of interest in this scaled limit is $E_1^{\mathrm{hard}}(s;a)$, the probability that the interval $(0,s)$ is free of eigenvalues. This was evaluated in terms of a Painlevé V transcendent in [19] and expressed as a τ-function for a $\mathrm{P_V}$ system in [31]. The latter result can be written

$$\frac{d}{dx}\log E_1^{\mathrm{hard}}(x^2;(a-1)/2) = \tilde{h}_V(x) \tag{5.37}$$

where

$$\sigma_V(x) := x\tilde{h}_V(x) + \frac{1}{4}x^2 - \frac{(a-1)}{2}x + \frac{a(a-1)}{4}$$

satisfies (4.8) with $t \mapsto 2x$ and

$$v_1 = -v_2 = \frac{1}{4}(a-1), \quad v_3 = -v_4 = \frac{1}{4}(a+1). \tag{5.38}$$

On the other hand, if follows from Proposition 19 that

$$u\Big(2ix; \big(\frac{1}{4}(a+1), 0\big), \frac{1}{4}(a-1); \xi\Big) - \frac{1}{2}(a-1)x + \frac{1}{4}a(a-1)$$

also satisfies (4.8) with $t \mapsto 2x$ and parameters (5.38). Hence

$$x\frac{d}{dx}\log E_1^{\mathrm{hard}}(x^2;(a-1)/2) + \frac{1}{4}x^2 \doteq u\Big(2ix; \big(\frac{1}{4}(a+1), 0\big), \frac{1}{4}(a-1); \xi\Big), \tag{5.39}$$

where we use the symbol $\doteq$ to mean that both sides satisfy the same differential equation.

In the special case $(a-1)/2 = m$, $m \in \mathbb{Z}_{\ge 0}$, if follows from the definition of $E_1^{\mathrm{hard}}(s;(a-1)/2)$ that it is analytic in s. With this value of $(a-1)/2$, the right hand side of (5.39) reads $u(2ix;((m+1)/2,0),m/2;\xi)$. But we know from (4.13) that u with these parameters, analytic in x, is given by an m-dimensional integral, thus leading to the following result.

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PROPOSITION 21. *For $m \in \mathbb{Z}_{\geq 0}$,*

$$E_1^{hard}(s^2;m) = e^{-s^2/8+ms}\big\langle e^{-2s\sum_{j=1}^m x_j}\big\rangle_{\mathrm{JUE}_m}\big|_{\lambda_1=\lambda_2=1/2}. \tag{5.40}$$

Proof. The only point which remains to be checked is that for u in (5.39) given by (4.13), both sides have the same small x behaviour, thus determining the boundary condition in their characterisation as the solution of the same differential equation. This is the same as showing that both sides of (5.40) have the same small s behaviour. Now, with the scaled density of the LOE ensemble (5.36) denoted $\rho_a(X)$, it's easy to see from the definitions that

$$\frac{d}{ds}E_1^{\mathrm{hard}}(s^2,m) \sim -2s\rho_{2m}(s^2).$$

But we know that [26]

$$\rho_a(s^2) = K^{\mathrm{hard}}(s^2,s^2) + \frac{J_{a+1}(s)}{4s}\Big(1 - \int_0^s J_{a-1}(v)\,dv\Big),$$
$$K^{\mathrm{hard}}(s^2,s^2) := \frac{1}{4}\Big((J_a(s))^2 - J_{a+1}(s)J_{a-1}(s)\Big).$$

For small s the term $J_{a+1}(s)/4s$ dominates, implying the result

$$\frac{d}{ds}E_1^{\mathrm{hard}}(s^2,m) \sim -\frac{s^{2m+1}}{2^{2m+2}\Gamma(2m+2)}. \tag{5.41}$$

To determine the small s behaviour of the right hand side of (5.40), we note that the JUE_m average with $\lambda_1 = \lambda_2 = 1/2$ coincides with an average over the unitary symplectic group $\mathrm{USp}(m)$ upon the change of variables $\lambda_j = \frac{1}{2}(\cos\theta_j + 1)$. Explicitly (see e.g. [30, eq. (1.26)])

$$e^{ms}\big\langle e^{-2s\sum_{j=1}^m x_j}\big\rangle_{\mathrm{JUE}_m}\big|_{\lambda_1=\lambda_2=1/2} = \big\langle e^{(s/2)\,\mathrm{Tr}\,U}\big\rangle_{U\in\mathrm{USp}(m)}. \tag{5.42}$$

Furthermore, it is known from the work of Rains [59] (see also [8]) that

$$\big\langle e^{(s/2)\,\mathrm{Tr}\,U}\big\rangle_{U\in\mathrm{USp}(m)} = \sum_{n=0}^{\infty}\frac{(s/2)^{2n}}{2^{2n}n!}\frac{f_{nm}^{\mathrm{inv}}}{(2n-1)!!} \tag{5.43}$$

where f_{nm}^{inv} denotes the number of fixed point free involutions of $\{1,2,\dots,2n\}$ constrained so that the length of the maximum decreasing subsequence is less than or equal to $2m$. Following [1], the small s behaviour of (5.43) follows by noting that from the definition, for $m \le n$ we have $f_{nm}^{\rm inv} = 2^n(2n-1)!!$ (the number of fixed point free involutions without constraint) while $f_{m+1\,m}^{\rm inv} = 2^{m+1}(2m+1)!! - 1$. Hence

$$\Big\langle e^{(s/2)\,{\rm Tr}\,U} \Big\rangle_{U \in {\rm USp}(m)} = \exp\bigg\{ s^2/8 - \frac{(s/2)^{2(m+1)}}{2^{m+1}(m+1)!(2m+1)!!} + O(s^{2(m+2)}) \bigg\}.$$

The derivative of this expression multiplied by $e^{-s^2/8}$ has leading order behaviour (5.41), as required. ∎

Note that substituting (5.42) in (5.40) gives the identity

$$E_1^{\rm hard}((2s)^2; m) = e^{-s^2/2} \Big\langle e^{s\,{\rm Tr}\,U} \Big\rangle_{U \in {\rm USp}(m)}. \tag{5.44}$$

Consider next the Laguerre symplectic ensemble (LSE) which refers to the PDF

$$\frac{1}{C} \prod_{l=1}^N x_l^a e^{-x_l} \prod_{1 \le j < k \le N} (x_k - x_j)^4, \quad x_l \ge 0.$$

The corresponding scaled hard edge gap probability $E_4^{\rm hard}(s;a)$ has been shown in [31] to have the τ-function evaluation

$$E_4^{\rm hard}(x^2; a+1) = \frac{1}{2}\Big(\tilde{\tau}_V(x) + \tilde{\tau}_V^*(x) \Big) \tag{5.45}$$

where

$$\frac{d}{dx} \log \tilde{\tau}_V(x) = \tilde{h}_V(x) \quad \text{and} \quad \frac{d}{dx} \log \tilde{\tau}_V^*(x) \doteq \tilde{h}_V(x)$$

(recall the meaning of $\doteq$ from (5.39)). Thus

$$\tilde{\tau}_V(x) = E_1^{\rm hard}(x^2; (a-1)/2) \tag{5.46}$$

and the logarithmic derivative of both τ-functions in (5.45) satisfy the same differential equation, differing only in the boundary condition. From [31] we require

$$x \frac{d}{dx} \log \tilde{\tau}_V^*(x) \underset{x \to 0^+}{\sim} \frac{x}{2} J_a(x) \sim \frac{x^{a+1}}{2^{a+1}\Gamma(a+1)}. \tag{5.47}$$

Analogous to the results (5.40) and (5.44), $\tilde{\tau}_V^*(x)$ with $(a-1)/2 = m$, $m \in \mathbb{Z}_{\ge 0}$, can be written as an average over the JUE, or equivalently as an

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average over a classical group. To see this, we first note from Proposition 19 that

$$u(it;(\omega_1,0),\mu;\xi) \doteq u(it;(\mu,0),\omega_1;\xi),$$

and use this together with (5.39) and (5.46) to deduce

$$x\frac{d}{dx}\log\tilde{\tau}^*_V(x)+\frac{1}{4}x^2 \doteq u\Big(2ix;\big(\frac{1}{4}(a-1),0\big),\frac{1}{4}(a+1);\xi\Big).$$

But for $(a-1)/2=m$ and thus $(a+1)/4=(m+1)/2$, we see from (4.13) that

$$\begin{aligned} u\Big(2ix;\big(\frac{1}{2}(m-1),0\big),\frac{1}{2}(m+1);\xi^*\Big) \\ \doteq x\frac{d}{dx}\log\Big(e^{(m+1)x}\big\langle e^{-2x\sum_{j=1}^{m+1}x_j}\big\rangle_{\mathrm{JUE}_{m+1}}\big|_{\lambda_1=\lambda_2=-1/2}\Big). \end{aligned} \tag{5.48}$$

Because, as will be checked below, the small x behaviour of (5.48) coincides with (5.47), we can read off the expression for $\tilde{\tau}^*_V(x)$ as an average over the JUE.

Proposition 22. *For $(a-1)/2=m$, $m\in\mathbb{Z}_{\ge 0}$,*

$$\tilde{\tau}^*_V(x)=e^{-x^2/8+(m+1)x}\big\langle e^{-2x\sum_{j=1}^{m+1}x_j}\big\rangle_{\mathrm{JUE}_{m+1}}\big|_{\lambda_1=\lambda_2=-1/2}. \tag{5.49}$$

Proof. We have to verify the assertion that (5.49) is consistent with (5.47). Again following [1], we proceed in an analogous fashion to the proof of Proposition 21. Now, in the case $\lambda_1=\lambda_2=-1/2$, the JUE_{m+1} average coincides with an average over matrices in $O^+(2m+2)$ — $(2m+2)\times(2m+2)$ orthogonal matrices with determinant $+1$. Analogous to (5.42) we have

$$e^{(m+1)x}\big\langle e^{-2x\sum_{j=1}^{m+1}x_j}\big\rangle_{\mathrm{JUE}_{m+1}}\big|_{\lambda_1=\lambda_2=-1/2}=\big\langle e^{(x/2)\,\mathrm{Tr}\,U}\big\rangle_{U\in O^+(2m+2)}. \tag{5.50}$$

But we know from [1] (see also [67]) that

$$\big\langle e^{x\,\mathrm{Tr}\,U}\big\rangle_{U\in O^{\pm}(l)}=\exp\Big(\frac{x^2}{2}\pm\frac{x^l}{l!}+O(x^{l+1})\Big).$$

Thus

$$\tilde{\tau}^*_V(x)=\exp\Big(\frac{(^1\!/_2x)^{2m+2}}{(2m+2)!}+O(x^{2m+3})\Big),$$

which, recalling $(a-1)/2=m$, is consistent with (5.47). ∎

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We can express $E_4^{\rm hard}(x^2; a+1)$ in a form analogous to (5.44). For this purpose we recall that the joint PDF of the eigenvalues from the group $\mathrm{USp}(l)$, coincides with the joint PDF of the eigenvalues from the group $\mathrm{O}^-(2l+2)$ (not including the two fixed eigenvalues $\lambda = \pm 1$). Using this fact to rewrite the average in (5.44), substituting in (5.46), substituting (5.50) in (5.49), and substituting the results in (5.45) shows

$$
\begin{aligned}
(5.51)\quad & E_4^{\rm hard}(x^2; 2m+2) \\
&= \frac{1}{2} e^{-x^2/8} \Big(\big\langle e^{(x/2)\,\mathrm{Tr}\,U} \big\rangle_{U \in O^+(2m+2)} + \big\langle e^{(x/2)\,\mathrm{Tr}\,U} \big\rangle_{U \in O^-(2m+2)} \Big) \\
&= e^{-x^2/8} \big\langle e^{(x/2)\,\mathrm{Tr}\,U} \big\rangle_{U \in O(2m+2)}.
\end{aligned}
$$

5.3. Last passage percolation

Johansson [40] has introduced the following probabilistic model. Define on each site (i,j) of the lattice $\mathbb{Z}^2_{\geq 0}$ a non-negative integer variable $w(i,j)$, chosen independently with the geometric distribution $g(q^2)$ (thus $\Pr(g(q^2) = k) = (1-q^2)q^{2k}$, $k = 0, 1, \ldots,$). Let $(0,0)$ wu/wr $(M-1, N-1)$ denote the set of all weakly up / weakly right paths (meaning a sequence $\{(i_r, j_r)\}_{r=1}^l$ with $i_1 \leq i_2 \leq \cdots \leq i_l$ and $j_1 \leq j_2 \leq \cdots \leq j_l$) on the lattice starting at $(0,0)$ and finishing at $(M-1, N-1)$, moving only upwards or to the right. A quantity of interest is the random variable

$$
(5.52)\quad G^{\rm wu/wr}((M-1, N-1); q^2) := \max_{\pi \in (0,0)\ \mathrm{wu/wr}\ (M-1,\, N-1)} \sum_{(i,j) \in \pi} w(i,j)
$$

which with the integer variables $w(i,j)$ regarded as waiting times represents the last passage time of directed percolation paths $(0,0)$ wu/wr $(M-1, N-1)$. The relevance to the present study is that Baik and Rains [6], [4] have shown

$$
\begin{aligned}
(5.53)\qquad & \Pr\Big(G^{\rm wu/wr}((M-1, N-1); q^2) \leq l \Big) \\
&= (1-q^2)^{MN} \Big\langle \prod_{j=1}^{l} (1 + q^2 z_j)^N (1 + 1/z_j)^M \Big\rangle_{\mathrm{CUE}_l}.
\end{aligned}
$$

Using the results of this paper, (5.53) can be expressed both as a ${}_2F_1^{(1)}$ hypergeometric function and as a τ-function for a $\mathrm{P_{VI}}$ system.

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PROPOSITION 23. *We have*

$$\begin{aligned}(5.54)\qquad \Pr\Big(G^{\mathrm{wu/wr}}((M-1,N-1);q^2)\le l\Big) &= (1-q^2)^{MN}\,{}_2F_1^{(1)}(-N,-M;l;t_1,\dots,t_l)\big|_{t_1=\cdots=t_l=q^2}\\ &= (1-q^2)^{MN}\frac{J_N(M-N,0)}{J_N(M-N,l)}\\ &\qquad\times {}_2F_1^{(1)}(-l,M;M+N;1-t_1,\dots,1-t_N)\big|_{t_1=\cdots=t_N=q^2}\\ &= (1-q^2)^{MN}\tau_3(q^2;\hat{\mathbf{b}})\big|_{\substack{\mu=l,\,a=0\\ b=-M-N-l}}\end{aligned}$$

where the second equality is valid for $l\ge N$.

Proof. The average in (5.53) coincides with the CUE average (1.21) provided

$$(N,a',b',\mu,t)=(l,0,M,N,q^2).$$

With a, b specified in terms of a', b' in the line below (1.21), these values imply $(a,b)=(-l-M,M)$. The ${}_2F_1^{(1)}$ evaluations now follow from the specification of the right hand side of (5.53) with the above parameters, and the results (3.49) and (3.51). For the normalisation in the second equality we have used the general formula [73]

$${}_2F_1^{(1)}(r,N+\lambda_1;2N+\lambda_1+\lambda_2;t_1,\dots,t_N)\big|_{t_1=\cdots=t_N=1}=\frac{J_N(\lambda_1,\lambda_2-r)}{J_N(\lambda_1,\lambda_2)}.$$

The final equality follows from (3.53). □

The second equality in (5.54) is well suited to taking the exponential limit

$$(5.55)\qquad q=e^{-1/2L},\quad l=Ls,\quad L\to\infty$$

in which the discrete geometric random variables $w(i,j)$ in (5.52) tend to continuous exponential variables, with unit mean, and each random variable again confined to the lattice sites.

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PROPOSITION 24. *We have*

$$
\begin{aligned}
(5.56)\quad & g^{\mathrm{wu/wr}}((M-1,N-1);s) \\
&:= \lim_{L\to\infty} \Pr\Big(G^{\mathrm{wu/wr}}((M-1,N-1);e^{-1/L}) \le Ls\Big) \\
&= \Big(\prod_{j=0}^{N-1} \frac{\Gamma(1+j)}{\Gamma(M+1+j)}\Big) s^{MN} {}_1F_1^{(1)}(M;M+N;s_1,\dots,s_N)\Big|_{s_1=\cdots=s_N=-s} \\
&= \frac{1}{\prod_{j=0}^{N-1}(M-N+j)!j!} \int_0^s dx_1\, x_1^{M-N} e^{-x_1} \cdots \int_0^s dx_N\, x_N^{M-N} e^{-x_N} \\
&\qquad \times \prod_{j<k}(x_k-x_j)^2
\end{aligned}
$$

where in the final equality it is assumed $M > N$. Furthermore

$$
s\frac{d}{ds}\log g^{\mathrm{wu/wr}}((M-1,N-1);s) = U_N^{\mathrm{L}}(t;M-N,0) \tag{5.57}
$$

where $U_N^{\mathrm{L}}(t;a,0)$ satisfies (4.8) *with*

$$
\nu_0 = 0, \quad \nu_1 = 0, \quad \nu_2 = N+a, \quad \nu_3 = N.
$$

Proof. We see from (1.25) that in the limit (5.55)

$$
(1-q^2)^{MN}\frac{J_N(M-N,0)}{J_N(M-N,l)} \sim s^{MN}\prod_{j=0}^{N-1}\frac{\Gamma(1+j)}{\Gamma(M+1+j)},
$$

while it follows from (4.15) that

$$
\begin{aligned}
&{}_2F_1^{(1)}(-l,M;M+N;1-t_1,\dots,1-t_N)\Big|_{t_1=\cdots=t_N=q^2} \\
&\qquad \sim {}_1F_1^{(1)}(M;M+N;s_1,\dots,s_N)\Big|_{s_1=\cdots=s_N=-s}.
\end{aligned}
$$

To obtain the final equality in (5.56) we make use of (4.16) and (1.25) in the equality before. The formula (5.57) follows from the first equality in (5.56) and (4.17). □

The final equality in (5.56) was first derived by Johansson [40] using a different method. In words this equality says that the limiting last passage time cumulative distribution is equal to the probability of there being no eigenvalues in the interval (s,∞) of the Laguerre unitary ensemble with

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$a = M - N$ (the general Laguerre weight being $x^a e^{-x}$). With this identity established, (5.57) is equivalent to a result first derived in [66].

Formulas analogous to (5.53) are known for certain variations of the original Johansson model. One such variation involves 0, 1 Bernoulli random variables with distribution $b(q^2)$ at each site (thus $\Pr(b(q^2) = 0) = q^2/(1 + q^2)$, $\Pr(b(q^2) = 1) = 1/(1 + q^2)$). Furthermore, the set of weakly up / weakly right paths is replaced by the set of weakly up / strictly right paths (to be denoted wu/sr), meaning a sequence $\{(i_r, j_r)\}_{r=1}^{l}$ with $i_1 < i_2 < \cdots < i_l$ and $j_1 \le j_2 \le \cdots \le j_l$, or the set of strictly up / strictly right paths (to be denoted su/sr), meaning a sequence $\{(i_r, j_r)\}_{r=1}^{l}$ with $i_1 < i_2 < \cdots < i_l$ and $j_1 < j_2 < \cdots < j_l$. Thus with $B^{\mathrm{wu/sr}}$ and $B^{\mathrm{su/sr}}$ defined analogously to (5.52), one has [6], [34], [4]

$$\begin{aligned} &\Pr\Big(B^{\mathrm{wu/sr}}((M-1,N-1);q^2) \le l\Big) \\ &\quad = \frac{1}{(1+q^2)^{MN}} \Big\langle \prod_{j=1}^{l} (1+1/z_l)^{M} (1 - q^2 z_l)^{-N} \Big\rangle_{\mathrm{CUE}_l} \end{aligned} \tag{5.58}$$

$$\begin{aligned} &\Pr\Big(B^{\mathrm{su/sr}}((M-1,N-1);q^2) \le l\Big) \\ &\quad = (1-q^2)^{MN} \Big\langle \prod_{j=1}^{l} (1+1/z_l)^{-M} (1 + q^2 z_l)^{-N} \Big\rangle_{\mathrm{CUE}_l} \end{aligned} \tag{5.59}$$

where in (5.59) the contours of integration in the CUE_l average must be deformed so that the point $z = -1$ lies inside the contour while $z = -1/q^2$ lies outside.

Proposition 25. *We have*

$$\begin{aligned} &\Pr\Big(B^{\mathrm{wu/sr}}((M-1,N-1);q^2) \le l\Big) \\ &\quad = \frac{1}{(1+q^2)^{MN}}\, {}_2F_1^{(1)}(N,-M;l;t_1,\dots,t_l)\Big|_{t_1=\cdots=t_l=-q^2} \\ &\quad = \frac{1}{(1+q^2)^{MN}}\, \tau_3(-q^2;\hat{\mathbf{b}})\Big|_{\substack{\mu=l,\, N\mapsto -N \\ a=0,\, b=N-M-l}} \end{aligned} \tag{5.60}$$

$$\begin{aligned} &\Pr\Big(B^{\mathrm{su/sr}}((M-1,N-1);q^2) \le l\Big) \\ &\quad = (1-q^2)^{MN}\, {}_2F_1^{(1)}(N,M;l;t_1,\dots,t_l)\Big|_{t_1=\cdots=t_l=q^2} \\ &\quad = (1-q^2)^{MN} \tau_3(q^2;\hat{\mathbf{b}})\Big|_{\substack{\mu=l,\, N\mapsto -N \\ a=0,\, b=N+M-l}}. \end{aligned} \tag{5.61}$$

Proof. The averages (5.58) and (5.59) result from the CUE average (1.21) by setting

$$(N, a', b', \mu, t) = \begin{cases} (l, 0, M, -N, -q^2) & \text{wu/sr} \\ (l, 0, -M, -N, q^2) & \text{su/sr.} \end{cases} \tag{5.62}$$

With a', b' so specified, it follows from the equations in the line below (1.21) that (a, b) equals $(-l - M, M)$ and $(-l + M, -M)$ respectively. The ${}_2F_1^{(1)}$ evaluations now follow from the specification of the left hand side of (5.53) with the above parameters, and the results (3.49) and (3.51), and the τ-function evaluations in turn follow from (3.53). ∎

We remark that for $l \geq N$ it follows from the definitions that

$$\Pr\Big(B^{\text{wu/sr}}((M-1, N-1); q^2) \leq l\Big) = \Pr\Big(B^{\text{su/sr}}((M-1, N-1); q^2) \leq l\Big) = 1.$$

These equalities for $l = N$ can be read off from the above ${}_2F_1^{(1)}$ evaluations by noting the general formula [73]

$${}_2F_1^{(1)}(a, b; a; t_1, \ldots, t_m) = {}_1F_0^{(1)}(b; t_1, \ldots, t_m) = \prod_{j=1}^{m} (1 - t_j)^{-b}.$$

Underlying the Johansson probabilistic model and its generalisations is the Knuth correspondence between integer matrices, generalised permutations and semi-standard tableaux (see [40, Section 2.1]). In fact the random variable (5.52) in the Johansson model is equivalent to the random variable specifying the length of the longest increasing subsequence of random generalised permutations with a certain probability measure, or alternatively the length of the first row of a certain class of random Young diagrams. The notion of measures on Young diagrams also occurs in the work of Borodin and Olshanski on representations of the infinite symmetric group (see e.g. [13], [15]). In particular, for random Young diagrams distributed according to a so called z-measure, the probability that the first row length $Y(z;\xi)$ is less than or equal to l is shown to be given by

$$\Pr\Big(Y(z;\xi) \leq l\Big) = (1-\xi)^{|z|^2} \Big\langle \prod_{j=1}^{l} (1 + \sqrt{\xi} z_j)^{z} (1 + \sqrt{\xi}/z_j)^{\bar{z}} \Big\rangle_{\mathrm{CUE}_l}. \tag{5.63}$$

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Note that with $z = N$, $\xi = q^2$, this coincides with the case $M = N$ of (5.53). Comparing (5.63) with (1.18), we see that in the case z real

$$\Pr\Big(Y(z; e^{-t}) \le l\Big) = (1 - e^{-t})^{z^2} \widetilde{E}_l^{\mathrm{cJ}}(-it; (0,0), z; 0). \tag{5.64}$$

Hence we have the $\mathrm{P_{VI}}$ σ-function evaluation (3.25). Complimentary to this result, we draw attention to the recent work of Borodin [10] (see [11] for still more recent developments), who through a newly developed theory of discrete integral operators and discrete Riemann-Hilbert problems, has derived a difference equation for $\Pr\big(Y(z; e^{-t}) \le l\big)$ in the discrete variable l involving two auxiliary quantities which satisfy a particular case of the discrete $\mathrm{P_V}$ system (2.71), (2.72).

5.4. Symmetrised last passage percolation and distribution of the largest eigenvalue in the finite LOE and LSE

Consider the original Johansson probabilistic model introduced in the previous section. Choose the weights $w(i,j)$ for $i < j$ as independent geometric random variables with distribution $g(q^2)$, and impose the symmetry that $w(i,j) = w(j,i)$ for $i > j$. Choose the weights $w(i,i)$ on the diagonal independently with distribution $g(q)$, and consider the random variable (5.52) with $M = N$. By the symmetry constraint the set of paths in (5.52) can be restricted to the triangular region $i \le j$, so we denote (5.52) in this case by $G^{\mathrm{wu/wr}}_{\mathrm{triangle}}(N; q^2)$.

Baik and Rains [6] have shown that

$$\begin{aligned}\Pr\Big(G^{\mathrm{wu/wr}}_{\mathrm{triangle}}(N; q^2) \le l\Big) \\ = \frac{1}{2}(1-q)^N (1-q^2)^{N(N-1)/2} \Big\langle \det\big[(1+U)(1+qU)^N\big] \Big\rangle_{O^+(l)}.\end{aligned} \tag{5.65}$$

The significance of (5.65) from the viewpoint of the present study is that it can be written in the form (1.7). To see this we recall that for l even all eigenvalues of $O^+(l)$ come in complex conjugate pairs $e^{\pm i\theta}$, while for l odd one eigenvalue of $O^+(l)$ is equal to $+1$ and the rest come in complex conjugate pairs. Hence

$$\det\big[(1+U)(1+qU)^N\big] = \chi_{l,N} \prod_{j=1}^{[l/2]} (2 + 2\cos\theta_j)(1 + q^2 + 2q\cos\theta_j)^N$$

where $\chi_{l,N} = 1$, l even, $\chi_{l,N} = 2(1+q)^N$, l odd. Introducing the variables

$$\lambda_j = \frac{1}{2}(\cos\theta_j + 1) \tag{5.66}$$

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this reads

$$\det\big[(1+U)(1+qU)^N\big] = 2^{2(N+1)[l/2]} q^{N[l/2]} \chi_{l,N} \prod_{j=1}^{[l/2]} \lambda_j \Big(\frac{1}{4q}(1-q)^2+\lambda_j\Big)^N.$$

Furthermore, in terms of the variables (5.66), the PDF of the eigenvalues $\{e^{\pm i\theta_j}\}_{j=1,\dots,[l/2]}$ for matrices $U \in O^+(l)$ is of the form (1.1) with N replaced by N^*, $w(x) = x^a(1-x)^b$ (Jacobi weight) and parameters

$$(N^*, a, b) = ([l/2], -1/2, (-1)^{l-1}/2).$$

Thus we can rewrite (5.65) to read

$$\begin{aligned}(5.67)\qquad &\Pr\Big(G^{\mathrm{wu/wr}}_{\mathrm{triangle}}(N;q^2) \le l\Big)\\ &= C_{l,N}(q)\Big\langle \prod_{j=1}^{[l/2]} \lambda_j \Big(\frac{1}{4q}(1-q)^2+\lambda_j\Big)^N \Big\rangle_{\mathrm{JUE}_{[l/2]}}\Big|_{\substack{a=-1/2\\ b=(-1)^{l-1}/2}}\end{aligned}$$

where

$$(5.68)\qquad C_{l,N}(q) := 2^{2(N+1)[l/2]} q^{N[l/2]} \chi_{l,N} \frac{1}{2}(1-q)^N (1-q^2)^{N(N-1)/2}$$

The result (3.52) can be used to express (5.67) as a generalised hypergeometric function.

Proposition 26. *With $C_{l,N}(q)$ specified by (5.68) and $J_n(a,b)$ specified by (1.25)*

$$\begin{aligned}(5.69)\quad \Pr\Big(G^{\mathrm{wu/wr}}_{\mathrm{triangle}}(N;q^2) \le l\Big) &= C_{l,N}(q)\frac{J_{[l/2]}(1/2+N,(-1)^{l-1}/2)}{J_{[l/2]}(-1/2,(-1)^{l-1}/2)}\\ &\times {}_2F_1^{(1)}\Big(-[l/2], N+[l/2]+\frac{1+(-1)^{l-1}}{2}; N+\frac{1}{2};\\ &\qquad t_1,\dots,t_N\Big)\Big|_{t_1=\cdots=t_N=-(1-q)^2/4q}\end{aligned}$$

valid for $[l/2] \ge N$.

The expression (5.69) is well suited to taking the exponential limit (5.55).

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PROPOSITION 27. *We have*

$$g^{\mathrm{wu/wr}}_{\mathrm{triangle}}(N;s) := \lim_{L\to\infty} \Pr\Big(G^{\mathrm{wu/wr}}_{\mathrm{triangle}}(N;e^{-1/L}) \le Ls\Big) = \prod_{j=0}^{N} \frac{\Gamma(j+1)}{\Gamma(2j+1)} s^{N(N+1)/2} e^{-Ns/4} \times {}_0F_1^{(1)}(\_;N+1/2;t_1,\dots,t_N)\big|_{t_1=\cdots=t_N=s^2/64} \tag{5.70}$$

Proof. Suppose for definiteness that l is even. In the limit (5.55) we can check from the definition (5.68) and the evaluation (1.25), making use of Stirling's formula, that

$$C_{l,N}(q)\frac{J_{[l/2]}(1/2+N,(-1)^{l-1}/2)}{J_{[l/2]}(-1/2,(-1)^{l-1}/2)} \sim \prod_{j=0}^{N} \frac{\Gamma(j+1)}{\Gamma(2j+1)} s^{N(N+1)/2} e^{-Ns/4}.$$

Also, analogous to the result (4.15), we can see from the series definition (3.48) that in the limit (5.55)

$$ {}_2F_1^{(1)}\Big(-[l/2], N+[l/2]+\frac{1+(-1)^{l-1}}{2}; N+\frac{1}{2}; t_1,\dots,t_N\Big)\Big|_{t_1=\cdots=t_N=-(1-q)^2/4q} \sim {}_0F_1^{(1)}(\_;N+1/2;t_1,\dots,t_N)\big|_{t_1=\cdots=t_N=s^2/64}. \tag{5.71}$$

$\square$

In [23] the generalised hypergeometric function

$${}_0F_1^{(1)}(\_;N+\mu;t_1,\dots,t_N)\big|_{t_1=\cdots=t_N=t},$$

for $\mu=0$ and $\mu=2$ has been evaluated as an N-dimensional determinant. Generalising the working therein gives that for general μ

$${}_0F_1^{(1)}(\_;N+\mu;t_1,\dots,t_N)\big|_{t_1=\cdots=t_N=s^2/4} = \prod_{j=0}^{N-1} \frac{\Gamma(\mu+j+1)}{\Gamma(j+1)} \Big(\frac{2}{s}\Big)^{N\mu} \det\big[I_{j-k+\mu}(s)\big]_{j,k=0,\dots,N-1}. \tag{5.72}$$

Furthermore, we know from [30] that

$$\prod_{j=0}^{N-1} \frac{\Gamma(\mu+j+1)}{\Gamma(j+1)} \Big(\frac{2}{\sqrt{t}}\Big)^{N\mu} \det\big[I_{j-k+\mu}(\sqrt{t})\big]_{j,k=0,\dots,N-1} = \exp \int_0^t \frac{v(s;N,\mu)+s/4}{s}\,ds \tag{5.73}$$

where

$$v(t;N,\mu) = -\Big(\sigma_{\mathrm{III}'}(t) + \mu(\mu+N)/2\Big). \tag{5.74}$$

The function $\sigma_{\mathrm{III}'}(t)$ satisfies the differential equation

$$(t\sigma''_{\mathrm{III}'})^2 - v_1 v_2 (\sigma'_{\mathrm{III}'})^2 + \sigma'_{\mathrm{III}'}(4\sigma'_{\mathrm{III}'}-1)(\sigma_{\mathrm{III}'} - t\sigma'_{\mathrm{III}'}) - \frac{1}{4^3}(v_1-v_2)^2 = 0, \tag{5.75}$$

which is the Jimbo-Miwa-Okamoto σ-form of the $P_{\mathrm{III}'}$ equation (a simple change of variables relates P_{III} to $P_{\mathrm{III}'}$ [56]) with parameters

$$(v_1, v_2) = (\mu+N, -\mu+N) \tag{5.76}$$

and subject to the boundary conditions

$$\sigma_{\mathrm{III}'}(t) \underset{t\to\infty}{\sim} \frac{t}{4} - \frac{Nt^{1/2}}{2} + \Big(\frac{N^2}{4} - \frac{\mu^2}{2}\Big). \tag{5.77}$$

The boundary condition (5.77) does not distinguish solutions with $\mu \mapsto -\mu$ (note from (5.76) that under this mapping $v_1 \leftrightarrow v_2$, and this latter transformation leaves (5.75) unchanged). However, it follows from (5.72)–(5.74) that

$$\sigma_{\mathrm{III}'}(t) \underset{t\to 0^+}{\sim} -\frac{\mu(\mu+N)}{2} + \frac{\mu}{2(\mu+N)} t + O(t^2). \tag{5.78}$$

This boundary condition does distinguish solutions with $\mu \mapsto -\mu$.

Collecting together the above results allows the evaluation (5.70) of $g^{\mathrm{wu/wr}}_{\mathrm{triangle}}(N;s)$ to be further developed.

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PROPOSITION 28. *We have*

(5.79)

$$g^{\mathrm{wu/wr}}_{\mathrm{triangle}}(N;s) = 2^{3N/2}\frac{\Gamma(N+1)}{\Gamma(2N+1)}\Big(\prod_{j=0}^{N-1}\frac{\Gamma(j+3/2)}{\Gamma(2j+1)}\Big)s^{N^2/2}e^{-Ns/4} \times \det\big[I_{j-k+1/2}(s/4)\big]_{j,k=0,\dots,N-1}$$
$$= \prod_{j=0}^{N}\frac{\Gamma(j+1)}{\Gamma(2j+1)}s^{N(N+1)/2}\exp\int_0^{(s/4)^2}\Big(v(t;N,1/2)+\frac{t}{4}-\frac{Nt^{1/2}}{2}\Big)\frac{dt}{t}$$
$$= \exp\Big(-\int_{(s/4)^2}^{\infty}\Big(-\sigma_{\mathrm{III}'}(t)\Big|_{\substack{v_1=N+1/2\\ v_2=N-1/2}}+\frac{t}{4}-\frac{Nt^{1/2}}{2}+\frac{N^2-1/2}{4}\Big)\frac{dt}{t}\Big)$$

In the Okamoto τ-function theory of the $\mathrm{P}_{\mathrm{III}'}$ system [56], the transcendent $-\sigma_{\mathrm{III}'}(t)/t$ is an auxiliary Hamiltonian, being equal to the original Hamiltonian plus a function of t. Changing the function of t doesn't alter this property, so we have that

$$g^{\mathrm{wu/wr}}_{\mathrm{triangle}}(N;s) = \tau_{\mathrm{III}'}((s/4)^2)\Big|_{\substack{v_1=N+1/2\\ v_2=N-1/2}} \tag{5.80}$$

where

$$t\frac{d}{dt}\log\tau_{\mathrm{III}'}(t)\Big|_{\substack{v_1=N+1/2\\ v_2=N-1/2}} \doteq -\sigma_{\mathrm{III}'}(t)\Big|_{\substack{v_1=N+1/2\\ v_2=N-1/2}}+\frac{t}{4}-\frac{Nt^{1/2}}{2}+\frac{N^2-1/2}{4}. \tag{5.81}$$

The symmetrised Johansson model has been generalised by Baik and Rains [6], [7] to include a parameter α in the geometric distribution of the waiting times $w(i,i)$ of the diagonal sites. Thus these waiting times are chosen to have distribution $g(\alpha q)$, with the case $\alpha = 1$ corresponding to the original symmetrised Johansson model. With the random variable (5.52) now denoted $G^{\mathrm{wu/wr}}_{\mathrm{triangle}}(N;\alpha,q^2)$, one then has [4]

$$\Pr\Big(G^{\mathrm{wu/wr}}_{\mathrm{triangle}}(N;\alpha,q^2)\le l\Big) = \frac{1}{2}(1-\alpha q)^N(1-q^2)^{N(N-1)/2} \times\Big(\big\langle\det\big[(1+\alpha U)(1+qU)^N\big]\big\rangle_{U\in O^+(l)}+\big\langle\det\big[(1+\alpha U)(1+qU)^N\big]\big\rangle_{U\in O^-(l)}\Big) \tag{5.82}$$

The analogue of Propositions 26, 27 and 28 can be obtained for the case $\alpha = 0$.

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PROPOSITION 29. *Let* $\chi^+_{l,N} = 1$, l *even and* $\chi^+_{l,N} = (1+q)^N$, l *odd, and put*

$$C^+_{l,N}(q) = 2^{2N[l/2]} q^{N[l/2]} \chi^+_{l,N} \frac{1}{2}(1-q^2)^{N(N-1)/2}.$$

Let $\chi^-_{l,N} = (1-q^2)^N$, l *even and* $\chi^-_{l,N} = (1-q)^N$, l *odd, and put*

$$C^-_{l,N}(q) = 2^{2N[(l-1)/2]} q^{N[(l-1)/2]} \chi^-_{l,N} \frac{1}{2}(1-q^2)^{N(N-1)/2}.$$

We have

$$\begin{aligned}
(5.83)\quad \Pr\Big(G^{\mathrm{wu/wr}}_{\mathrm{triangle}}(N;0,q^2) \le l\Big) &= C^+_{l,N}(q) \frac{J_{[l/2]}(-1/2+N,(-1)^{l-1}/2)}{J_{[l/2]}(-1/2,(-1)^{l-1}/2)} \\
&\quad \times {}_2F_1^{(1)}\Big(-[l/2], N+[l/2]+\frac{(-1)^{l-1}-1}{2}; N-\frac{1}{2}; \\
&\qquad\qquad t_1,\dots,t_N\Big)\Big|_{t_1=\cdots=t_N=-(1-q)^2/4q} \\
&\quad + C^-_{l,N}(q) \frac{J_{[(l-1)/2]}(1/2+N,(-1)^l/2)}{J_{[l/2]}(1/2,(-1)^l/2)} \\
&\quad \times {}_2F_1^{(1)}\Big(-[(l-1)/2], N+[(l-1)/2]+\frac{1+(-1)^l}{2}; N+\frac{1}{2}; \\
&\qquad\qquad t_1,\dots,t_N\Big)\Big|_{t_1=\cdots=t_N=-(1-q)^2/4q}.
\end{aligned}$$

valid for $[l/2] \ge N$.

Proof. The rewrite of the average over $O^+(l)$ in (5.82) is done in the same way as in going from (5.65) to (5.69). To rewrite the average over $O^-(l)$ in (5.82) we must first recall that for l even there are eigenvalues ± 1, while the remaining eigenvalues come in complex conjugate pairs $e^{\pm i\theta_j}$, and for l odd there is an eigenvalue -1, with the remaining eigenvalues also coming in complex conjugate pairs $e^{\pm i\theta_j}$. Furthermore, in terms of the variables (5.66), the PDF of the eigenvalues $\{e^{\pm i\theta_j}\}_{j=1,\dots,[(l-1)/2]}$ is of the form (1.1) with N replaced by N^*, $w(x) = x^a(1-x)^b$ and parameters

$$(N^*, a, b) = ([(l-1)/2], 1/2, (-1)^l/2).$$

Using these facts, we proceed as in the rewrite of the $O^+(l)$ average. ∎

PROPOSITION 30. *We have*

(5.84)

$$\begin{aligned}
g^{\mathrm{wu/wr}}_{\mathrm{triangle}}(N;0,s) &:= \lim_{L\to\infty} \Pr\Big(G^{\mathrm{wu/wr}}_{\mathrm{triangle}}(N;0,e^{-1/L}) \le Ls\Big) \\
&= \frac{1}{2}\Bigg\{\prod_{j=0}^{N} \frac{\Gamma(j+1)}{\Gamma(2j+1)} s^{N(N+1)/2} e^{-Ns/4} \\
&\qquad \times {}_0F_1^{(1)}(_;N+1/2;t_1,\dots,t_N)\big|_{t_1=\cdots=t_N=s^2/64} \\
&\quad + \prod_{j=1}^{N} \frac{\Gamma(j)}{\Gamma(2j-1)} s^{N(N-1)/2} e^{-Ns/4} \\
&\qquad \times {}_0F_1^{(1)}(_;N-1/2;t_1,\dots,t_N)\big|_{t_1=\cdots=t_N=s^2/64}\Bigg\} \\
&= \frac{1}{2}\Bigg\{\exp\Bigg(-\int_{(s/4)^2}^{\infty}\Big(-\sigma_{\mathrm{III}'}(t)\big|_{\substack{v_1=N+1/2\\ v_2=N-1/2}} + \frac{t}{4} - \frac{Nt^{1/2}}{2} + \frac{N^2-1/2}{4}\Big)\frac{dt}{t}\Bigg) \\
&\quad + \exp\Bigg(-\int_{(s/4)^2}^{\infty}\Big(-\sigma_{\mathrm{III}'}(t)\big|_{\substack{v_1=N-1/2\\ v_2=N+1/2}} + \frac{t}{4} - \frac{Nt^{1/2}}{2} + \frac{N^2-1/2}{4}\Big)\frac{dt}{t}\Bigg)\Bigg\}
\end{aligned}$$

Proof. These formulas follow from Proposition 29 by following the working which led from Proposition 26 to Propositions 27 and 28. The two terms in (5.84) correspond to the two terms in (5.83) but in the reverse order. ∎

Baik [5] has recently evaluated the scaled limit of (5.82) for general α (the variable α is also involved in the scaling). The evaluation is given in terms of the solution of a Riemann-Hilbert problem. It is pointed out in [5] that this has consequence with regard to the distribution of the largest eigenvalue in an interpolating Laguerre ensemble, which passes continuously from the LOE$_N$ to the LSE$_{N/2}$. This follows from the identity [6] (see also [32])

(5.85)
$$\begin{aligned}
g^{\mathrm{wu/wr}}_{\mathrm{triangle}}(N;A,s) &:= \lim_{L\to\infty} \Pr\Big(G^{\mathrm{wu/wr}}_{\mathrm{triangle}}(N;e^{A/2L},e^{-1/L}) \le Ls\Big) \\
&= \frac{1}{C}\int_0^s dx_1 \cdots \int_0^s dx_N\, \chi_{x_1>x_2>\cdots>x_N\ge 0} \\
&\qquad \times e^{-(1/2)\sum_{j=1}^N x_j} e^{(A/2)\sum_{j=1}^N (-1)^{j-1}x_j} \prod_{j<k}(x_j-x_k)
\end{aligned}$$

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where $\chi_T = 1$ if T is true and $\chi_T = 0$ otherwise. The case $A = 0$ corresponds to $\alpha = 1$. Now, with $A = 0$ the integral in (5.85) is precisely the gap probability $E_1((s,\infty);N;0)$ (no eigenvalues in the interval (s,∞)) for the LOE_N with parameter $a = 0$. Equating (5.85) in this case with (5.80) then gives the τ-function evaluation

$$E_1^{\mathrm{LOE}}((s,\infty),N;0) = \tau_{\mathrm{III}'}((s/4)^2)\Big|_{\substack{v_1=N+1/2\\ v_2=N-1/2}}. \tag{5.86}$$

Taking the $A \to -\infty$ limit in (5.85) corresponds to $\alpha = 0$. We see from (5.85) that for N even

$$\lim_{A\to-\infty} g^{\mathrm{wu/wr}}_{\mathrm{triangle}}(N;A,s) = E_4^{\mathrm{LSE}}((s,\infty);N/2;0),$$

where $E_4^{\mathrm{LSE}}((s,\infty);n;a)$ denotes the probability that there are no eigenvalues in the LSE_n with weight $x^a e^{-x}$. Equating (5.85) in this case with (5.80) then gives

$$E_4^{\mathrm{LSE}}((s,\infty);N/2;0) = \frac{1}{2}\Big(\tau_{\mathrm{III}'}((s/4)^2)\Big|_{\substack{v_1=N+1/2\\ v_2=N-1/2}} + \tau_{\mathrm{III}'}((s/4)^2)\Big|_{\substack{v_1=N-1/2\\ v_2=N+1/2}}\Big). \tag{5.87}$$

As the differential equation (5.75) is unchanged by the interchange $v_1 \leftrightarrow v_2$ the Hamiltonians for both τ-functions in (5.87) satisfy the same differential equation. It is interesting to compare (5.87) with the structural formula [27]

$$E_4^{\mathrm{LSE}}((s,\infty);N/2;0) = \frac{1}{2}\Big(E_1^{\mathrm{LOE}}((s,\infty);N;0) + \frac{E_2^{\mathrm{LUE}}((s,\infty);N;0)}{E_1^{\mathrm{LOE}}((s,\infty);N;0)}\Big). \tag{5.88}$$

Doing this tells us that

$$E_2^{\mathrm{LUE}}((s,\infty);N;0) = \tau_{\mathrm{III}'}((s/4)^2)\Big|_{\substack{v_1=N+1/2\\ v_2=N-1/2}} \tau_{\mathrm{III}'}((s/4)^2)\Big|_{\substack{v_1=N-1/2\\ v_2=N+1/2}}. \tag{5.89}$$

On the other hand we know from [65] that

$$\begin{aligned} E_2^{\mathrm{LUE}}((s,\infty);N;0) &= \exp\Big(-\int_s^\infty \sigma(t)\Big|_{\substack{\nu_0=\nu_1=0\\ \nu_2=\nu_3=N}} \frac{dt}{t}\Big) \\ &= \tau_V(s)\Big|_{\substack{\nu_0=\nu_1=0\\ \nu_2=\nu_3=N}}, \end{aligned} \tag{5.90}$$

where $\sigma(t)$ satisfies (4.8), and so

$$\tau_V(s)\Big|_{\substack{\nu_0=\nu_1=0\\ \nu_2=\nu_3=N}} = \tau_{\mathrm{III}'}((s/4)^2)\Big|_{\substack{v_1=N+1/2\\ v_2=N-1/2}} \tau_{\mathrm{III}'}((s/4)^2)\Big|_{\substack{v_1=N-1/2\\ v_2=N+1/2}}. \tag{5.91}$$

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Similar τ-function identities to (5.91) have occurred in the works [25], [70]. The Painlevé transcendent evaluations of $E_1^{\rm LOE}((s,\infty);N;0)$ and $E_4^{\rm LSE}((s,\infty);N/2;0)$ given in [5] differ from (5.86) and (5.87), involving instead of τ-functions for the PIII' system, the transformed PV transcendent found in [65] to be simply related to the derivative of $\sigma(t)\Big|_{\substack{\nu_0=\nu_1=0\\ \nu_2=\nu_3=N}}$.

5.5. Diagonal-diagonal correlation in the $2d$ Ising model

The square lattice Ising model has one of two possible states, $\sigma_{ij}=\pm 1$, on each site (i,j) of the two-dimensional square lattice (see e.g. [9]). The infinite square lattice of states is achieved by considering a sequence of finite lattices, of dimension $(2N+1)\times(2N+1)$ say, centred about the origin. The joint probability density function for a particular configuration of states in the finite system is given by

$$\frac{1}{Z_{2N+1}}\exp\Big[K_1\sum_{j=-N}^{N}\sum_{i=-N}^{N-1}\sigma_{ij}\sigma_{i+1\,j}+K_2\sum_{i=-N}^{N}\sum_{j=-N}^{N-1}\sigma_{ij}\sigma_{i\,j+1}\Big],$$

where Z_{2N+1} is the normalisation. Thus there is a coupling between nearest neighbours in the horizontal and vertical direction. In the limit $N\to\infty$, an unpublished result of Onsager (see [44]) gives that the diagonal spin-spin correlation has the Toeplitz form

$$\langle\sigma_{00}\sigma_{nn}\rangle=\det[a_{i-j}]_{i,j=1,\dots,n} \tag{5.92}$$

where

$$a_p=\frac{1}{2\pi}\int_0^{2\pi}d\theta\,e^{-ip\theta}\Big[\frac{1-(1/k)e^{-i\theta}}{1-(1/k)e^{i\theta}}\Big]^{1/2},\quad k:=\sinh 2K_1\sinh 2K_2. \tag{5.93}$$

Regarding the Fourier integral (5.93) as a contour integral, deforming the contour of integration and then writing the Toeplitz determinant as an average of the CUE we see that

$$\langle\sigma_{00}\sigma_{nn}\rangle=\begin{cases}\Big\langle\prod_{l=1}^{n}z_l^{1/4}|1+z_l|^{-1/2}(1+(1/k^2)z_l)^{1/2}\Big\rangle_{{\rm CUE}_n}, & 1/k^2\le 1\\ k^{-n}\Big\langle\prod_{l=1}^{n}z_l^{-3/4}|1+z_l|^{-1/2}(1+k^2z_l)^{1/2}\Big\rangle_{{\rm CUE}_n}, & k^2\le 1.\end{cases} \tag{5.94}$$

We can now read off from (3.7) the following results.

PROPOSITION 31. *Let $s = 1/k^2$. We have*

$$-\frac{1}{4}(1+n^2)s + \frac{1}{8} + s(s-1)\frac{d}{ds}\log\langle\sigma_{00}\sigma_{nn}\rangle \\ = U_n^{\mathrm{J}}(s; 1/2-n, 1/2, -1/2) \tag{5.95}$$

where U_n^{J} satisfies the $\mathrm{P_{VI}}$ *equation in σ-form* (1.27) *with parameters*

$$\mathbf{b} = \Big(\frac{1}{2}(n-1), \frac{n}{2}, \frac{1}{2}(n+1), -\frac{n}{2}\Big).$$

Let $t = k^2$. We have

$$-\frac{n^2}{4}t - \frac{1}{8} + t(t-1)\frac{d}{dt}\log\langle\sigma_{00}\sigma_{nn}\rangle = U_n^{\mathrm{J}}(t; -1/2-n, 1/2, -1/2) \tag{5.96}$$

where U_n^{J} satisfies the $\mathrm{P_{VI}}$ *equation in σ-form* (1.27) *with parameters*

$$\mathbf{b} = \Big(\frac{n}{2}, \frac{n-1}{2}, \frac{n}{2}, -\frac{n+1}{2}\Big).$$

It is straightforward to check from (5.96) and (1.27) that

$$\sigma(t) := t(t-1)\frac{d}{dt}\log\langle\sigma_{00}\sigma_{nn}\rangle - \frac{1}{4}$$

satisfies the differential equation

$$\Big(t(t-1)\frac{d^2\sigma}{dt^2}\Big)^2 \\ = n^2\Big((t-1)\frac{d\sigma}{dt} - \sigma\Big)^2 - 4\frac{d\sigma}{dt}\Big((t-1)\frac{d\sigma}{dt} - \sigma - \frac{1}{4}\Big)\Big(t\frac{d\sigma}{dt} - \sigma\Big).$$

This is a result due to Jimbo and Miwa [37]. Our point therefore is not a new characterisation of $\langle\sigma_{00}\sigma_{nn}\rangle$, but rather the fact that the known characterisation fits our development of the Okamoto τ-function theory of $\mathrm{P_{VI}}$. We note too that a recent result of Borodin [10] can be used to give a recurrence for $\langle\sigma_{00}\sigma_{nn}\rangle$ as a function of n. This recurrence applies to all Toeplitz determinants

$$q_n^{(z,z',\xi)} := (1-\xi)^{zz'}\det[g_{i-j}]_{i,j=1,\dots,n} \\ g_p = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, e^{-ip\theta}(1-\sqrt{\xi}e^{i\theta})^z(1-\sqrt{\xi}e^{-i\theta})^{z'}. \tag{5.97}$$

Comparison with (5.92) and (5.93) shows

$$\langle \sigma_{00}\sigma_{nn}\rangle = \frac{1}{1-(1/k)^2} q_n^{(-1/2,1/2,1/k^2)}.$$

Recent work of Adler and van Moerbeke [2] gives a different recurrence relation for (5.97) to that of Borodin, and thus a further recurrence for $\langle \sigma_{00}\sigma_{nn}\rangle$. Explicit computation of $\langle \sigma_{00}\sigma_{nn}\rangle$ as a power series in k^2, for which both the differential and difference equation characterisations are well suited, is a crucial step in recent very high precision numerical studies of the zero field susceptibility [47], [48], [57], [58].

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