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ON A THEOREM OF COHEN AND LYNDON ABOUT FREE BASES FOR NORMAL SUBGROUPS

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1. Introduction. Let $S(\neq 1)$ be a subgroup of a group G. We consider the question: when are the conjugates of S "as independent as possible"? Specifically, suppose S^G (the normal subgroup generated by S in G) is the free product $\Pi^* S^{g_\alpha}$ where $S^{g_\alpha} = g_\alpha S g_\alpha^{-1}$ and g_α ranges over a subset J of G. Then J must be part of a (left) coset representative system for G mod $S^G \cdot N$ where N is the normalizer of S in G. (For, $g \in S^G g_\alpha N$ implies S^g is conjugate to S^{g_α} in S^G; however, distinct non-trivial free factors of a free product are never conjugate.)

We say that S^G *is the free product of maximally many conjugates of S in G* if $S^G = \Pi^* S^{g_\alpha}$ where g_α ranges over a (complete) left coset representative system for G mod $S^G N$ (or equivalently, g_α ranges over a double coset representative system for G mod (S^G, N)); in this case we say briefly that S has the *fpmmc* property in G.

If S has the *fpmmc* property in G and $S^G = \Pi^* S^{h_\beta}$, then h_β must range over a double coset representative system for G mod (S^G, N). (For, if $h_\beta \in S^G g_\beta N$, then S^G is the normal subgroup of itself generated by all S^{g_β}, and so the g_β must include all the g_α.)

It is easy to see that if $S^G = \Pi^* S^{g_\alpha}$ then S has the *fpmmc* property in G if and only if for every $g \in G$, S^g is conjugate in S^G to some S^{g_α}.

Theorem 1 gives a necessary and sufficient condition for certain subgroups of a free product with amalgamated subgroup or of an HNN group to have the *fpmmc* property in the whole group. The proof is based on the subgroup theorems in [**3**] and [**4**]. From Theorem 1 we derive the "Main Theorem" of Cohen and Lyndon [**2**] which (in the above terminology) states that a cyclic subgroup has the *fpmmc* property in a free group.

2. A lemma. It is shown in [**3**] that if H is a subgroup of $(A * B; U)$ and H has trivial intersection with each conjugate of U, then H is the free product of a free group F and the factors $D_\alpha A D_\alpha^{-1} \cap H$, $D_\beta B D_\beta^{-1} \cap H$ where D_α and D_β range over double coset representative systems for G mod (H, A) and G mod (H, B) respectively (see Corollary 3 of Theorem 5 in [**3**]); moreover if H is generated by its intersection with conjugates of A and B, then $F = 1$. The analogous result for HNN groups is the following, which is a refinement of Theorem 6 of [**4**]:

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LEMMA 1. *Let G be an* HNN *group*

(1) $$G = \langle t_1, t_2, \ldots, K; \text{rel } K, t_1L_1t_1^{-1} = M_1, t_2L_2t_2^{-1} = M_2, \ldots \rangle,$$

and let H be a subgroup of G having trivial intersection with each conjugate of each L_i. Then H is the free product of a free group F and the factors $gKg^{-1} \cap H$ where g ranges over a double coset representative system for G mod (H, K). Moreover, if H is generated by its intersections with conjugates of K, then $F = 1$.

Proof. By Theorem 6 of [**4**], H is the free product of a free group F and factors different from 1 of the form $g_iKg_i^{-1} \cap H$ where g_i ranges over a certain subset of G; moreover, by Corollary 3 to Theorem 1 of [**4**], each subgroup $gKg^{-1} \cap H$ $(\neq 1)$, $g \in G$, is conjugate in H to one of the subgroups $g_iKg_i^{-1} \cap H$. Furthermore, by the last part of Lemma 3 of [**4**], if $gKg^{-1} \cap H = hg_iKg_i^{-1}h^{-1} \cap H \neq 1$ for some $h \in H$, then $HgK = Hg_iK$. Hence if g_α ranges over a double coset representative system for G mod (H, K), then for each g_α such that $g_\alpha Kg_\alpha^{-1} \cap H \neq 1$ there is an unique g_i defining the same double coset; and so supplementing the g_i with those g_α such that $g_\alpha Kg_\alpha^{-1} \cap H = 1$, we have that H is the free product of F and subgroups $gKg^{-1} \cap H$ where g ranges over a double coset representative system for G mod (H, K).

Finally, if H is generated by its intersections with conjugates of K, then H is contained within the normal subgroup of H generated by its free factors $g_iKg_i^{-1} \cap H$, so that $F = 1$.

3. An fpmmc theorem for generalized free products and HNN groups.

THEOREM 1. *Let G be a generalized free product $(A * K; U)$ or an* HNN *group as in* (1). *Suppose S is a subgroup of K such that S^K has trivial intersection with U or with each L_i, M_i. Then S has the fpmmc property in G if and only if S has the property in K.*

Proof. We may assume $S \neq 1$.

We first show that the conclusion of the theorem holds for any group $G > K > S$ if G, K, S satisfy the following conditions:

(i) $gSg^{-1} \cap K \neq 1$ implies $g \in K$;

(ii) $S^G \cap K = S^K$;

(iii) $S^G = \prod^* (gKg^{-1} \cap S^G)$ where g ranges over a double coset representative system for G mod (S^G, K).

Condition (i) implies that if S has the *fpmmc* property in G, then it has it in K. For, clearly (i) implies that the normalizer N of S in K is the normalizer of S in G. Let $S^G = \prod^* S^{g_\alpha}$ where g_α ranges over a double coset representative system for G mod (S^G, N). Now S^K is generated by all S^k, $k \in K$, and so S^K is generated by its intersection in S^G with conjugates of the factors S^{g_α}. Hence by the Kurosh subgroup theorem,

$$S^K = \prod_\alpha {}^* (\prod_j {}^* (D_{\alpha j} S^{g_\alpha} D_{\alpha j}{}^{-1} \cap S^K))$$

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where for each g_α, $D_{\alpha j}$ runs over a double coset representative system for S^G mod (S^K, S^{g_α}). But (i) implies $D_{\alpha j}S^{g_\alpha}D_{\alpha j}^{-1} \cap S^K$ is 1 unless $D_{\alpha j}g_\alpha \in K$; moreover, if $D_{\alpha j}g_\alpha \in K$, then $D_{\alpha j}S^{g_\alpha}D_{\alpha j}^{-1} \cap S^K = S^{D_{\alpha j}g_\alpha}$. Thus $S^K = \prod^* S^{D_{\alpha j}g_\alpha}$ where $D_{\alpha j}g_\alpha \in K$. Furthermore, since for each g_α, $S^G = \bigcup S^K D_{\alpha j} S^{g_\alpha}$, it follows that $S^G g_\alpha N = \bigcup S^K D_{\alpha j} g_\alpha N$ for each g_α. Hence $G = \bigcup S^G g_\alpha N = \bigcup S^K D_{\alpha j} g_\alpha N$, so that $K = \bigcup S^K D_{\alpha j} g_\alpha N$ where $D_{\alpha j} g_\alpha \in K$. Consequently, S has the *fpmmc* property in K.

On the other hand, if (i), (ii), and (iii) hold and S has the *fpmmc* property in K, then S has the property in G. For, since S^G is normal in G, properties (ii) and (iii) imply $S^G = \prod^* g_\alpha S^K g_\alpha^{-1}$ where g_α ranges over a double coset representative system for G mod (S^G, K). By hypothesis, $S^K = \prod^* S^{k_j}$ where k_j ranges over a double coset representative system for K mod (S^K, N). Therefor $S^G = \prod^* S^{g_\alpha k_j}$ and $G = \bigcup S^G g_\alpha K = \bigcup S^G g_\alpha S^K k_j N = \bigcup S^G g_\alpha k_j N$. Consequently, S has the *fpmmc* property in G.

To establish conditions (i), (ii), and (iii), first suppose G is $(A * K; U)$. Since $S^K \cap U = 1$, condition (i) holds. Moreover, $G/S^G = (A * K/S^K; U)$, so that (ii) holds and $S^G \cap A = 1$. By the remark preceding Lemma 1, condition (iii) holds.

Finally, suppose G is an HNN group as in (1). Then G/S^G is also an HNN group with base K/S^K and the same free part and associated subgroups L_i, M_i as in G. Therefore condition (ii) holds, and S^G has trivial intersection with each L_i. Therefore by Lemma 1, condition (iii) follows. Moreover, by the last part of Lemma 3 of [**4**], if $g \notin K$ then $K \cap gSg^{-1}$ is contained in some conjugate of L_i, and so condition (ii) also holds.

Clearly, *if G, K, S are as in Theorem* 1, *and S is normal in K, then S has the fpmmc property in G.*

As an illustration of Theorem 1, we show that in a Fuchsian group G with positive periods $\gamma_1, \ldots, \gamma_t$ and positive genus n,

$$G = \langle c_1, \ldots, c_t, a_1, b_1, \ldots, a_n, b_n; c_1^{\gamma_1}, \ldots, c_t^{\gamma_t}, c_1^{-1} \ldots c_t^{-1}[a_1, b_1] \ldots [a_n, b_n]\rangle,$$

that the subgroup $S = \mathrm{gp}(c_1, \ldots, c_t)$ has the *fpmmc* property in G. Write G as an HNN group

$$\langle a_1, c_1, \ldots, c_t, b_1, \ldots, a_n, b_n; c_1^{\gamma_1}, \ldots, c_t^{\gamma_t}, \quad a_1 b_1 a_1^{-1} = c_t \ldots c_1[b_n, a_n] \ldots [b_2, a_2]b_1\rangle,$$

with free part $\mathrm{gp}(a_1)$, base K, the free product of $S = \prod^*\langle c_i; c_i^{\gamma_i}\rangle$ and the free group $\langle b_1, a_2, b_2, \ldots, a_n, b_n\rangle$, and associated subgroups $L = \mathrm{gp}(b_1)$ and $M = \mathrm{gp}(c_t \ldots c_1[b_n, a_n] \ldots [b_2, a_2]b_1)$. Clearly, S^K has trivial intersection with L and M. Since S has the *fpmmc* property in K, S has the property in G.

COROLLARY. *Let $G > K > S > H$. Suppose S has the fpmmc property in G. Then S has the fpmmc property in K provided $gSg^{-1} \cap K \neq 1$ implies $g \in K$. Moreover, the image of S in G/H^G has the fpmmc property in G/H^G.*

Proof. The first conclusion was established in the course of the proof of Theorem 1.

To establish the last assertion, let $S^G = \prod^* S^{g_\alpha}$ where g_α ranges over a double coset representative system for G mod (S^G, N) and N is the normalizer of S in G. Let φ be the natural homomorphism of G onto G/S^G. We may assume that H is normal in N (otherwise replace H by H^N). Then H^g, $g \in G$, is conjugate in S^G to some H^{g_α}, and so $S^G/H^G = \prod^* (\varphi(S))^{\varphi(g_\alpha)} \simeq \prod^* (S/H)^{\varphi(g_\alpha)}$. Moreover, every conjugate of $\varphi(S)$ in G/H^G is conjugate in $\varphi(S^G)$ to one of the factors $(\varphi(S))^{\varphi(g_\alpha)}$. Hence $\varphi(S)$ has the *fpmmc* property in G/H^G.

The theorem of Cohen and Lyndon, which we derive in the next section from Theorem 1, states that a cyclic subgroup has the *fpmmc* property in a free group. Using the last part of the above corollary it then follows that in a one-relator group

$$P = \langle a, b, \ldots ; r^n \rangle, \quad n > 1,$$

where r is not a proper power, the gp(r) has the *fpmmc* property in P. G. Baumslag [**1**] has used this last result to show that the group P is hopfian if the group obtained by dividing out the elements of finite order, namely,

$$P_1 = \langle a, b, \ldots ; r \rangle$$

is hopfian.

Using his method of proof we obtain the following generalization. *Let S be the free product of finitely many finite groups S_i, $1 \leqq i \leqq m$. Suppose S has the fpmmc property in G, the normalizer of S in G is S itself, and S^G contains all elements of finite order in G. Then G is hopfian if G/S^G is hopfian.*

For, suppose G/S^G is hopfian and β is an endomorphism of G onto itself. Since S^G is generated by all the elements of finite order, β maps S^G into S^G. Moreover, since β induces an automorphism on G/S^G, it follows that $\beta^{-1}(S^G) = S^G$. Thus the kernel of β is in S^G and β maps S^G onto itself. Now $S^G = \prod^* S_i^{g_\alpha}$ where g_α ranges over a coset representative system for G mod S^G. We show β is a one-one mapping of S_i into S^G. For, $\beta(S_i) < S_{j_i}^{h_i}$, $h_i \in G$; since S^G is not the normal subgroup of G generated by a proper subset of $S_1, \ldots, S_m$ (any S_i^g is conjugate in S^G to $S_i^{g_\alpha}$ for some g_α), the range of j_i consists of the integers from 1 to m. Replacing β by its $m!$-th power, we may assume that β maps each S_i into a conjugate of itself. If $\beta(S_i) = T_i^{h_i}$ where $T_i < S_i$, then $S^G = \text{gp}(S_i^{g_\alpha}) = \text{gp}(T_i^{\beta(g_\alpha)h_i})$. For each i, $\beta(g_\alpha)h_i$ ranges over a coset representative system for G mod S^G, so that $S^G = \text{gp}(T_i^{h_{\alpha i} g_\alpha})$ where $h_{\alpha i} \in S^G$. But then mapping S^G into the direct product of the $S_i^{g_\alpha}$, we have that $T_i^{h_{\alpha i} g_\alpha}$ goes into $U_i^{g_\alpha}$ where U_i is conjugate to T_i in S_i; therefore $T_i = S_i$. Thus β maps S_i one-one into S^G. Since distinct subgroups in $\{\beta(S_i^{g_\alpha})\}$ are conjugate in S^G to distinct free factors $S_i^{g_\alpha}$, the collection $\{\beta(S_i^{g_\alpha})\}$ generates its free product. Consequently, β maps S^G one-one onto S^G, and so β is an automorphism of G.

For example, the group

$$G = \langle a, b, c_1, \ldots, c_t, d; c_1^{\gamma_1}, \ldots, c_t^{\gamma_t}, c_1^{-1} \ldots c_t^{-1}[a, b], d^k = a^m b^n \rangle$$

where $km \neq 0$ or $kn \neq 0$ is hopfian. For, letting $S = \text{gp}(c_1, \ldots, c_t)$, we have that S is its own normalizer, has the *fpmmc* property in G, and S is a free product of finitely many finite cyclic groups. Moreover, G/S^G is the free product of the free abelian group on a, b and the infinite cyclic group on d with an infinite cyclic group amalgamated. Now it is easy to show that this last group is residually finite and therefore hopfian. Hence G is hopfian.

4. The theorem of Cohen and Lyndon.

THEOREM 2. *Let F be the free group on $a, b, c, \ldots$ and let r^m be an element of F where r is not a true power. Then $\text{gp}(r^m)$ has the fpmmc property in F, i.e., $\text{gp}(r^m)^F = \prod^* \text{gp}(r^m)^{f_\alpha}$ where f_α ranges over a double coset representative system for F* mod $(\text{gp}(r^m)^F, \text{gp}(r))$.

Proof. The proof is by induction on the length of r. If r has syllable length one, then r is a free generator of F, and $\text{gp}(r^m)$ is normal in $\text{gp}(r)$, which in turn is a free factor of F. Hence by Theorem 1, $\text{gp}(r^m)$ has the *fpmmc* property in F.

Suppose r has syllable length greater than one. If some generator occurs in r with zero exponent sum, take $P = F$. Otherwise we enlarge F using the standard Magnus method so as to introduce a generator on which r will have zero exponent sum. Specifically, if r involves the free generators a, b of F with non-zero exponent sums α, β respectively, then we form

$$P = (\langle x \rangle * F; x^\beta = a), \tag{2}$$

introduce the generator $y = bx^\alpha$, and replace b by $yx^{-\alpha}$. Suppose that when $r(a, b, c, \ldots)$ is rewritten in terms of $x, y, c, \ldots$ we obtain $r'(x, y, c, \ldots)$, which when rewritten in terms of the conjugates $y_i = x^i y x^{-i}$, $c_i = x^i c x^{-i}, \ldots$ of the generators $y, c, \ldots$ of P becomes $r_0(y_i, c_j, \ldots)$.

Then the free group P can be written as an HNN group

$$\langle x, K; xLx^{-1} = M \rangle \tag{3}$$

where K is the free group on $y_i, c_j, \ldots$ where i ranges from the smallest to the largest subscript occurring on y in r_0 and is zero if no subscript occurs on y in r_0; similarly for the range of $j, \ldots$; L and M are freely generated by the subsets of the generators of K which exclude those generators $y_i, c_j, \ldots$ with largest and smallest subscripts respectively. By the Magnus Freiheitssatz (see, e.g., [**5**]), L, M have trivial intersection with $\text{gp}(r_0^m)^K$. Since r_0 has shorter length than r, it follows that $\text{gp}(r_0^m)$ has the *fpmmc* property in K, and therefore by Theorem 1, $\text{gp}((r')^m)$ has this property in P. Since $\text{gp}(r^m)^F$ has trivial intersection with $\text{gp}(a)$, Theorem 1 implies $\text{gp}(r^m)$ has the *fpmmc* property in F.

References

1. G. Baumslag, *Residually finite one-relator groups*, Bull. Amer. Math. Soc. *73* (1967), 618–620.
2. D. E. Cohen and R. C. Lyndon, *Free bases for normal subgroups of free groups*, Trans. Amer. Math. Soc. *108* (1963), 526–537.
3. A. Karrass and D. Solitar, *The subgroups of a free product of two groups with an amalgamated subgroup*, Trans. Amer. Math. Soc. *150* (1970), 227–255.
4. ——— *Subgroups of* HNN *groups and groups with one defining relation*, Can. J. Math. *23* (1971), 627–643.
5. W. Magnus, A. Karrass, and D. Solitar, *Combinatorial group theory*, Pure and Appl. Math., Vol. 13 (Interscience, New York 1966).

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