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A NOTE ON THE JENSEN-GOULD CONVOLUTIONS

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ABSTRACT. With the aid of a recent result obtained by the first author, an expression is derived which unifies the well-known Jensen and Gould formulas.

Jensen [4] gave the well-known convolution

$$\sum_{k=0}^{n}\binom{\alpha+\beta k}{k}\binom{\gamma-\beta k}{n-k}=\sum_{k=0}^{n}\binom{\alpha+\gamma-k}{n-k}\beta^{k}. \tag{1}$$

Gould [3] proved the Abel-type analog of (1)

$$\sum_{k=0}^{n}\frac{(\alpha-\beta k)^{k}}{k!}\frac{(\gamma-\beta k)^{n-k}}{(n-k)!}=\sum_{k=0}^{n}\frac{(\alpha+\gamma)^{k}}{k!}\beta^{n-k}. \tag{2}$$

Furthermore, Carlitz [1] established that under certain specified conditions if

$$\sum_{k=0}^{n}Q_{k}(\alpha+\beta k)Q_{n-k}(\gamma-\beta k)=\sum_{k=0}^{n}\beta^{k}Q_{n-k}(\alpha+\gamma-k) \tag{3}$$

then

$$Q_{n}(\alpha)=\binom{\alpha}{n}\qquad(n=0,1,2,\ldots)$$

and if

$$\sum_{k=0}^{n}Q_{k}(\alpha+\beta k)Q_{n-k}(\gamma-\beta k)=\sum_{k=0}^{n}\beta^{k}Q_{n-k}(\alpha+\gamma) \tag{4}$$

then

$$Q_{n}(\alpha)=\frac{\alpha^{n}}{n!}\qquad(n=0,1,2,\ldots).$$

The purpose of the present note is to present a result which gives as special cases the expressions (1) and (2).

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THEOREM. *For a, b, β, μ, s, σ complex numbers and n, m nonnegative integers, then*

$$\sum_{k=0}^{n}\sum_{p=0}^{m} A_k(a+sk+\sigma p)A_{n-k}(b-a-sk-\sigma p)$$

(5)
$$\times B_p(-\beta+sk+\sigma p)B_{m-p}(\mu+\beta-sk-\sigma p)$$
$$=\sum_{k=0}^{n}\sum_{p=0}^{m}\binom{k+p}{p}s^k\sigma^p A_{n-k}(b-k)B_{m-p}(\mu)$$

where

$$A_n(\alpha)=\binom{\alpha}{n},\ B_n(\alpha)=\frac{\alpha^n}{n!}.$$

Proof. Equating equations (2.7) and (2.9) in Cohen [2] and multiplying both sides of the resulting equation by $(1-z)^{-\lambda}\exp(\mu y)$, and replacing α by $-a-1$, s by $-s$, s' by $-\sigma$, one obtains

$$\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-z)^k(-y)^p\exp(y\beta+\mu y-ysk-y\sigma p)(\beta-sk-\sigma p)^p(-a-sk-\sigma p)_k}{k!p!(1-z)^{-a-sk-\sigma p+k+\lambda}} \tag{6}$$

$$=\frac{(1-z)^{-\lambda}\exp(\mu y)}{(1-sz-\sigma y)} \tag{7}$$

$$=\sum_{n,m,k,p=0}^{\infty}\frac{(\lambda)_n z^n}{n!}\frac{\mu^m y^m}{m!}\frac{(k+p)!s^k\sigma^p z^k y^p}{k!p!} \tag{8}$$

$$=\sum_{n,m=0}^{\infty} z^n y^m \sum_{k=0}^{n}\sum_{p=0}^{m}\frac{(\lambda)_{n-k}\mu^{m-p}(k+p)!s^k\sigma^p}{(n-k)!k!(m-p)!p!} \tag{9}$$

Now, consider equation (6), which may be expanded to give

$$\sum_{n,m,k,p=0}^{\infty}\frac{z^n y^m z^k y^p(-\beta+sk+\sigma p)^p(\mu+\beta-sk-\sigma p)^m(-a+\lambda-sk-\sigma p+k)_n}{k!\,p!\,n!\,m!\,(a+1+sk+\sigma p)_{-k}} \tag{10}$$

where $(\alpha)_k=\Gamma(\alpha+k)/\Gamma(\alpha)$, quotient of two gamma functions. Equation (10) may be expressed as

$$\sum_{n,m=0}^{\infty} z^n y^m\sum_{k=0}^{n}\sum_{p=0}^{m}\frac{(-\beta+sk+\sigma p)^p(\mu+\beta-sk-\sigma p)^{m-p}(-a+\lambda-sk+k-\sigma p)_{n-k}}{k!\,p!\,(n-k)!\,(m-p)!\,(a+1+sk+\sigma p)_{-k}}$$
(11)

Now equating coefficients of (9) and (11), putting $\lambda=b-n+1$, and some simplification gives the required result (5).

It may be noted that by putting $m=0$ in (5), one obtains essentially the Jensen formula (1), and in symbolic form, the equation (3). Similarly, $n=0$ in (5) gives the Gould formula (2) and (4).

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