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Matching of Weighted Orbital Integrals for Metaplectic Correspondences

Paul Mezo

Abstract. We prove an identity between weighted orbital integrals of the unit elements in the Hecke algebras of $\mathrm{GL}(r)$ and its n-fold metaplectic covering, under the assumption that n is relatively prime to any proper divisor of every $1 \leq j \leq r$.

1 Introduction

In order to successfully compare trace formulas of two groups, one must match orbital integrals of functions in the two corresponding Hecke algebras. The basic step in this endeavour is the matching at the units of the Hecke algebras. This is, for instance, the basic step in proving Langlands' "fundamental lemma". When comparing the Arthur-Selberg trace formulas of two groups, one requires an additional matching. The truncation process involved in the Arthur-Selberg trace formula leads to the matching of *weighted* orbital integrals.

The two groups considered here are $\mathrm{GL}(r)$ and an n-fold metaplectic covering thereof [11]. The matching of invariant orbital integrals at the units has been completely solved through the combined work of Flicker, Kazhdan, Patterson and Waldspurger (see the Appendix of [5]). We prove a matching of weighted orbital integrals at the units, under the assumption that for any $1 \leq j \leq r$ and any proper divisor i of j, n and i are relatively prime. Our proof relies heavily on the work of the above authors. Given more information about metaplectic coverings, one should be able remove the assumption by using an inductive trace formula argument as in [12].

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2 Statement of Theorem

Let $r \geq 2$ and n be positive integers. Let F be a $\mathfrak{p}$-adic field which contains the group μ_n of n-th roots of unity. We also assume that $|n| = 1$. Set R to be the ring of integers of F and q to be the cardinality of its residual field.

Put $G = \mathrm{GL}(r, F)$ and $K = \mathrm{GL}(r, R)$. We follow the definition in [6, Section 2] of an n-fold metaplectic covering,

$$1 \to \mu_n \xrightarrow{\mathbf{i}} \tilde{G} \underset{\mathbf{s}}{\overset{\mathbf{p}}{\rightleftarrows}} G \to 1,$$

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using the 2-cocycle τ_{mv}, for fixed $0 \le m \le n-1$. The maps in this short exact sequence of topological groups are given by

$$\mathbf{i}(\zeta) = (1, \zeta), \quad \mathbf{p}(\gamma, \zeta) = \gamma, \quad \text{and} \quad \mathbf{s}(\gamma) = (\gamma, 1),$$

where $\gamma \in G$ and $\zeta \in \mu_n$. Given a subgroup H of G, we write $\tilde{H}$ in place of $\mathbf{p}^{-1}(H)$. The cocycle τ_{mv} is trivial on $K \times K$. This implies that $\tilde{K}$ is equal to the group $K \times \mu_n$.

We define a map, $G_0 \xrightarrow{\prime} \tilde{G}$, on a dense subset G_0 of G as in [6, Section 4]. This map preserves conjugacy classes and is the "orbit map" which affords a comparison of G with $\tilde{G}$. If n is odd then $G = G_0$ and this map is given by

$$\gamma' = \mathbf{s}(\gamma)^n, \quad \gamma \in G.$$

If n is even then the map satisfies

$$\gamma' = \mathbf{i}(\pm 1)\mathbf{s}(\gamma)^n, \quad \gamma \in G_0.$$

Fix a Haar measure on G such that K has measure 1. The centralizer of $\tilde{\gamma} \in \tilde{G}$ in $\tilde{G}$ is denoted by $\tilde{G}_{\tilde{\gamma}}$. Fix Haar measures on G_γ for $\gamma \in G$. To $\tilde{K}$ and $\tilde{G}_{\tilde{\gamma}}$ we assign the Haar measures which, when composed with $\mathbf{s}$, coincide with those of K and $G_{\mathbf{p}(\tilde{\gamma})}$ respectively.

We take M_0 to be the diagonal subgroup of G. The set of Levi subgroups of G containing M_0 is denoted by $\mathcal{L}$. Throughout, M denotes an element in $\mathcal{L}$. All remaining notation, such as $\mathcal{L}(M)$, $\mathcal{P}(M)$, $\mathfrak{a}_M$, *etc.* is adopted from [3, Sections 1–2]. Of course, we specialize this notation to the case $G = \mathrm{GL}(r, F)$.

Define the function $f^0_{\tilde{M}}$ on $\tilde{M}$ by

$$\tilde{f}^0_M(\gamma, \zeta) = \left\{ \begin{array}{ll} 0, & \text{if } \gamma \notin K \cap M \\ \zeta^{-1}, & \text{otherwise} \end{array} \right\}, \quad \gamma \in M, \quad \zeta \in \mu_n.$$

In particular f^0_M is just the characteristic function of $M \cap K$. The weighted orbital integral, $J^{\tilde{L}}_{\tilde{M}}(\tilde{\gamma}, f^0_{\tilde{L}})$ at $\tilde{\gamma} \in \tilde{M}$, is defined as

$$\left| D^L\big(\mathbf{p}(\tilde{\gamma})\big) \right|^{1/2} \int_{\tilde{L}_{\tilde{\gamma}} \backslash \tilde{L}} f^0_{\tilde{L}}(x^{-1}\tilde{\gamma}x) v^L_M\big(\mathbf{p}(x)\big)\, dx, \quad L \in \mathcal{L}(M).$$

Define μ^M_n to be the group of those matrices in the center of M whose diagonal entries all lie in μ_n. It is simple to see that

$$J^L_M(\eta\gamma, f^0_L) = J^L_M(\gamma, f^0_L), \quad \eta \in \mu^L_n,$$

as v^L_M is left-invariant under M and μ^L_n is contained in both K and M.

We have the following matching of weighted orbital integrals.

Theorem *Suppose $\gamma \in M$ is semisimple and γ^n is regular in G. Suppose further that, for any $1 \le j \le r$ and any proper divisor i of j, n and i are relatively prime. Then*

$$J^{\tilde{L}}_{\tilde{M}}(\gamma', f^0_{\tilde{L}}) = \sum_{\eta \in \mu^M_n / \mu^L_n} J^L_M(\eta\gamma, f^0_L), \quad L \in \mathcal{L}(M).$$

As mentioned in the introduction, the case $M = L$ of this theorem has been proven for all positive integers n. We shall therefore take this case for granted and mention it no further. The proof of this theorem shall consume the rest of this paper.

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3 At the Regular Elliptic Elements

We first recall some results of Kazhdan in [7, Section 3]. According to [7, Lemma 2, Section 3], any element γ of K has a topological Jordan decomposition. That is, there exist unique commuting elements s and u in K and an integer c, relatively prime to q, such that $s^c = 1$, $\lim_{N\to\infty} u^{q^N} = 1$ and $\gamma = su$. We call s topologically semisimple and u topologically unipotent. [7, Lemma 3, Section 3] states that

$$\{x \in G : x^{-1}\gamma x \in K\} \subset G_s K.$$

This lemma is of great consequence for the Theorem in the case $L = G$. For G-regular semisimple $\gamma \in M \cap K$, the integrand of $J_M(\gamma, f_G^0)$ vanishes outside of the subset above. It therefore suffices to consider the integral over $G_\gamma \setminus G_s$.

We proceed by considering the case $L = G$ and examining the structure of G_s. For the remainder of this section, we take γ to belong to $K \cap M$, to be elliptic in M and assume that γ^n is regular in G. It is well-known that M may be decomposed as a product of subgroups,

$$M_1 \times \cdots \times M_k,$$

where $M_i \cong \mathrm{GL}(r_i, F)$, $1 \le i \le k$ and $\sum_{i=1}^k r_i = r$. We will often identify M_i with $\mathrm{GL}(r_i, F)$. Accordingly, we represent the elements γ, s and u by the respective ordered k-tuples, $(\gamma_1, \ldots, \gamma_k)$, $(s_1, \ldots, s_k)$ and $(u_1, \ldots, u_k)$. Clearly, the topological Jordan decomposition of γ_i is $s_i u_i$, $1 \le i \le k$.

Lemma 1 *Suppose that $s_1 = \cdots = s_k$, and n is relatively prime to r_1. Then $G_s = G_{s^n}$, and there exists an extension F_1' of F and an integer r_1' such that $G_s \cong \mathrm{GL}(kr_1', F_1')$.*

Proof Clearly, $s_1 u_1, \ldots, s_1 u_k$ are regular and elliptic in $\mathrm{GL}(r_1, F)$. We know that s_1 commutes with these elements, hence it too is elliptic in $\mathrm{GL}(r_1, F)$. In particular s_1 and s_1^n are semisimple. According to [9, pp. 164–165], the centralizer of s_1^n in $\mathrm{GL}(r_1, F)$ is isomorphic to $\prod_{i=1}^{\ell} \mathrm{GL}(r_i', F_i')$, where $F_1', \ldots, F_\ell'$ are extensions of F and $\sum_{i=1}^{\ell} r_i'[F_i' : F] = r_1$. Since this group contains an elliptic torus of $\mathrm{GL}(r_1, F)$, we must have $\ell = 1$. In particular, s_1^n may be regarded as an element of F_1'. We repeat the above procedure with s_1 in place of s_1^n to obtain a field extension F_1'' such that the centralizer of s_1 in $\mathrm{GL}(r_1, F)$ is isomorphic to $\mathrm{GL}(r_1'', F_1'')$. Clearly, F_1'' is the field obtained from F_1' by adjoining an n-th root of s_1^n. That is, $F_1'' = F_1(s)$. By [10, Theorem 6.2 (ii) Chapter VI], $[F_1'' : F_1']$ divides n. At the same time,

$$r_1 = r_1''[F_1'' : F_1'][F_1' : F].$$

As n and r_1 are relatively prime, we conclude that $[F_1'' : F_1'] = 1$. It follows that the F-subalgebras generated by s and s^n in $\mathrm{Mat}_r(F)$ are both equal to $\prod_{i=1}^{kr_1'} F_1'$. Appealing once again to [9, pp. 164–165] we obtain the lemma. ■

The proof of the following corollary is a detailed application of parts of [6, Section 12] to weighted orbital integrals.

Corollary 1 *Suppose* $s_1, \dots, s_k$ *all belong to the same conjugacy class of* $\mathrm{GL}(r_1, F)$*, and* n *is relatively prime to* r_1*. Then*

$$J_{\tilde{M}}\big(\mathbf{s}(\gamma)^n, f^0_{\tilde{G}}\big) = |D^G(\gamma)|^{1/2} \int_{G_\gamma \backslash G_s} f^0_G(x^{-1}ux)v_M(x)\,dx = J_M(\gamma, f^0_G).$$

Proof As weighted orbital integrals are constant on conjugacy classes, we may assume $s_1 = \cdots = s_k$. [4, Lemma 2 Section 1.1] implies that $\tilde{G}_{\mathbf{s}(\gamma)^n} = \widetilde{G_\gamma}$. We may identify $\widetilde{G_\gamma} \setminus \tilde{G}$ with $G_\gamma \setminus G$ as measure spaces via the map $\mathbf{s}$. Consequently $J_{\tilde{M}}\big(\mathbf{s}(\gamma)^n, f^0_{\tilde{G}}\big)$ is equal to

$$|D^G(\gamma^n)|^{1/2} \int_{G_\gamma \backslash G} f^0_{\tilde{G}}\big(\mathbf{s}(x)^{-1}\mathbf{s}(\gamma)^n\mathbf{s}(x)\big)\, v_M(x)\,dx.$$

After applying Lemma 1 and integrating over K, we may replace this expression with

$$|D^G(\gamma^n)|^{1/2} \int_{G_\gamma \backslash G_s} f^0_G\big(\mathbf{s}(x)^{-1}\mathbf{s}(s)^n\mathbf{s}(u)^n\mathbf{s}(x)\big)\, v_M(x)\,dx.$$

By a variant of [8, Proposition 0.1.5] (*cf.* [9, p. 212] also), we have

$$\mathbf{s}(x)^{-1}\mathbf{s}(s^n)\mathbf{s}(x) = \mathbf{i}\Big(\big((\det s)^n, \det x\big)_F^{1+2m}(s_1^n, \det_1 x)_{F_1'}^{-1}\Big). \tag{1}$$

Here $(\cdot,\cdot)_F$ and $(\cdot,\cdot)_{F_1'}$ are the n-th Hilbert symbols of F and F_1' respectively, and $\det_1$ is the determinant map of $\mathrm{GL}(kr_1', F_1')$. Since $(\det s)^n$ and s^n are n-th powers in F and F_1' respectively, the Hilbert symbols on the right have value 1. Therefore $J_{\tilde{M}}\big(\mathbf{s}(\gamma)^n, f^0_{\tilde{G}}\big)$ is reduced to

$$|D^G(\gamma^n)|^{1/2} \int_{G_\gamma \backslash G_s} f^0_{\tilde{G}}\big(\mathbf{s}(x)^{-1}\mathbf{s}(u)^n\mathbf{s}(x)\big)\, v_M(x)\,dx.$$

Given $x \in G_s$, then $x^{-1}ux \in K$ if and only if $x^{-1}u^nx \in K$. Indeed, n and q are relatively prime, so for any integer N, there exist integers a and b such that $an+bq^N = 1$. Therefore,

$$x^{-1}ux = (x^{-1}u^nx)^a(x^{-1}u^{q^N}x)^b,$$

and the result follows from the topological unipotency of u. Now suppose $x^{-1}ux \in K$ and $\mathbf{s}(x)^{-1}\mathbf{s}(u)\mathbf{s}(x) = (x^{-1}ux, \zeta)$ for some $\zeta \in \mu_n$. Then

$$\lim_{N\to\infty}(x^{-1}u^{q^N}x, \zeta^{q^N}) = \lim_{N\to\infty}(x^{-1}ux,\zeta)^{q^N} = \mathbf{s}(x)^{-1}\big(\lim_{N\to\infty}\mathbf{s}(u)^{q^N}\big)\,\mathbf{s}(x) = 1.$$

Therefore, $\lim_{N\to\infty}\zeta^{q^N} = 1$. This implies that $\zeta = 1$, since n and q are relatively prime. We have shown that

$$f^0_{\tilde{G}}\big(\mathbf{s}(x)^{-1}\mathbf{s}(u)^n\mathbf{s}(x)\big) = f^0_G(x^{-1}ux).$$

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All that remains, is to deal with the Weyl discriminants. Taking $|\cdot|_{F_1'}$ to be the absolute value of the extension F_1', it is then a simple exercise to show that

$$|D^G(\gamma^n)| = |D^{G_s}(s^n u^n)|_{F_1'} = |D^{G_s}(u^n)|_{F_1'} = |D^{G_s}(su)|_{F_1'} = |D^G(\gamma)|. \qquad \blacksquare$$

Lemma 2 and Corollaries 2 and 3 describe what happens when the hypothesis of Corollary 1 does not hold.

Lemma 2 *Suppose $L \in \mathcal{L}(M)$ such that $L \neq G$ and $x \in L$. Then $v_M(x)$ vanishes.*

Proof Obviously, we may take L to be a maximal proper subgroup in $\mathcal{L}(M)$. It is well-known that L must then be equal to $L_1 \times L_2$, where $L_1 \cong \mathrm{GL}(d, F)$ and $L_2 \cong \mathrm{GL}(r-d, F)$ for some integer $1 \leq d \leq r-1$. Accordingly, we represent x by the ordered pair (x_1, x_2). Given $Q \in \mathcal{P}(M)$, we have

$$L = MN_{Q\cap L}(K \cap L).$$

Since v_M is left-invariant under M and right-invariant under K we may assume that x is unipotent. Given $P \in \mathcal{P}(M)$, we also have

$$L = N_{P\cap L}M(K \cap L).$$

It follows that

$$H_P(x) = H_{P\cap L}(x) = \big(H_{P\cap L_1}(x_1), H_{P\cap L_2}(x_2)\big) \in \mathfrak{a}_{M\cap L_1} \oplus \mathfrak{a}_{M\cap L_2} \cong \mathfrak{a}_M.$$

We may decompose x_j as $u_j m_j k_j$, where $u_j \in N_{P\cap L_j}$, $m_j \in M \cap L_j$ and $k_j \in K \cap L_j$, for $j = 1, 2$. Clearly,

$$\det m_1 = \det m_2 = 1,$$

as x_1 and x_2 are unipotent. This implies that

$$H_P(x) = \big(H_{P\cap L_1}(m_1), H_{P\cap L_2}(m_2)\big) \in \mathfrak{a}^{L_1}_{M\cap L_1} \oplus \mathfrak{a}^{L_2}_{M\cap L_2} \cong \mathfrak{a}^L_M.$$

That is, $\langle \lambda, H_P(x)\rangle$ vanishes for all $\lambda \in \mathfrak{a}^*_L$. Now, according to [1, (6.5)], there exists a constant C such that

$$v_M(x) = C \sum_{P\in\mathcal{P}(M)} \frac{\langle \lambda, H_P(x)\rangle^{\dim(A_M/A_G)}}{\prod_{\alpha\in\Delta_P} \lambda(\alpha^\vee)}, \quad \lambda \in i\mathfrak{a}^*_M.$$

The expression on the right is independent of $\lambda \in i\mathfrak{a}^*_M$. We know that the polynomial function,

$$\lambda \mapsto \langle \lambda, H_P(x)\rangle^{\dim(A_M/A_G)}, \quad \lambda \in i\mathfrak{a}^*_M,$$

has a zero of multiplicity $\dim(A_M/A_G)$ at any element in $i\mathfrak{a}^*_L \subset i\mathfrak{a}^*_M$. On the other hand, the polynomial function,

$$\lambda \mapsto \prod_{\alpha\in\Delta_P} \lambda(\alpha^\vee), \quad \lambda \in \mathfrak{a}^*_{M,\mathbf{C}},$$

has a zero of multiplicity strictly less than $|\Delta_P| = \dim(A_M/A_G)$ at any element of $i\mathfrak{a}^*_L$ which does not also lie in $i\mathfrak{a}^*_G$. The lemma follows. $\blacksquare$

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Corollary 2 *Suppose s_i^n and s_{i+1}^n do not lie in the same conjugacy class for some $1 \leq i \leq k-1$. Then $J_{\tilde{M}}\big(\mathbf{s}(\gamma)^n, f_{\tilde{G}}^0\big)$ vanishes.*

Proof After possibly conjugating s by a permutation matrix, we may assume that if s_j is not conjugate to s_{j+1} for some $1 \leq j \leq k-1$, then s_j is not conjugate to $s_{j'}$ for any $j+1 \leq j' \leq k-1$. As explained on [9, pp. 164–165], there exist an integer $\ell \geq 2$ and extensions, $F_1, \dots, F_\ell$, of F such that G_{s^n} is isomorphic to $\prod_{i=1}^{\ell} \mathrm{GL}(t_i, F_i)$ and $\sum_{i=1}^{\ell} t_i[F_i : F] = r$. Let $y \in G$ be the diagonal matrix whose first $t_1[F_1 : F]$ diagonal entries are 1 and whose remaining diagonal entries are -1. It is simple to verify that $G_y \in \mathcal{L}$ and $G_y \neq G$. At the same time, it is apparent that $G_s \subset G_y$. Hence, by [7, Lemma 3 Section 3] and Lemma 2,

$$J_{\tilde{M}}\big(\mathbf{s}(\gamma)^n, f_{\tilde{G}}^0\big) = |D^G(\gamma^n)|^{1/2} \int_{\tilde{G}_{\mathbf{s}(\gamma)^n} \backslash \widetilde{G_y}} f_{\tilde{G}}^0\big(x^{-1}\mathbf{s}(\gamma)^n x\big)\, v_M\big(\mathbf{p}(x)\big)\, dx = 0. \qquad \blacksquare$$

Corollary 3 *Suppose s_i^n and s_{i+1}^n do not lie in the same conjugacy class for some $1 \leq i \leq k-1$. Then*

$$J_{\tilde{M}}\big(\mathbf{s}(\gamma)^n, f_{\tilde{G}}^0\big) = \sum_{\eta \in \mu_n^M/\mu_n^G} J_M(\eta\gamma, f_G^0) = 0.$$

Proof Suppose $\eta \in \mu_n^M$. As η lies in the center of M and n is relatively prime to q, the topological Jordan decomposition of $\eta\gamma$ is $(\eta s)u = \big((\eta_1 s_1)u_1, \dots, (\eta_k s_k)u_k\big)$. The hypothesis, together with the fact that $\eta_j^n = 1$ for all $1 \leq j \leq k$, imply that $\eta_i s_i$ and $\eta_{i+1}s_{i+1}$ do not belong to the same conjugacy class. The corollary now follows from an application of Corollary 2 in the case $n = 1$. $\blacksquare$

Corollary 4 *Suppose $s_1^n, \dots, s_k^n$ all lie in the same conjugacy class, and n is relatively prime to r_1. Then*

$$J_{\tilde{M}}\big(\mathbf{s}(\gamma)^n, f_{\tilde{G}}^0\big) = \sum_{\eta \in \mu_n^M/\mu_n^G} J_M(\eta\gamma, f_G^0).$$

Proof Suppose η belongs to μ_n^M but does not belong to μ_n^G. In view of Corollary 1, it suffices to show that $J_M(\eta\gamma, f_G^0)$ vanishes. As in the proof of Corollary 3, the topological Jordan decomposition of $\eta\gamma$ is $(\eta s)u = \big((\eta_1 s_1)u_1, \dots, (\eta_k s_1)u_k\big)$. Evidently, $\eta_i \neq \eta_{i+1}$ for some $1 \leq i \leq k-1$, otherwise η would belong to μ_n^G. Consequently $\eta_i s_1$ and $\eta_{i+1}s_1$ do not belong to the same conjugacy class and an application of Corollary 2 yields the desired vanishing. $\blacksquare$

The proof of the following proposition is really just an indication of how to generalize the previous results to the case $L \neq G$ of the Theorem.

Proposition *Suppose that for any $1 \leq j \leq r$ and any proper divisor i of j, n is relatively prime to i. Then*

$$J_{\tilde{M}}^{\tilde{L}}\big(\mathbf{s}(\gamma)^n, f_{\tilde{L}}^0\big) = \sum_{\eta \in \mu_n^M/\mu_n^L} J_M^L(\eta\gamma, f_L^0), \quad L \in \mathcal{L}(M).$$

Proof Taken together, Corollary 3 and Corollary 4 provide a proof of the Proposition in the case $L = G$. Suppose $L \neq G$. We decompose L as the product,

$$L_1 \times \cdots \times L_\ell,$$

where $L_j \cong \mathrm{GL}(t_j, F)$ and $\sum_{j=1}^{\ell} t_j = r$. There exists a partition, $0 = k_0 < \cdots < k_\ell = k$, such that

$$M_{k_{j-1}+1} \times M_{k_{j-1}+2} \times \cdots \times M_{k_j} \subset L_j,$$

and $\sum_{i=k_{j-1}+1}^{k_j} r_i = t_j$, for all $1 \leq j \leq \ell$. Taking our hypothesis into account, we apply Lemma 1 to show that if $s_{k_{j-1}+1}, \ldots, s_{k_j}$ all belong to the same conjugacy class for all $1 \leq j \leq \ell$, then there exist extensions, $E_1, \ldots, E_\ell$, of F and positive integers, $t'_1, \ldots, t'_\ell$, such that $L_s = L_{s^n} \cong \prod_{j=1}^{\ell} \mathrm{GL}(t'_j, E_j)$. After changing equation (1) in the proof of Corollary 1 to

$$\mathbf{s}(x)^{-1}\mathbf{s}(s^n)\mathbf{s}(x) = \mathbf{i}\Big(\big((\det s)^n, \det x \big)_F^{1+2m} \prod_{j=1}^{\ell} (s_1^n, \det_j x_j)_{E_j}^{-1} \Big)$$

(*cf.* [9, p. 212]), the remaining lemmas and corollaries are easily seen to hold when G is replaced by L. ■

The Proposition is a proof of the Theorem in the elliptic case. Indeed, if $r \geq 4$ then n is odd by assumption and $\gamma' = \mathbf{s}(\gamma)^n$. In addition, $\gamma' = \mathbf{s}(\gamma)^n$ for $\gamma \in M_0$ [6, Section 4]. Therefore, the only case left to consider is the case that $r = 3$ and M is a maximal proper Levi subgroup. In this case, Corollary 3 implies

$$J_{\tilde{M}}(\gamma', f^0_{\tilde{G}}) = \pm J_{\tilde{M}}\big(\mathbf{s}(\gamma)^n, f^0_{\tilde{G}}\big) = 0 = \sum_{\eta \in \mu_n^M/\mu_n^G} J_M(\eta\gamma, f^0_G).$$

4 Proof of the Theorem

By the Proposition, it suffices to consider the case that $\gamma \in M$ is semisimple and γ^n is regular in G, but γ is not elliptic in M. Then, by Jordan canonical form, γ is conjugate to an elliptic element in $M_1 \in \mathcal{L}$ where $M_1 \subsetneq M$. This fact allows us to use a descent formula for weighted orbital integrals [2, (8.2)]. Namely,

$$J^{\tilde{L}}_{\tilde{M}}(\gamma', f^0_{\tilde{L}}) = \sum_{L_1 \in \mathcal{L}^L(M_1)} d^L_{M_1}(M, L_1) J^{\tilde{L}_1}_{\tilde{M}_1}(\gamma', f^0_{\tilde{L}_1}).$$

The constants, $d^L_{M_1}(M, L_1)$, $L_1 \in \mathcal{L}^L(M_1)$, are defined in [2, Section 7]. They vanish unless

$$\mathfrak{a}^L_{M_1} \cong \mathfrak{a}^M_{M_1} \oplus \mathfrak{a}^{L_1}_{M_1}.$$

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In particular $d^L_{M_1}(M, L) = 0$. This descent formula allows us to argue by induction. Assume inductively that the Theorem holds for $L_1 \in \mathcal{L}^L(M)$ with $L_1 \neq L$. Then

$$J^{\tilde{L}}_{\tilde{M}}(\gamma', f^0_{\tilde{L}}) = \sum_{L_1 \in \mathcal{L}^L(M_1)} d^L_{M_1}(M, L_1) J^{\tilde{L}_1}_{\tilde{M}_1}(\gamma', f^0_{\tilde{L}_1})$$
$$= \sum_{L_1 \in \mathcal{L}^L(M_1)} d^L_{M_1}(M, L_1) \sum_{\eta \in \mu_n^{M_1}/\mu_n^{L_1}} J^{L_1}_{M_1}(\eta\gamma, f^0_{L_1}).$$

We need a lemma to justify the next step.

Lemma 3 *Suppose* $\mathfrak{a}^L_{M_1} \cong \mathfrak{a}^M_{M_1} \oplus \mathfrak{a}^{L_1}_{M_1}$. *Then the canonical map,*

$$\mu_n^M/\mu_n^L \to \mu_n^{M_1}/\mu_n^{L_1},$$

is a bijection.

Proof The vector spaces $\mathfrak{a}^L_M$ and $\mathfrak{a}^L_{L_1}$ may be regarded as the respective orthogonal complements of $\mathfrak{a}^M_{M_1}$ and $\mathfrak{a}^{L_1}_{M_1}$ in $\mathfrak{a}^L_{M_1}$. As a consequence we also have

$$\mathfrak{a}^L_M \oplus \mathfrak{a}^L_{L_1} \cong \mathfrak{a}^L_{M_1}.$$

It is readily verified that the homomorphism,

$$H_{M_1} : M_1 \to \mathfrak{a}_{M_1},$$

passes to a homomorphism

$$H'_{M_1} : A_{M_1}/A_L \to \mathfrak{a}^L_{M_1}$$

such that $H'_{M_1}(A_M/A_L) \subset \mathfrak{a}^L_M$ and $H'_{M_1}(A_{L_1}/A_L) \subset \mathfrak{a}^L_{L_1}$. Accordingly,

$$H'_{M_1}\big((A_M \cap A_{L_1})/A_L\big) \subset \mathfrak{a}^L_M \cap \mathfrak{a}^L_{L_1} = \{0\}.$$

In other words, $|\xi(x)| = 1$ for all x belonging to the split torus $A_M \cap A_{L_1}$, and all characters $\xi \in X(M_1)$, which are trivial when restricted to L. This implies that $A_{L_1} \cap A_M \subset A_L$. As a result,

$$\mu_n^M/\mu_n^L \to \mu_n^{M_1}/\mu_n^{L_1},$$

is injective. It is also bijective as

$$|\mu_n^M/\mu_n^L| = n^{\dim(\mathfrak{a}^L_M)} = n^{\dim(\mathfrak{a}^L_{M_1}) - \dim(\mathfrak{a}^L_{L_1})} = |\mu_n^{M_1}/\mu_n^{L_1}|. \qquad \blacksquare$$

This lemma tells us that we may replace the previous decomposition of $J^{\tilde{L}}_{\tilde{M}}(\gamma', f^0_{\tilde{L}})$ with

$$\sum_{L_1 \in \mathcal{L}^L(M_1)} d^L_{M_1}(M, L_1) \sum_{\eta \in \mu_n^M/\mu_n^L} J^{L_1}_{M_1}(\eta\gamma, f^0_{L_1}) = \sum_{\eta \in \mu_n^M/\mu_n^L} J^L_M(\eta\gamma, f^0_L).$$

The proof of the Theorem is complete. ■

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Max-Planck-Institut für Mathematik Bonn
PB: 7280
D-53072 Bonn
Germany
email: mezo@mpim-bonn.mpg.de

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