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CORRIGENDUM

Corrigendum to 'Endoscopy for Hecke categories, character sheaves and representations'

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Abstract
We fix an error on a 3-cocycle in the original version of the paper 'Endoscopy for Hecke categories, character sheaves and representations'. We give the corrected statements of the main results.

1. The error

The published version of [1] contains an error that led to the wrong conclusion on a certain 3-cocycle that appears in the monoidal structure of the monodromic Hecke category.

Below we use notations from [1]. The statement [1, Lemma 10.10] is wrong; that is, the cohomology class of $(\lambda, \mu^\natural) \in \mathrm{H}^3(\Xi, \overline{\mathbb{Q}}_\ell^\times)$ is not always trivial. The mistake in the 'proof' is that, although the character sheaf $\mathcal{L}$ becomes trivial when restricted to $T(\mathbb{F}_q)$, the trivialisation cannot necessarily be made $W_\mathcal{L}$-equivariantly. Recall (λ, μ) comes from a 2-cocycle $c \in Z^2(W, T)$, which in turns comes from the extension

$$1 \to T \to N_G(T) \to W \to 1.$$

Namely, choose a lifting $\widetilde{w} \in N_G(T)$ for each $w \in W$, and let $c(w_1, w_2) \in T$ be such that $\widetilde{w}_1\widetilde{w}_2 = c(w_1, w_2)\widetilde{w_1 w_2}$. On the other hand, the datum of a character sheaf $\mathcal{L}$ on T gives an extension of abelian groups

$$1 \to \overline{\mathbb{Q}}_\ell^\times \to E_\mathcal{L} \to T \to 1 \tag{1.1}$$

where $E_\mathcal{L}$ consists of pairs (t, τ) where $t \in T$ and τ is a nonzero element of $\mathcal{L}_t$. This extension carries an action of $W_\mathcal{L}$. Taking $W_\mathcal{L}$-cohomology we get a connecting homomorphism

$$\delta_\mathcal{L} : \mathrm{H}^2(W_\mathcal{L}, T) \to \mathrm{H}^3(W_\mathcal{L}, \overline{\mathbb{Q}}_\ell^\times).$$

Then $(\lambda, \mu^\natural) = \delta_\mathcal{L}(c)$.

Now we can always arrange so that c takes values in $T[2]$ (using Tits liftings). Restricting (1.1) to $T[2]$ the short exact sequence splits, but not necessarily $W_{\mathcal{L}}$-equivariantly. Therefore, the composition

$$\mathrm{H}^2(W, T[2]) \to \mathrm{H}^2(W, T) \to \mathrm{H}^2(W_{\mathcal{L}}, T) \xrightarrow{\delta_{\mathcal{L}}} \mathrm{H}^3(W_{\mathcal{L}}, \overline{\mathbb{Q}}_\ell^\times) \tag{1.2}$$

is still not necessarily zero.

For example, when $G = \mathrm{SL}(2)$ and $\mathcal{L}$ has order 2, we have $W_{\mathcal{L}} = W \cong \mathbb{Z}/2\mathbb{Z}$, and the composition (1.2) is nonzero.

2. Correction

The 3-cocycle responsible for the convolution structure on the monodromic Hecke category is the product of two 3-cocycles: one is σ defined in [1, §5.8] and studied in [3], which is often nontrivial; the other one is the $\mu^\natural$ mentioned above, which can also be nontrivial. It turns out that the cohomology classes of these two cocycles cancel each other, so their product is cohomologically trivial.

In the new version of the paper [2], we give a construction of rigidified minimal IC sheaves that in particular imply the cancellation between σ and $\mu^\natural$ (although we no longer need σ and $\mu^\natural$ in the new version of the paper).

The idea is to consider a geometric Whittaker model

$${}_\psi\mathcal{M}_{\mathcal{L}} := D^b_m((U^-, \psi)\backslash G/(B, \mathcal{L}))$$

that is on the one hand a right module for the monodromic Hecke category and, on the other hand, equivalent to mixed sheaves on a point by taking stalks at the identity element $e \in G$. See [2, §5.9]. This allows us to rigidify minimal IC sheaves, denoted $\mathrm{IC}(w^\beta)^\dagger_{\mathcal{L}}$ for blocks β. In [2, Lemma 5.12] we show that there are canonical isomorphisms

$$\mathrm{IC}(w^\gamma)^\dagger_{\mathcal{L}'} \star \mathrm{IC}(w^\beta)^\dagger_{\mathcal{L}} \xrightarrow{\sim} \mathrm{IC}(w^{\gamma\beta})^\dagger_{\mathcal{L}}$$

for two composable blocks β and γ, and these isomorphisms are associative. More generally, in [2, Definition 6.14] we define a rigidified IC sheaf $\mathrm{IC}(w)^\dagger_{\mathcal{L}}$ for any w.

As a result, the main theorems in [1] involving nonneutral blocks can be simplified and no twisting by cocycles appear in the statements. We give the corrected statements below.

Let G be a connected reductive group over $k = \overline{\mathbb{F}}_q$. Let $\mathfrak{o} \subset \mathrm{Ch}(T)$. For $\mathcal{L} \in \mathfrak{o}$, let $H_{\mathcal{L}}$ be the endoscopic group attached to $\mathcal{L}$, equipped with a Borel subgroup $B^H_{\mathcal{L}}$ and a relative pinning with respect to G. For each $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$ one can define a $(H_{\mathcal{L}'}, H_{\mathcal{L}})$-bitorsor ${}_{\mathcal{L}'}\mathfrak{H}^\beta_{\mathcal{L}}$, so that the disjoint union of ${}_{\mathcal{L}'}\mathfrak{H}^\beta_{\mathcal{L}}$ for all $\mathcal{L}, \mathcal{L}' \in \mathfrak{o}$ and all blocks β form a groupoid compatible with the convolution of blocks (see [2, §10.3]). Restricting to $\mathcal{L}' = \mathcal{L}$, ${}_{\mathcal{L}}\mathfrak{H}_{\mathcal{L}} := \coprod_{\beta \in {}_{\mathcal{L}}\underline{W}_{\mathcal{L}}} {}_{\mathcal{L}}\mathfrak{H}^\beta_{\mathcal{L}}$ is an algebraic group with neutral component $H_{\mathcal{L}}$ and component group $\Omega_{\mathcal{L}}$. Let

$${}_{\mathcal{L}'}\mathcal{E}^\beta_{\mathcal{L}} := D^b_m(B^H_{\mathcal{L}'}\backslash {}_{\mathcal{L}'}\mathfrak{H}^\beta_{\mathcal{L}}/B^H_{\mathcal{L}}).$$

2.1. Theorem (Monodromic-endoscopic equivalence for all blocks, [2, Theorem 10.7]). *Under the above notations,*

1. *For $\mathcal{L}, \mathcal{L}' \in \mathfrak{o}$ and any block $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$, there is an equivalence of triangulated categories*

$${}_{\mathcal{L}'}\Psi^\beta_{\mathcal{L}} : {}_{\mathcal{L}'}\mathcal{E}^\beta_{\mathcal{L}} \xrightarrow{\sim} {}_{\mathcal{L}'}\mathcal{D}^\beta_{\mathcal{L}}$$

that sends $\Delta(w)^H_{\mathcal{L}}, \nabla(w)^H_{\mathcal{L}}$ and $IC(w)^H_{\mathcal{L}}$ in ${}_{\mathcal{L}'}\mathcal{E}^\beta_{\mathcal{L}}$ to $\Delta(w)^\dagger_{\mathcal{L}}, \nabla(w)^\dagger_{\mathcal{L}}$ and $IC(w)^\dagger_{\mathcal{L}}$ in ${}_{\mathcal{L}'}\mathcal{D}^\beta_{\mathcal{L}}$, for $w \in \beta$.

2. *The equivalences $\{{}_{\mathcal{L}'}\Psi^\beta_{\mathcal{L}}\}$ are compatible with convolutions.*

Let G be a connected reductive group over $k = \overline{\mathbb{F}}_q$. Let $\mathfrak{o} \subset \mathrm{Ch}(T)$ be a W-orbit and $\mathcal{L} \in \mathfrak{o}$. Let $\mathbf{c} \subset W^\circ_{\mathcal{L}}$ be a two-sided cell and $\Omega_{\mathbf{c}}$ its stabiliser in $\Omega_{\mathcal{L}}$. We have the abelian category of semisimple character sheaves $\underline{\mathcal{CS}}^{[\mathbf{c}]}_{\mathfrak{o}}(G)$ on G with semisimple parameter $\mathfrak{o}$ and belonging to the cell $[\mathbf{c}] \subset W \times \mathfrak{o}$ containing $\mathbf{c}$. Let H be the endoscopic group attached to $\mathcal{L}$. To each $\beta \in \Omega_{\mathcal{L}}$, we consider the β-twisted semisimple unipotent character sheaves category $\underline{\mathcal{CS}}^{\mathbf{c}}_u(H;\beta)$ on H, which carries a $\Omega_{\mathbf{c}}$-action. For more detail, see [2, §11.7,11.9].

2.2. Theorem (Monodromic-endoscopic equivalence for character sheaves, [2, Theorem 11.10]). *Under the above notations, there is a canonical equivalence of semisimple abelian categories*

$$\underline{\mathcal{CS}}^{[\mathbf{c}]}_{\mathfrak{o}}(G) \cong \bigoplus_{\beta \in \Omega_{\mathbf{c}}} \underline{\mathcal{CS}}^{\mathbf{c}}_u(H;\beta)^{\Omega_{\mathbf{c}}}.$$

Let G be a connected reductive group over $k = \overline{\mathbb{F}}_q$ with an $\mathbb{F}_q$-Frobenius structure $\epsilon : G \to G$. Let $\mathfrak{o} \subset \mathrm{Ch}(T)$ be a W-orbit stable under ϵ, and let $\mathcal{L} \in \mathfrak{o}$. Let $\mathbf{c}$, $[\mathbf{c}]$, $\Omega_{\mathbf{c}}$ and H be as in Theorem 2.2. Let $\mathfrak{B}_{\mathbf{c}} = \{\beta \in {}_{\mathcal{L}}\underline{W}_{\epsilon\mathcal{L}} | w^\beta \circ \epsilon$ preserves the cell $\mathbf{c}$ of $W^\circ_{\mathcal{L}}\}$, with the ϵ-twisted conjugation action of $\Omega_{\mathbf{c}}$ denoted by $\mathrm{Ad}_\epsilon(\Omega_{\mathbf{c}})$. Each $\beta \in \mathfrak{B}_{\mathbf{c}}$ defines a $\mathbb{F}_q$-Frobenius structure $\sigma_{\beta\epsilon}$ of H. Moreover, the category $\mathrm{Rep}^{\mathbf{c}}_u(H^{\sigma_{\beta\epsilon}})$ of unipotent representations of $H^{\sigma_{\beta\epsilon}}$ in the cell $\mathbf{c}$ carries an action of $\Omega_{\mathbf{c},\beta}$, the simultaneous stabiliser of $\mathbf{c}$ and β in $\Omega_{\mathcal{L}}$. For more detail, see [2, §12.5].

2.3. Theorem (Monodromic-endoscopic equivalence for representations, [2, Corollary 12.7]). *Choose a representative for each $Ad_\epsilon(\Omega_{\mathbf{c}})$-orbit of $\mathfrak{B}_{\mathbf{c}}$, and denote this set of representatives by $\dot{\mathfrak{B}}_{\mathbf{c}}$. There is a canonical equivalence of semisimple abelian categories*

$$Rep^{[\mathbf{c}]}_{\mathfrak{o}}(G^\epsilon) \cong \bigoplus_{\beta \in \dot{\mathfrak{B}}_{\mathbf{c}}} Rep^{\mathbf{c}}_u(H^{\sigma_{\beta\epsilon}})^{\Omega_{\mathbf{c},\beta}}.$$

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Conflict of Interest: None.

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