Bull. Aust. Math. Soc. **99** (2019), 508–520
doi:10.1017/S0004972719000145

EMBEDDING OF METRIC GRAPHS ON HYPERBOLIC SURFACES

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(Received 9 September 2018; accepted 31 December 2018; first published online 13 February 2019)

Abstract

An embedding of a metric graph (G, d) on a closed hyperbolic surface is *essential* if each complementary region has a negative Euler characteristic. We show, by construction, that given any metric graph, its metric can be rescaled so that it admits an essential and isometric embedding on a closed hyperbolic surface. The essential genus $g_e(G)$ of (G, d) is the lowest genus of a surface on which such an embedding is possible. We establish a formula to compute $g_e(G)$ and show that, for every integer $g \geq g_e(G)$, there is an embedding of (G, d) (possibly after a rescaling of d) on a surface of genus g. Next, we study minimal embeddings where each complementary region has Euler characteristic -1. The maximum essential genus $g_e^{\max}(G)$ of (G, d) is the largest genus of a surface on which the graph is minimally embedded. We describe a method for an essential embedding of (G, d), where $g_e(G)$ and $g_e^{\max}(G)$ are realised.

2010 *Mathematics subject classification*: primary 57M15; secondary 05C10.

Keywords and phrases: hyperbolic surface, fat graph, girth, Betti number, Betti deficiency.

1. Introduction

Graphs on surfaces play an important role in the study of topology and geometry of surfaces. A two-cell embedding of a graph G on a closed oriented *topological surface* S_g of genus g is a cellular decomposition of S_g, whose one-skeleton is isomorphic to G [6]. In topological graph theory, the characterisation of the surfaces on which a graph can be two-cell embedded is a famous problem and well-studied [13]. In this direction, Kuratowski showed that a graph is planar if and only if it does not contain $K_{3,3}$ (the complete bipartite graph with partitions of size 3 and 3) or K_5 (the complete graph with five vertices) as a minor. Hence, these are the only minimal nonplanar graphs.

The genus of a surface S is denoted by $g(S)$. The genus of a graph G is defined by $g(G) = \min\{g(S)\}$, where the minimum is taken over the surfaces S on which G is two-cell embedded. The maximum genus $g_M(G)$ is similarly defined [15].

The author has been supported by a Post Doctoral Fellowship funded by a J. C. Bose fellowship of Professor Mahan Mj.

In [6], Duke has shown that every finite graph G admits a two-cell embedding on a surface S_g of genus g for each g with $g(G) \leq g \leq g_M(G)$. A two-cell embedding of a graph realising its genus is called a minimal embedding. A maximal embedding is defined similarly. In [6, Theorem 3.1], Duke derived a sufficient condition for a two-cell embedding to be nonminimal and provided an algorithm to obtain an embedding on a lower genus surface.

The maximum genus problem was studied by Xuong in [15]. In [15, Theorem 3], Xuong obtained the formula $g_M(G) = \frac{1}{2}(\beta(G) - \zeta(G))$ for the maximum genus, where $\beta(G)$ and $\zeta(G)$ are the Betti number and Betti deficiency of G, respectively. Furthermore, in a maximal embedding, the number of two-cells in the cellular decomposition is $1 + \zeta(G)$. For more results on two-cell embeddings, we refer to [13] and [9].

In this paper, when we say surface, we will always mean a Riemannian surface with constant negative sectional curvature -1. Such a surface is called a hyperbolic surface.

Configurations of geodesics on surfaces have become increasingly important in the study of mapping class groups and the moduli spaces of surfaces through the *systolic* function [12] and *filling* pair length function [2]. The study of filling systems has its origin in the work of Thurston [14]. To construct a spine of the moduli space Thurston defined a set χ_g consisting of the closed hyperbolic surfaces of genus g whose systoles fill the surfaces [14]. We call χ_g the Thurston set of genus g (see [2]). Recently, Anderson *et al.* [1] studied the shape of χ_g, comparing it with the set Y_g of trivalent surfaces (the surfaces with a pants decomposition with all curves of length bounded above and below by positive constants independent of g, see [12, Section 4]) by giving a lower bound on the Hausdorff distance between χ_g and Y_g in the moduli space $\mathcal{M}_g$.

There is a natural connection between graphs and surfaces. For instance, given a system of curves on a surface, the union forms a so-called *fat graph*, where the intersection points are the vertices, sub-arcs between the intersections are the edges and the cyclic order on the set of edges incident at each vertex is determined by the orientation of the surface. In [3], Balacheff *et al.* studied the geometry and topology of Riemann surfaces by embedding a suitable graph on the surface which captures some of its geometric and topological properties.

A graph G on a closed surface S is *essential* if each component of $S \setminus G$ has a negative Euler characteristic. The definition of essential graphs is motivated by the definition of essential curves on closed surfaces with negative Euler characteristic. In [11], we studied essential systolic graphs on closed hyperbolic surfaces aiming to get a natural decomposition of the moduli space of hyperbolic surfaces by associating to a surface its systolic graph.

In this paper, we consider essential graphs on surfaces, whose edges are realised by geodesic segments. Furthermore, no two edges meet in their interior. Such a graph has a metric, where the distances between points on the graph are measured along a shortest path in the induced metric on the graph.

By a graph, we always mean a *finite* and *connected* graph, but the graphs under consideration need not be simplicial. In particular, we will be looking at (finite and connected) metric graphs.

Definition 1.1. A metric graph is a pair (G, d) consisting of a graph G and a positive real valued function $d : E \to \mathbb{R}_+$ on the set E of edges of G.

The central questions in this paper can be summarised in the following way.

Question 1.2. Let (G, d) be a metric graph.

(1) Does there exist a closed hyperbolic surface on which (G, d) can be essentially embedded?
(2) Characterise the surfaces on which such an embedding of (G, d) is possible and find the lowest genus of such a surface.

While studying embeddings of metric graphs, we are of course interested in isometric embeddings, that is, an injective map $\Phi : (G, d) \to S$ which preserves the lengths of the edges. An embedding $\Phi : (G, d) \to S$ is called *essential* if $\Phi(G)$ is essential on S.

Scaling of the metric. Given a metric graph (G, d) and a positive real number t,

$$d_t : E \to \mathbb{R}_+ \quad \text{is defined by } d_t(e) = td(e) \text{ for all } e \in E.$$

Then d_t is the metric obtained from d scaling by t.

From the *collar lemma* [7, Lemma 13.6], if α and β are two intersecting closed geodesics on a hyperbolic surface, then $l(\beta) \geq 2\sinh^{-1}(1/\sinh(l(\alpha)/2))$, which implies two intersecting closed geodesics cannot both be arbitrarily short. Thus it is natural to consider Question 1.2 up to scaling. Therefore, the general question is as follows: Given a metric graph (G, d), does there exists a $t > 0$ such that (G, d_t) can be essentially and isometrically embedded on a closed hyperbolic surface?

Remark 1.3. Henceforth, *a metric embedding of a metric graph* is an isometric embedding up to scaling its metric.

The first result we obtain answers Question 1.2(1).

Theorem 1.4. *Given a metric graph (G, d) with degree of each vertex at least three, there exists a closed surface S_g of genus $g = |E| + \beta(G)$ on which (G, d) is essentially and metrically embedded, where $\beta(G)$ and $|E|$ are the Betti number and the number of edges of G, respectively.*

Notation. We denote by $\mathcal{S}(G, d)$ the set of closed surfaces on which (G, d) admits an essential metric embedding.

Now, we focus on the genera of the surfaces in $\mathcal{S}(G, d)$.

Definition 1.5. The essential genus of (G, d) is $g_e(G) = \min\{g(S) \mid S \in \mathcal{S}(G, d)\}$.

If T is a spanning (or maximal) tree of a graph G, then $\xi(G, T)$ denotes the number of components in $G \setminus E(T)$ with an odd number of edges, where $E(T)$ is the set of edges of T.

DEFINITION 1.6. The Betti deficiency of a graph G is defined by

$$\zeta(G) = \min\{\xi(G, T) \mid T \text{ is a spanning tree of } G\}. \tag{1.1}$$

In contrast to Xuong's formula [15, Theorem 3], we prove the following theorem, which computes the essential genus of a metric graph and thus answers Question 1.2(2).

THEOREM 1.7. *The essential genus of a metric graph (G, d) is given by*

$$g_e(G) = \frac{1}{2}(\beta(G) - \zeta(G)) + 2q + r,$$

where $\beta(G), \zeta(G)$ are the Betti number and Betti deficiency of G, respectively, and q, r are the unique integers satisfying $\zeta(G) + 1 = 3q + r$, $0 \le r < 3$. Furthermore, for any given $g \ge g_e(G)$, there exists a surface F of genus g on which (G, d) admits an essential metric embedding.

An embedding of (G, d) on a surface S is the simplest if the Euler characteristic of each complementary component is -1. We therefore define *minimal embedding* as follows.

DEFINITION 1.8. An embedding $\Phi : (G, d) \to S$ is called minimal if $\chi(\Sigma) = -1$ for each component Σ in $S \setminus \Phi(G)$.

Given a metric graph, there exists a minimal embedding where the essential genus is realised. Note that, the essential genus can also be realised by a nonminimal embedding. For instance, the complement might contain a torus with two boundary components. The set of closed surfaces on which (G, d) can be minimally and metrically embedded is denoted by $S_m(G, d)$. The genera of the surfaces in $S_m(G, d)$ are bounded from below by $g_e(G)$ and this bound is sharp. We define the maximum essential genus of (G, d) by

$$g_e^{\max}(G) = \max\{g(S) \mid S \in S_m(G, d)\}.$$

It follows from Euler's equation that $g_e^{\max}(G) \le 1/2(\beta(G) + 1 + 2|E|/T(G))$ (see, for example, [4]). Here, $T(G)$ is the *girth* of the graph G.

Next, we focus on an explicit construction of minimal (or maximal) embeddings. To embed a graph on a surface minimally (or maximally), the crucial part is to find a suitable *fat graph structure* which gives the minimum (or maximum) number of boundary components among all possible fat graph structures on the graph. For a definition of fat graphs, we refer to Definition 2.2. For a fat graph structure σ_0 on G, the number of boundary components in (G, σ_0) is denoted by $\#\partial(G, \sigma_0)$.

We prove the following proposition which leads to an algorithm for minimal and maximal embeddings. Given any integer g satisfying $g_e(G) \le g \le g_e^{\max}(G)$, there exists a closed hyperbolic surface of genus g on which (G, d) can be minimally and metrically embedded, answering Question 1.2.

Proposition 1.9. *Let $G = (E, \sim, \sigma_1)$ be any graph with degree of each vertex at least three. Suppose $\sigma_0 = \prod_{v \in V} \sigma_v$ is a fat graph structure on G such that there is a vertex v which is common in b (≥ 3) boundary components. Then there exists a fat graph structure σ'_0 on G, such that*

$$\#\partial(G, \sigma'_0) = \#\partial(G, \sigma_0) - 2.$$

2. Preliminaries

In this section, we recall some graph theory and geometric notions. Also, we develop a lemma which is essential in the subsequent sections.

2.1. Fat graph. Before going to the formal definition of fat graph (ribbon graph), we recall a definition of graph and a few graph parameters. The definition of graph we use here is not the standard one which is used in ordinary graph theory. But, it is straightforward to see that this definition is equivalent to the standard one. We use this definition because it is a convenient starting point to describe fat graphs.

Definition 2.1. A finite graph is a triple $G = (E_1, \sim, \sigma_1)$, where E_1 is a finite, nonempty set with an even number of elements, σ_1 is a fixed-point free involution on E_1 and $\sim$ is an equivalence relation on E_1.

In ordinary language, E_1 is the set of directed edges, $E := E_1/\sigma_1$ is the set of undirected edges and $V := E_1/\sim$ is the set of vertices. The involution σ_1 maps a directed edge to its reverse directed edge. If $\vec{e} \in E_1$, we say that $\vec{e}$ is emanating from the vertex $v = [\vec{e}]$, the equivalence class of $\vec{e}$. The degree of a vertex $v \in V$ is defined by $\deg(v) = |v|$.

The *girth* $T(G)$ of a graph G is the length of a shortest nontrivial simple cycle, where the length of a cycle is the number of edges it contains. The girth of a tree (graph without a simple cycle) is defined to be infinity. The *Betti number* of a graph G is defined by $\beta(G) = -|V| + |E| + 1$.

Now, we define fat graphs. Informally, a *fat graph* is a graph equipped with a cyclic order on the set of directed edges emanating from each vertex. If the degree of a vertex is less than three, then the cyclic order is trivial. Therefore, we consider the graphs with degree of each vertex at least three.

Definition 2.2. A fat graph is a quadruple $G = (E_1, \sim, \sigma_1, \sigma_0)$, where

(1) $(E_1, \sim, \sigma_1)$ is a graph; and
(2) σ_0 is a permutation on E_1 so that each cycle corresponds to a cyclic order on the set of oriented edges emanating from a vertex.

For a vertex v of degree d, $\sigma_v = (e_{v,1}, e_{v,2}, \ldots, e_{v,d})$ represents a cyclic order on v, where $e_{v,i}, i = 1, 2, \ldots, d$, are the directed edges emanating from the vertex v and $\sigma_0 = \prod_{v \in V} \sigma_v$. Given a fat graph G, we construct an oriented topological surface $\Sigma(G)$ with boundary by thickening its edges. The number of boundary components in $\Sigma(G)$ is the number of disjoint cycles in $\sigma_1 * \sigma_0^{-1}$ (see [10, Section 2.1]). For more details on fat graphs, we refer to [10, 11] and [8].

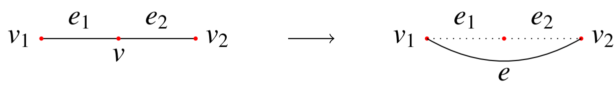

FIGURE 1. Replacement of two edges by a single edge and removal of a vertex.

2.2. Pair of pants. A hyperbolic three-holed sphere is called a pair of pants. It is a fact in hyperbolic geometry that given any three positive real numbers l_1, l_2 and l_3, there exists a unique pair of pants with boundary geodesics of lengths l_1, l_2 and l_3 (see [5, Section 3.1]). Let P_x be the pair of pants with boundary geodesics of lengths 1, 1 and $2x$, where $x \in \mathbb{R}_+$. We define a function f, where for $x \in \mathbb{R}_+$, $f(x)$ is the distance between the two boundary components of P_x of length 1.

LEMMA 2.3. *The function f is continuous and strictly monotonically increasing with*

$$f(\mathbb{R}_+) = (f_{\min}, \infty), \quad \textit{where } f_{\min} = \cosh^{-1}\left(\frac{\cosh^2 \frac{1}{2} + 1}{\sinh^2 \frac{1}{2}}\right).$$

PROOF. The distance $f(x)$ between the boundary components of length 1 is realised by the common perpendicular geodesic segment to these boundary geodesics.

The common perpendicular geodesic segments between a pair of distinct boundary components of P_x decompose it into two isometric right-angled hexagons with alternate sides of lengths $\frac{1}{2}, \frac{1}{2}$ and x. Now, using formula (i) in [5, Theorem 2.4.1],

$$f(x) = \cosh^{-1}\left(\frac{\cosh^2 \frac{1}{2} + \cosh x}{\sinh^2 \frac{1}{2}}\right),$$

which implies the lemma. □

3. Essential embedding of a metric graph

In this section, we prove Theorem 1.4. Note that, if a graph G is a cycle, then we find an embedded geodesic loop with length equal to the total length of the cycle, and populate it with vertices according to the lengths of each edge. Therefore, in the remaining part in this section, we exclude the case where the graph is a cycle.

DEFINITION 3.1. A metric graph is called geometric if it can be essentially and metrically embedded on a closed surface.

Let (G, d) be a metric graph with degree of each vertex at least two, where $G = (E_1, \sim, \sigma_1)$. If $v = \{\vec{e}_i : i = 1, 2\}$ is a vertex with degree 2, then we define a new graph $G' = (E'_1, \sim', \sigma'_1)$ with metric d' by removing the vertex v and replacing two edges $e_i = \{\vec{e}_i, \overleftarrow{e}_i\}, i = 1, 2$, by a single edge $e = \{\vec{e}, \overleftarrow{e}\}$ in G (see Figure 1). The metric d' is defined by, $d'(x) = d(x)$, for all $x \in E' \setminus \{e\}$ and $d'(e) = d(e_1) + d(e_2)$.

LEMMA 3.2. *A graph (G, d) is geometric if and only if (G', d') is geometric. Moreover, the essential genera of these graphs are the same, that is, $g_e(G) = g_e(G')$.*

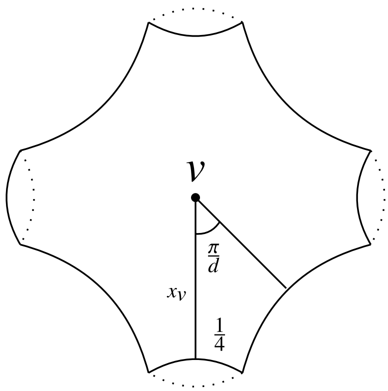

FIGURE 2. Hyperbolic d-holed sphere, $d = 4$.

PROOF. Suppose (G', d') is geometric and $\Phi : (G', d'_t) \to S$ is a metric embedding for some $t > 0$. Considering the point in the interior of e (according to the lengths of e_1 and e_2) as a vertex, we get a metric embedding of (G, d_t) on S. Similarly, a metric embedding of (G, d) gives the metric embedding of (G', d') by forgetting the vertex v. Hence, the lemma follows. □

In the light of Lemma 3.2, from now on, we assume that the degree of each vertex of the graph G is at least three, that is, $\deg(v) \geq 3$, for all $v \in V$.

PROOF OF THEOREM 1.4. Let (G, d) be a given metric graph with degree of each vertex at least three. For each vertex v, we assign a hyperbolic $\deg(v)$-holed sphere $S(v)$ with each boundary geodesic of length 1. Namely, we construct $2\deg(v)$-sided right-angled hyperbolic polygons $P(v)$ with v pairwise nonadjacent edges of length $1/2$ by attaching $2\deg(v)$ copies of $Q(\pi/\deg(v))$. Here, $Q(\theta)$ denotes a sharp corner (also known as a Lambert quadrilateral), whose only angle not equal to a right angle is θ and a side opposite to this angle is of length $\frac{1}{4}$ (see Figure 2). Then consider two copies of $P(v)$ and glue them in an obvious way by isometries to obtain $S(v)$.

Consider the central point on each copy of $P(v)$ on $S(v)$ and connect it to the boundary components by distance realising geodesic segments. These geodesic segments meet boundary components orthogonally. Now, applying formula (vi) in [5, Theorem 2.3.1], on the sharp corner $Q(\pi/\deg(v))$ as indicated in Figure 2, we find the length of the perpendicular geodesic segment is

$$x_v = \sinh^{-1}\left(\coth(1/4)\coth(\pi/\deg(v))\right).$$

For an edge e with the ends u and v, we define $l(e) = x_u + x_v$. Now, we choose a positive real number t such that $d_t(e) > l(e) + f_{\min}$ for all edges e in G, where $f_{\min}$ is given in Lemma 2.3.

To each edge e, we assign a pair of pants P_{x_e} (as in Section 2.2, Lemma 3.2), where $x_e \in \mathbb{R}$ satisfies $f(x_e) = d_t(e) - l(e)$.

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Now, glue the surfaces $S(v), v \in V$ and $P_{x_e}, e \in E$, along the boundaries of length 1 according to the graph G with twists so that all of the distance-realising orthogonal geodesic segments meet. Thus, we obtain a surface, denoted by $\Sigma_\partial(G, d)$, with boundary on which (G, d_t) is isometrically embedded. We turn our surface into a closed surface $\Sigma(G, d)$ by attaching one-holed tori to the boundary components.

Finally, we count the genus g of $\Sigma(G, d)$ by counting the number of pairs of pants in a pants decomposition which gives $2g - 2 = \sum_{v\in V}(\deg(v) - 2) + 2|E|$. This equation and the relation $\sum_{v\in V} \deg(v) = 2|E|$ conclude the proof. □

4. Fat graph structures and embeddings

In this section, for a given fat graph structure σ_0 on (G, d), we construct an essential metric embedding on a closed surface $S(G, d, \sigma_0)$, where the fat graph structure is realised and the genus is minimised.

A graph G on an oriented surface S has a natural fat graph structure σ_0 determined by the orientation of S. Conversely, if σ_0 is any given fat graph structure on G, then there exists an essential metric embedding of (G, d) on a closed hyperbolic surface S of genus $g = |E| + \beta(G)$ (see Theorem 1.4), where the fat graph structure σ_0 is realised. Construction of such an embedding follows the same procedure as in the proof of Theorem 1.4. The only difference is that one needs to glue the $\deg(v)$-holed spheres $S(v)$, $v \in V$, and $P_{x_e}, e \in E$, according to the fat graph structure σ_0.

4.1. Embedding on a surface with totally geodesic boundary. Let $N_\epsilon(G, d, \sigma_0)$ be the regular (tubular) ϵ-neighbourhood of G on S, where $\epsilon > 0$ is sufficiently small. Let β' be a boundary component of $N_\epsilon(G, d, \sigma_0)$. Then β' is an essential simple closed curve on S as the graph is essentially embedded (in particular, no complementary region is a disc). Therefore, there is a unique geodesic representative β (simple and closed) in its free homotopy class. Note that, the geodesic representatives β of the boundary components of $N_\epsilon(G, d, \sigma_0)$ are disjoint from the embedding of the graph on the surface S (see, for example, [11, Section 7]). We obtain the surface $\Sigma_0(G, d, \sigma_0)$ with totally geodesic boundary by cutting S along the geodesics in the free homotopy classes of the boundary components of $N_\epsilon(G, d, \sigma_0)$.

LEMMA 4.1. *The metric graph (G, d) is metrically embedded on the surface $\Sigma_0(G, d, \sigma_0)$ with totally geodesic boundary. Furthermore, the number of boundary components of $\Sigma_0(G, d, \sigma_0)$ is the number of orbits of $\sigma_1 * \sigma_0^{-1}$.*

PROOF. By the construction described above, (G, d) is metrically embedded on the surface $\Sigma_0(G, d, \sigma_0)$ which is a subsurface of S. Finally, $\Sigma_0(G, d, \sigma_0)$ is homeomorphic to $\Sigma(G, \sigma_0)$. Therefore, the number of boundary components of $\Sigma_0(G, d, \sigma_0)$ is the number of orbits of $\sigma_1 * \sigma_0^{-1}$. □

4.2. Embedding on a closed surface. In this subsection, we cap the surface $\Sigma_0(G, d, \sigma_0)$ by surfaces with boundary to obtain a closed surface. Equivalently, we embed $\Sigma_0(G, d, \sigma_0)$ metrically and essentially on a closed surface. We describe two gluing procedures below.

4.2.1. Glue I. In this gluing procedure, we assume that $\Sigma_0(G, d, \sigma_0)$ has at least three boundary components and choose any three of them, say β_1, β_2 and β_3. We consider a pair of pants Y, with boundary geodesics b_1, b_2 and b_3 of lengths $l(\beta_1), l(\beta_2)$ and $l(\beta_3)$, respectively. We glue β_i with b_i, $i = 1, 2, 3$, by hyperbolic isometries. In this gluing, the resulting surface has genus two more than the genus of $\Sigma_0(G, d, \sigma_0)$ and number of boundary components is three less than that of $\Sigma_0(G, d, \sigma_0)$.

4.2.2. Glue II. Let β be a boundary geodesic of $\Sigma_0(G, d, \sigma_0)$. We glue a hyperbolic one-holed torus with boundary length $l(\beta)$ to the surface $\Sigma_0(G, d, \sigma_0)$ along β by an isometry. The resulting surface has genus one more than that of $\Sigma_0(G, d, \sigma_0)$ and the number of boundary components is one less than that of $\Sigma_0(G, d, \sigma_0)$.

Now, assume that $\Sigma_0(G, d, \sigma_0)$ has b boundary components. Using the division algorithm, there are unique integers q and r such that $b = 3q + r$, where $0 \leq r \leq 2$. Then following the gluing procedure Glue I (see Section 4.2.1) for q times and Glue II (see Section 4.2.2) for r times, we obtain the desired closed hyperbolic surface denoted by $S(G, d, \sigma_0)$.

REMARK 4.2. The genus of $S(G, d, \sigma_0)$ depends upon the fat graph structure σ_0.

5. Minimum genus problem

In this section, our goal is to prove Theorem 1.7.

Let (G, d) be a metric graph with degree of each vertex at least three and let $\chi(G)$ denote the Euler characteristic of G. We consider a fat graph structure σ_0 on G and denote by $\Sigma_0(G, d, \sigma_0)$ the surface with geodesic boundary obtained in Section 4. As G is a spine of $\Sigma_0(G, d, \sigma_0)$,

$$\chi(\Sigma_0(G, d, \sigma_0)) = \chi(G), \tag{5.1}$$

where $\chi(\Sigma_0(G, d, \sigma_0))$ denotes the Euler characteristic of $\Sigma_0(G, d, \sigma_0)$. The assumption $\deg(v) \geq 3$ and the relation $2|E| = \sum_{v \in V} \deg(v) \geq 3|V|$ implies that $\chi(G) < 0$.

LEMMA 5.1. *Let σ_0 and σ'_0 be two fat graph structures on (G, d). Then the difference between the number of boundary components of $\Sigma_0(G, d, \sigma_0)$ and $\Sigma_0(G, d, \sigma'_0)$ is an even integer, that is, $\#\partial\Sigma_0(G, d, \sigma_0) - \#\partial\Sigma_0(G, d, \sigma'_0)$ is divisible by 2.*

PROOF. Let the genera of $\Sigma(G, d, \sigma_0)$ and $\Sigma(G, d, \sigma'_0))$ be g and g'. By Euler's equation, $2 - 2g - \#\partial\Sigma_0(G, d, \sigma_0) = 2 - 2g' - \#\partial\Sigma_0(G, d, \sigma'_0)$ which implies the lemma. □

The number of boundary components of a surface F is denoted by $\#\partial F$. The genus of a fat graph (G, σ_0) is the genus of the associated surface and denoted by $g(G, \sigma_0)$. Similarly, we denote the number of boundary components of a fat graph by $\#\partial(G, \sigma_0)$.

LEMMA 5.2. *Let σ_0 and σ'_0 be two fat graph structures on a metric graph (G, d) such that $\#\partial(\Sigma_0(G, d, \sigma_0)) - \#\partial(\Sigma_0(G, d, \sigma'_0)) = 2$. Then*

$$g(S(G, d, \sigma'_0)) \leq g(S(G, d, \sigma_0)).$$

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Proof. Suppose that the genus and the number of boundary components of $\Sigma_0(G, d, \sigma_0)$ are g and b, respectively. Then by Euler's formula and (5.1), we have $2 - 2g - b = \chi(G)$ which implies that $b = 2 - 2g - \chi(G)$. For the integer b, by the division algorithm, there exist unique integers q and r such that $b = 3q + r$ where $0 \le r < 3$. Therefore, by construction (see Section 4), the genus of $S(G, d, \sigma_0)$ is

$$g(S(G, d, \sigma_0)) = g + 2q + r.$$

Now, assume that the genus and number of boundary components of $\Sigma_0(G, d, \sigma'_0)$ are g' and b', respectively. Then Euler's formula and (5.1) give $b' = 2 - 2g' - \chi(G)$. The hypothesis $b' = b - 2$ of the lemma implies $g' = g + 1$.

Now, we compute the genus of the closed surface $S(G, d, \sigma'_0)$. There are three cases to consider as $b' = 3q + r - 2$ with $r \in \{0, 1, 2\}$.

Case 1: $r = 0$. In this case $b' = 3(q - 1) + 1$. Thus the genus of $S(G, d, \sigma'_0)$ is $g' + 2(q - 1) + 1 = g + 2q$ which is equal to the genus of $S(\Sigma, d, \sigma_0)$. Therefore, the lemma holds with equality.

Case 2: $r = 1$. In this case $b' = 3(q - 1) + 2$. Therefore, the genus of $S(G, d, \sigma'_0)$ is $g + 1 + 2(q - 1) + 2 = g + 2q + 1$ which is equal to the genus of $S(\Sigma, d, \sigma_0)$. Therefore, the lemma holds with equality.

Case 3: $r = 2$. In this case, the genus of $S(G, d, \sigma_0)$ is $g + 2q + 2$. Now, $b' = b - 2 = 3q$ implies that the genus of $S(G, d, \sigma'_0)$ is $g' + 2q = g + 1 + 2q$. Therefore,

$$g(S(G, d, \sigma'_0)) = g(S(G, d, \sigma_0)) - 1 < g(S(G, d, \sigma_0)). \qquad \square$$

Proof of Theorem 1.7. To find the essential genus of (G, d), we consider a fat graph structure σ_0 on G which gives the maximum genus of $\Sigma_0(G, d, \sigma_0)$, or equivalently, the minimum number of boundary components (see Lemma 5.2). For such a fat graph structure σ_0, from [15, Theorem 3], the genus of $\Sigma_0(G, d, \sigma_0)$ is $\frac{1}{2}(\beta(G) - \zeta(G))$. Moreover, the number of boundary components of the fat graph (G, σ_0) is $1 + \zeta(G)$, which is equal to the number of boundary components of $\Sigma_0(G, d, \sigma_0)$. By the division algorithm, for the integer $1 + \zeta(G)$, there are unique integers q and r such that

$$1 + \zeta(G) = 3q + r, \quad \text{where } 0 \le r < 3.$$

Therefore, the genus of $S(G, d, \sigma_0)$ is $g_e(G) = \frac{1}{2}(\beta(G) - \zeta(G)) + 2q + r$, as follows from the construction in Section 4. This proves the first part of the theorem.

Now, we focus on the proof of the remaining part of the theorem, that is we show that for any $g \ge g_e(G)$ the graph (G, d) can be metrically embedded on a closed surface of genus g. We define $g' = g - g_e(G)$. Let us consider the surface $S(G, d, \sigma_0)$ of genus $g_e(G)$ constructed above. Now, there are two possibilities.

Case 1. If the number of boundary components of $\Sigma_0(G, d, \sigma_0)$ is divisible by 3, then we have a Y-piece, denoted by $Y(\beta'_1, \beta'_2, \beta'_3)$, attached to $\Sigma_0(G, d, \sigma_0)$ along the boundary components $\beta_1, \beta_2, \beta_3$ by hyperbolic isometries in the construction of $S(G, d, \sigma_0)$ (see the construction in Section 4). We replace this Y-piece from

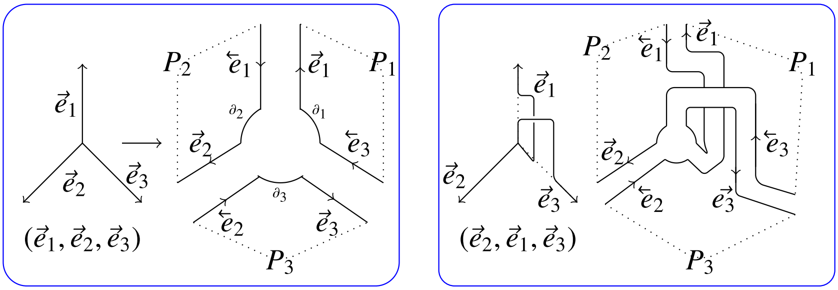

Figure 3. Change of cyclic order at a vertex.

$S(G, d, \sigma_0)$ by a hyperbolic surface $F_{g',3}$ of genus g' and three boundary components, again denoted by β_1', β_2' and β_3', of lengths $l(\beta_1)$, $l(\beta_2)$ and $l(\beta_3)$, respectively. We denote the new surface by $S_{g'}(G, d, \sigma_0)$.

Case 2. In this case, we consider the possibility that the number of boundary components of $\Sigma_0(G, d, \sigma_0)$ is not divisible by 3. Then there is a subsurface $F_{1,1}$ of genus 1 and a single boundary component β', which we have attached to $\Sigma_0(G, d, \sigma_0)$ along the boundary component β to obtain $S(G, d, \sigma_0)$. Now, we replace $F_{1,1}$ by $F_{g'+1,1}$, a hyperbolic surface of genus $g' + 1$ and a single boundary component β' of length $l(\beta)$, in $S(G, d, \sigma_0)$. We denote the resulting surface by $S_{g'}(G, d, \sigma_0)$.

The surface $S_{g'}(G, d, \sigma_0)$ has genus g and (G, d) is metrically embedded on it. □

6. Algorithm: minimal embedding with minimum/maximum genus

In this section, we study minimal essential embeddings and prove Proposition 1.9. We conclude this section with Remark 6.2 which provides an algorithm for minimal embedding with minimum and maximum genus.

Let us consider a trivalent fat graph (Γ, σ_0) with a vertex v which is shared by three distinct boundary components. We construct a new fat graph structure to reduce the number of boundary components.

Lemma 6.1. *Let (Γ, σ_0) be a three-regular fat graph. If Γ has a vertex which is common in three boundary components, then there is a fat graph structure σ_0' such that*

$$\#\partial(\Gamma, \sigma_0') = \#\partial(\Gamma, \sigma_0) - 2.$$

Proof. Suppose the vertex v is in three distinct boundary components of Γ. Assume that $v = \{\vec{e}_i,\ i = 1, 2, 3\}$ with $\sigma_v = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ (see Figure 3, left). Suppose ∂_i, $i = 1, 2, 3$, are the boundary components given by

$$\partial_1 = \vec{e}_1 P_1 \overleftarrow{e}_3, \quad \partial_2 = \vec{e}_2 P_2 \overleftarrow{e}_1 \quad \text{and} \quad \partial_3 = \vec{e}_3 P_3 \overleftarrow{e}_2,$$

where the P_i's are finite (possibly empty) paths in the graph and $\overleftarrow{e}_i = \sigma_1(\vec{e}_i)$ (see Figure 3, right). We replace the order $\sigma_v = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ by $\sigma_v' = (\vec{e}_2, \vec{e}_1, \vec{e}_3)$ to obtain a new fat graph structure σ_0'. Then the boundary components of (Γ, σ_0') are given by

$$\partial(\Gamma, \sigma_0') = (\partial(\Gamma, \sigma_0) \setminus \{\partial_i | i = 1, 2, 3\}) \cup \{\partial\},$$

https://doi.org/10.1017/S0004972719000145 Published online by Cambridge University Press

where $\partial = \vec{e}_2P_2\overleftarrow{e}_1\vec{e}_3P_3\overleftarrow{e}_2\vec{e}_1P_1\overleftarrow{e}_3 = \partial_2 * \partial_3 * \partial_1$. Here, $*$ is the usual concatenation operation. Therefore, the number of boundary components in (Γ, σ_0') is the same as the number of boundary components in (Γ, σ_0) minus two. □

Proof of Proposition 1.9. Let $v_0 = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_k\}$ be a vertex which is in at least three boundary components. We assume that the cyclic order at v_0 is given by

$$\sigma_{v_0} = (\vec{e}_1, \vec{e}_2, \vec{e}_3, \ldots, \vec{e}_i, \vec{e}_{i+1}, \ldots, \vec{e}_k),$$

where $k \geq 3$, $3 \leq i \leq k$ and $\vec{e}_{k+1} = \vec{e}_1$. We can choose three boundary components $b_i, i = 1, 2, 3$, such that there is an edge $e_2 = \{\vec{e}_2, \overleftarrow{e}_2\}$ with $\vec{e}_2$ in b_1 and $\overleftarrow{e}_2$ in b_2. We can write $b_1 = \vec{e}_2P_1\overleftarrow{e}_1, b_2 = \vec{e}_3P_2\overleftarrow{e}_2$ and $b_3 = \vec{e}_{i+1}P_3\overleftarrow{e}_i$, where the P_j's are some paths (possibly empty) in the fat graph. Now, we consider a new cyclic order at v, given by $\sigma_v' = (\vec{e}_1, \vec{e}_3, \ldots, \vec{e}_i, \vec{e}_2, \vec{e}_{i+1}, \ldots, \vec{e}_k)$. Then, in the new fat graph structure σ_0', the boundary components of (G, σ_0') are $(\partial(G, \sigma_0) \setminus \{b_1, b_2, b_3\}) \cup \{b\}$, where $b = b_1 * b_2 * b_3$. Therefore,

$$\#\partial(G, \sigma_0') = \#\partial(G, \sigma_0) - 2. \qquad \square$$

Remark 6.2. One can obtain a minimal embedding applying Proposition 1.9. let us consider a fat graph (G, σ_0). If v is a vertex shared by $k \leq 2$ boundary components, then it follows from Lemmas 5.1 and 5.2 that there is no replacement of the cyclic order σ_v (keeping the cyclic order on other vertices unchanged) to reduce the number of boundary components. If there is a vertex v which is common in at least three boundary components, then one can replace the fat graph structure by applying Proposition 1.9 to reduce the number of boundary components by two. Therefore, by repeated application of Proposition 1.9, we can obtain a fat graph structure on the graph G that provides the essential genus $g_e(G)$.

Using Proposition 1.9, in the reverse way, we can obtain a fat graph structure which provides the maximal genus $g_e^{\max}(G)$ of a minimal embedding. Namely, if there is a vertex v with cyclic order $\sigma_v = (\vec{e}_2, \vec{e}_3 \ldots, \vec{e}_i, \vec{e}_1, \vec{e}_{i+1}, \ldots \vec{e}_k)$ and a boundary component ∂ of the form $\partial = \vec{e}_2P_1\overleftarrow{e}_1\vec{e}_3P_2\overleftarrow{e}_2e_{i+1}P_3\overleftarrow{e}_i$, where the P_j's are some paths in G, only then, we can replace σ_0 by $\sigma_v' = (\vec{e}_1, \vec{e}_2, \vec{e}_3, \ldots, \vec{e}_i, \vec{e}_{i+1}, \ldots, \vec{e}_k)$ to obtain a new fat graph structure σ_0', such that

$$\#\partial(G, \sigma_0') = \#\partial(G, \sigma_0) + 2.$$

By repeated use of Proposition 1.9, we can obtain a fat graph structure on G which provides $g_e^{\max}(G)$.

It would be interesting to study quasi-essential metric embeddings, where some of the complementary components are allowed to be simply connected. Note that, for such a component it is necessary that the corresponding boundary component satisfies the polygonal inequality: for an n-sided polygon the total length of any $(n-1)$ sides is strictly greater than the length of the remaining side. We hope to address this question in the future.

Acknowledgement

The author would like to thank the referee for several helpful comments and suggestions.

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