LEGENDRE POLYNOMIALS IN IRRATIONALITY PROOFS

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It is shown that a simple trick involving Legendre polynomials
readily yields the irrationality of e^{α} , $\alpha \in \mathbb{Q}$, π^2 , and of
the zeros of Bessel functions of integer order. Generalisation
of this idea yields the irrationality of $\zeta(3)$.

1.

In this paper we give simple proofs for the irrationality of e^{α}
$(\alpha \in \mathbb{Q})$, π^2 and the zeros of Bessel functions of integer order. Of
course these results are not new. Lamberti [7] showed the irrationality
of π and e^{α} $(\alpha \neq 0, \alpha \in \mathbb{Q})$ about two hundred years ago. Legendre [8]
showed the irrationality of π^2 and the irrationality of the zeros of
Bessel functions are implicit in his 'Note IV'. These classical proofs use
continued fraction expansions of certain analytic functions. Other proofs
were given for example by Niven [9] (for π) and Gerritzen [6] (for the
zeros of Bessel functions). Also Siegel [10] produces some irrationality
proofs (pp. 8, 9, 10) and in the same book one finds transcendence proofs
for the numbers mentioned.

The proofs that we give here make use of a simple trick involving
Legendre-polynomials. It is possible to generalize this method to other
orthogonal polynomials with non-trivial weights. Also, one can obtain
irrationality measures for certain numbers by this method, but for the sake

of simplicity we do not exploit these possibilities. For irrationality
measures one might look at Alladi and Robinson [1], Bundschuh [3] or
Choodnovski [4]. Finally, it is possible to give two- and three-
dimensional generalisations of our method to produce irrationality proofs
for $\zeta(2)$ and $\zeta(3)$; see Beukers [2].

2.

The Legendre-polynomial for the interval $[0, 1]$ is defined by

$$(1) \qquad P_n(x) = (1/n!)(d/dx)^n x^n (1-x)^n .$$

After working out the differentiations we find

$$P_n(x) = \sum_{m=0}^{n} \binom{n}{m} \binom{n+m}{n} (-x)^m .$$

Notice that the degree of $P_n(x)$ is n and that $P_n(x) \in \mathbb{Z}[x]$. These
are the only things we need in our computations. For more facts about
Legendre-polynomials and other orthogonal polynomials see, for example, [5]
or [11].

THEOREM 1. *Let* $a \in \mathbb{Q}$, $a \neq 0$. *Then* $e^a \notin \mathbb{Q}$.

Proof. It is sufficient to prove our theorem for $a \in \mathbb{N}$. Consider

$$I_n = \int_0^1 e^{at} P_n(t) dt .$$

Insert the expression (1) for $P_n(t)$ and perform a repeated partial
integration until we obtain

$$I_n = (-1)^n (a^n/n!) \int_0^1 e^{at} t^n (1-t)^n dt .$$

Since the integrand is positive, we have $I_n \neq 0$. The following estimate
is obvious;

$$(2) \qquad |I_n| < |a/n|^n (e^a/n!) .$$

On the other hand I_n consists of terms of the type

$$\int_0^1 e^{at} t^m dt = 1/a^{m+1} \int_0^a e^t t^m dt \quad (m \leq n) \ .$$

By partial integration it is easy to see that

$$\int_0^a e^t t^m dt = C + De^a \ ,$$

where $C, D \in \mathbb{Z}$. Hence

$$I_n = \left(1/a^{n+1}\right)\left(R_n + S_n e^a\right) \ ,$$

where $R_n, S_n \in \mathbb{Z}$. Now suppose $e^q = p/q$. Since I_n is nonzero it can be estimated below by

$$1/qa^{n+1} \leq \left(1/a^{n+1}\right)|R_n + S_n(p/q)| = |I_n| \ ,$$

which contradicts the upper bound (2) for I_n if n is sufficiently large. Hence e^a is irrational.

THEOREM 2. π^2 *is irrational.*

Proof. Consider

$$I_n = \int_0^1 (\sin \pi t) P_{2n}(t) dt \ .$$

By substituting the expression (1) for P_n and after $2n$-fold partial integration we obtain

$$I_n = (-1)^n \left(\pi^{2n}/(2n)!\right) \int_0^1 (\sin \pi t) t^{2n} (1-t)^{2n} dt$$

and hence the estimate

$$|I_n| < (\pi/4)^{2n}\left(1/(2n)!\right) \ .$$

Also note that $I_n \neq 0$. On the other hand I_n consists of terms of the shape

$$\int_0^1 t^m (\sin \pi t)dt \ , \quad m \leq 2n \ .$$

By partial integration we see that

$$\int_0^1 t^m (\sin \pi t)dt = (1/\pi) - \left(m(m-1)/\pi^2\right) \int_0^1 t^{m-2}(\sin \pi t)dt \ .$$

Hence $I_n = \pi^{-1} A_n \left(\pi^{-2}\right)$, where A_n is an $(n-1)$st degree polynomial with integral coefficients. Now suppose $\pi^2 = p/q$. Since I_n is nonzero it can be estimated below by

$$1/\pi p^n \leq \left|(1/\pi)A_n(q/p)\right| = \left|I_n\right| \ ,$$

contradicting the upper bound for I_n if n is sufficiently large. Hence π^2 is rational.

The following theorem deals essentially with the Bessel-function

$$J_r(z) = \sum_{k=0}^{\infty} (-1)^n (z/2)^{r+2n}/n!(n+r)!$$

where $r = 0, 1, 2, \ldots$. In order to make computations simpler however, we shall state the theorem in terms of the simplified function

$$L_r(z) = \sum_{n=0}^{\infty} z^n/n!(n+r)! \ .$$

The connection between $J_r(z)$ and $L_r(z)$ is given by

$L_r\left(-z^2/2\right) = z^{-r} J_r(z)$. Also, we shall make use of the modified polynomial

(3) $P_{n,r}(x) = \left(x^{-r}/n!\right)(d/dx)x^{r+n}(1-x)^n \ .$

It is easy to see that $P_{n,r}(x)$ has degree n and integral coefficients.

THEOREM 3. *The zeros of $L_r(z)$ are irrational.*

Proof. Consider

$$I_n(z) = z^n \int_0^1 t^r L_r(zt) P_{n,r}(t) dt \ .$$

Substitute (3) and perform an n-fold partial integration. Then

(4) $$I_n(z) = z^{2n}\left((-1)^n/n!\right) \int_0^1 D^n L_r(zt) t^{n+r}(1-t)^n dt \ ,$$

where D denotes differentiation. From the Taylor expansion of $L_r(z)$ it is easy to see that

$$\left| D^n L_r(z) \right| \le e^{|z|}$$

for any $z \in \mathbb{C}$ and any $n \ge 0$. Hence

(5) $$|I_n| < |z/2|^{2n}(2/n!)e^{|z|} \ .$$

On the other hand $I_n(z)$ consists of terms of the shape

$$z^n \int_0^1 L_r(zt) t^{m+r} dt = z^{n-m-r-1} \int_0^z L_r(t) t^{m+r} dt \ , \quad m \le n \ .$$

By using the differential equation $z^{-r} D\left(z^{r+1} DL_r\right) = L_r$ and partial integration, we see that

$$\int_0^z L_r(t) t^{m+r} dt = z^{m+r+1} L_r'(z) - mz^{m+r} L_r(z) + m(m+r) \int_0^z t^{m+r-1} L_r(t) dt$$

and hence

$$\int_0^z L_r(t) t^{m+r} dt = U_m(z) z^{r+1} L_r'(z) + V_m(z) z^{r+1} L_r(z)$$

where $U_m(z)$, $V_m(z)$ are polynomials of degree $m, m-1$ respectively with integral coefficients. Notice also that $V_m(0) \ne 0$. Collecting terms we find that

(6) $$I_n(z) = A_n(z) L_r'(z) + B_n(z) L_r(z) \ ,$$

where $A_n(z)$, $B_n(z)$ are of degree $n, n-1$ respectively and have integral coefficients. Notice that $B_n(0) \ne 0$ for all n .

Now suppose α is a rational zero of $L_r(z)$. It is clear that $L_r'(\alpha) \neq 0$ because otherwise the differential equation would imply $L_r(z) \equiv 0$. Inequality (5) now implies

$$|A_n(\alpha)L_r'(\alpha)| < |\alpha/2|^{2n+1}(2/n!)e^{|\alpha|} \ .$$

Since α is rational we have either $A_n(\alpha) = 0$ or $|A_n(\alpha)| \geq b^{-n}$, where b is the denominator of α . For sufficiently large n however, the second possibility contradicts the upper bound we have for $A_n(\alpha)$. Therefore $A_n(\alpha) = 0$ for all sufficiently large n , and so is $I_n(\alpha)$. We will show that this is not possible, and thus prove our theorem.

Consider (6) for n and $n - 1$, and eliminate $L_r(z)$. We obtain

(7) $$D_n(z)L_r'(z) = B_{n-1}(z)I_n(z) - B_n(z)I_{n-1}(z) \ ,$$

where we have put $D_n(z) = A_n(z)B_{n-1}(z) - A_{n-1}(z)B_n(z)$. From (4) we see that $D_n(z)L_r'(z)$ is divisible by z^{2n-2} , and since $L_r'(0) \neq 0$, the polynomial $D_n(z)$ is divisible by z^{2n-1} . Since $D_n(z)$ has degree at most $2n - 2$, we infer $D_n(z) = c_n z^{2n-2}$ for some constant c_n . Substitute this in (7), divide by z^{2n-2} and put $z = 0$. Then we end up with

$$c_n L_r'(0) = -B_n(0)\left((-1)^{n-1}/(n-1)!\right) \int_0^1 D^{n-1} L_r(0) t^{n+r}(1-t)^n dt$$

$$= -B_n(0)\left((-1)^{n-1}/(n-1)!\right) D^{n-1} L_r(0)\left((n+r)!n!/(2n+r+1)!\right) \ .$$

Since all factors on the right-hand side are nonzero we conclude that $c_n \neq 0$ for all n . Hence $D_n(\alpha) \neq 0$, and from expression (7) with $z = \alpha$ we see that at least one of $I_n(\alpha), I_{n-1}(\alpha)$ is not zero, which concludes the proof of our theorem.

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