Appendix B Goldstone's theorem and the pi mesons

It is easy to prove Goldstone's theorem for theories with fundamental scalar fields. But the theorem is more general than that, and some of its most interesting applications are in theories without fundamental scalars. We can illustrate this with QCD. In the limit where there are two massless quarks (i.e. in the limit where we neglect the masses of the u and d quarks), we can write the QCD Lagrangian in terms of spinors

$$\Psi = \begin{pmatrix} u \\ d \end{pmatrix} \tag{B1}$$

as

$$\mathcal{L} = \bar{\Psi} i \gamma^{\mu} D_{\mu} \Psi - \frac{1}{4} F_{\mu\nu}^2. \tag{B2}$$

This Lagrangian has symmetries

$$\Psi \to e^{i\omega^a \tau^a/2} \Psi, \quad \Psi \to e^{i\omega^a \tau^a - \gamma^5/2} \Psi \tag{B3}$$

(the τ_s^a are the Pauli matrices). In the limit where the two quarks are massless, QCD is thus said to have the symmetry $SU(2)_{\mathrm{L}} \times SU(2)_{\mathrm{R}}$.

So, writing a general four-component fermion as

$$\Psi = \begin{pmatrix} q \\ \bar{q}^* \end{pmatrix}, \tag{B4}$$

the Lagrangian has the form:

$$\mathcal{L} = i\Psi \sigma^{\mu} D_{\mu} \Psi^* + i\bar{\Psi} \sigma^{\mu} D_{\mu} \bar{\Psi}^*. \tag{B5}$$

In this form, we have two separate symmetries:

$$\Psi \to \exp\left(i\omega_{\mathrm{L}}^a \frac{\tau^a}{2}\right)\Psi, \quad \bar{\Psi} \to \exp\left(i\omega_{\mathrm{R}}^a \frac{\tau^a}{2}\right)\bar{\Psi}. \tag{B6}$$

Written in this way, it is clear why the symmetry is called $SU(2)_{\mathrm{L}} \times SU(2)_{\mathrm{R}}$.

Now, it is believed that in QCD the operator $\overline{\Psi}\Psi$ has a non-zero vacuum expectation value, i.e.

$$\langle \overline{\Psi}\Psi \rangle \approx (0.3\,\mathrm{GeV})^3 \delta_{ff'}. \tag{B7}$$

This is in four-component language; in two-component language this becomes:

$$\langle \bar{\Psi}_f \Psi_{f'} + \bar{\Psi}_f^* \Psi_{f'}^* \rangle \neq 0. \tag{B8}$$

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This leaves ordinary isospin, the transformation without the γ_5 in four-component language, or with $\omega^a_{\mathrm{L}} = -\omega^a_{\mathrm{R}}$, unbroken, in two-component language.

But there are three broken symmetries. Correspondingly, we expect that there are three Goldstone bosons. To prove this, write

$$\mathcal{O} = \bar{\Psi}\Psi, \quad \mathcal{O}^a = \bar{\Psi}\gamma^5 \frac{\tau^a}{2}\Psi. \tag{B9}$$

Under an infinitesimal transformation,

$$\delta\mathcal{O} = 2i\omega^a\mathcal{O}^a, \quad \delta\mathcal{O}^a = i\omega^a\mathcal{O}. \tag{B10}$$

In the quantum theory these give the commutation relations

$$[Q^a, \mathcal{O}] = 2i\mathcal{O}^a, \quad [Q^a, \mathcal{O}^b] = i\delta^{ab}\mathcal{O}. \tag{B11}$$

Here Q^a is the integral of the time component of a current. To see that there must be a massless particle, we study

$$0 = \int d^4x\, \partial_\mu \left[\langle\Omega|T(j^{\mu a}(x)\mathcal{O}^b(0))|\Omega\rangle e^{-ip\cdot x}\right] \tag{B12}$$

(this follows because the integral of a total derivative is zero). We can evaluate the right-hand side, carefully writing out the time-ordered product in terms of ϑ-functions and noting that the action of ∂_0 on the ϑ-functions gives δ-functions:

$$0 = \int d^4x\, \langle\Omega|\,[j^{oa}(x), \mathcal{O}^b(0)]\delta(x^0)|\Omega\rangle e^{-ip\cdot x} - ip_\mu \int d^4x\, \langle\Omega|T(j^{\mu a}(x)\mathcal{O}^b(0))|\Omega\rangle. \tag{B13}$$

Now consider the limit $p^\mu = 0$. The first term on the right-hand side becomes the matrix element of $[Q^a, \mathcal{O}^b(0)] = \mathcal{O}(0)$. This is non-zero. The second term must be singular, then, if the equation is to hold. This singularity, as we will now show, requires the presence of a massless particle. For this we use the spectral representation of the Green's function. In general a pole can arise at zero momentum only from a massless particle. To understand this singularity we introduce a complete set of statesand, say for $x^0 > 0$, write it as

$$\sum_\lambda \int \frac{d^3p}{2E_p(\lambda)} \langle\Omega|j^{\mu a}(x)|\lambda_p\rangle\langle\lambda_p|\mathcal{O}^b(0)|\Omega\rangle. \tag{B14}$$

In the sum we can separate the term coming from the massless particle. Call this particle π^b. On Lorentz-invariance grounds,

$$\langle\Omega|j^{\mu a}|\pi^b(p)\rangle = f_\pi p^\mu \delta^{ab}. \tag{B15}$$

Set

$$\langle\lambda_q|\mathcal{O}^a(x)|\pi^b(p)\rangle = Z\delta^{ab}e^{-ip\cdot x} \tag{B16}$$

Adding the contribution from the time ordering $x_0 < 0$, we obtain for the left-hand side a massless scalar propagator i/p^2 multiplied by $Zf_\pi p^\mu$, so the equation is now consistent:

$$\langle\bar{\Psi}\Psi\rangle = \frac{p^2}{p^2} f_\pi Z. \tag{B17}$$

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It is easy to see that, in this form, Goldstone's theorem generalizes to any theory without fundamental scalars in which a global symmetry is spontaneously broken.

Returning to QCD, what about the fact that the quarks are massive? The quark mass terms break the symmetries explicitly. But if these masses are small, we should be able to think of the potential as "tilted", i.e. almost, but not quite, symmetric as in Section 5.3.1. This gives rise to small masses for the pions. We could compute these by studying, again, correlation functions of derivatives of currents. A simpler procedure is to consider the symmetry-breaking terms in the Lagrangian:

$$\mathcal{L}_{\text{sb}} = \bar{\Psi} M \Psi, \tag{B18}$$

where M is the quark mass matrix,

$$M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}. \tag{B19}$$

Since the π mesons are, by assumption, light, we can focus on these. If we have a non-zero pion field, we can think of the fermions as being given by:

$$\Psi = \exp\left(i \frac{\pi^a}{f_\pi} \gamma^5 \frac{\tau^a}{2} \right) \Psi. \tag{B20}$$

In other words, the pion fields behave like symmetry transformations of the vacuum (and everything else).

Now assume that there is an "effective interaction" for the pions, containing kinetic terms $(1/2)(\partial_\mu \pi^a)^2$. Taking the form above for Ψ, the pions obtain a potential from the fermion mass terms. To work out this potential one substitutes this form for the fermions into the Lagrangian and replaces the fermion bilinear form by its vacuum expectation value. This gives

$$V(\pi) = \langle \bar{q} q \rangle \, \text{Tr}\big[\exp\big(i \omega^a \gamma_5 \tau^a \big) M \big]; \tag{B21}$$

one can now expand to second order in the pion fields, obtaining:

$$m_\pi^2 f_\pi^2 = (m_u + m_d) \langle \bar{q} q \rangle. \tag{B22}$$

Exercises

(1) Verify Eq. (B13).

(2) Derive Eq. (B22), known as the Gell-Mann–Oakes–Renner formula.

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