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Discreteness For the Set of Complex Structures On a Real Variety

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Abstract. Let X, Y be reduced and irreducible compact complex spaces and S the set of all isomorphism classes of reduced and irreducible compact complex spaces W such that $X \times Y \cong X \times W$. Here we prove that S is at most countable. We apply this result to show that for every reduced and irreducible compact complex space X the set $S(X)$ of all complex reduced compact complex spaces W with $X \times X^\sigma \cong W \times W^\sigma$ (where A^σ denotes the complex conjugate of any variety A) is at most countable.

1 Introduction

For any reduced complex compact space X let X^σ be its complex conjugate in the sense of [10], Section 2. For any integral algebraic variety X over Spec($\mathbf{C}$), let X^σ be its complex conjugate in the sense of Weil ([11], Section 1.3, or [1] or [7]). If X is projective, say $X \subseteq \mathbf{P}^N(\mathbf{C})$, one can define $X^\sigma \subseteq \mathbf{P}^N(\mathbf{C})$ taking as defining equations the complex conjugations of the homogeneous equations defining X (see [1], Section 2, or [7], Section 2). Set ${}_{\mathbf{R}}X = X \times X^\sigma$. The variety ${}_{\mathbf{R}}X$ is related to the variety obtained from X by the restriction of scalars $\mathbf{C} \setminus \mathbf{R}$ in the sense of Weil (see [11], Section 1.3, or [5], Exp. 195, Section C2).

Theorem 1 *Let X be a reduced and irreducible compact complex space. Let $S(X)$ be the set of all biholomorphism classes of reduced and irreducible compact complex spaces W such that ${}_{\mathbf{R}}X \cong {}_{\mathbf{R}}W$. Then $S(X)$ is at most countable.*

By [6], Corollary 1.4, for every integer k there is an elliptic curve X such that $\mathrm{card}(S(X)) > k$. To prove Theorem 1 we will prove the following result related to the cancellation problem for compact complex spaces or for projective varieties studied in [3], [4], [6] and [8].

Theorem 2 *Let X, Y be reduced and irreducible compact complex spaces. Let S be the set of all biholomorphism classes of reduced and irreducible compact complex spaces W such that $X \times Y \cong X \times W$. Then S is at most countable.*

There are many examples of compact complex manifolds which do not have the cancellation property (see [2], [3], [8], [9] and references therein) and the same is true in the category of complex projective manifolds, but we do not know examples

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related to Theorem 2 with card(S) infinite. By [6], Corollary 1.3, for large classes of projective varieties we have card$(S) = 1$ (*e.g.* projective manifolds of general type or with no non-constant map to a complex torus).

2 The Proofs

Proof of Theorem 2 Set $Z = X \times Y$ and $n = \dim(Y)$. Fix $W \in S(X)$. For each $P \in W$ the closed subspace $X \times \{P\}$ of Z has trivial normal bundle $N_{X\times\{P\},Z}$ and hence $h^0(X \times \{P\}, N_{X\times\{P\},Z}) = n$. The Douady space $D(Z)$ of all compact complex subspaces of Z has $H^0(X \times \{P\}, N_{X\times\{P\},Z})$ as Zariski tangent space at Z. Thus we see that the subset $D(Z, W) = \{X \times \{P\}\}_{P\in W}$ of $D(Z)$ covers a connected component of $D(Z)$. We have $D(X, W) \cap D(X, W') = \varnothing$ if W and W' are not biholomorphic. Since $D(Z)$ has only countably many irreducible components, each of them with countable topology ([4]), we conclude.

Proof of Theorem 1 Set $Z = {}_{\mathbf{R}}X$ and fix $W \in S(X)$. Copy the proof of Theorem 2, just writing $W \times \{P\}$, $P \in W$, instead of $X \times \{P\}$.

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