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AN APPLICATION OF HOMOGENIZATION THEORY TO HARMONIC ANALYSIS: HARNACK INEQUALITIES AND RIESZ TRANSFORMS ON LIE GROUPS OF POLYNOMIAL GROWTH

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ABSTRACT. We prove a homogenization formula for a sub-Laplacian $L = -(E_i^2 + \cdots + E_p^2)$ ($E_1, \ldots, E_p$ are left invariant Hörmander vector fields) on a connected Lie group G of polynomial growth. Then using a rescaling argument inspired from M. Avellaneda and F. H. Lin [2], we prove Harnack inequalities for the positive solutions of the equation $(\partial/\partial t + L)u = 0$. Using these inequalities and further exploiting the algebraic structure of G we prove that the Riesz transforms $E_iL^{-\frac{1}{2}}$, $L^{-\frac{1}{2}}E_i$, $1 \leq i \leq p$, are bounded on L^q, $1 < q < +\infty$ and from L^1 to weak-L^1.

RÉSUMÉ. On démontre une formule de homogènéisation pour un sous-Laplacien $L = -(E_i^2 + \cdots + E_p^2)$ ($E_1, \ldots, E_p$ sont des champs de vecteurs de Hörmander invariants à gauche) sur un group de Lie G connexe, à croissance polynômiale du volume. Après, en utilisant un argument de rescalarisation inspiré de M. Avellaneda et F. H. Lin [2], on démontre des inégalités de Harnack pour les solutions positives de l'equation $(\partial/\partial t + L)u = 0$. En utilisant ces inégalités et en exploitant la structure algébrique de G, on démontre que les transformés de Riesz $E_iL^{-\frac{1}{2}}$, $L^{-\frac{1}{2}}E_i$, $1 \leq i \leq p$ sont bornés sur L^q, $1 < q < +\infty$ et de L^1 dans L^1-faible.

0. **Introduction.** Let G be a connected Lie group of polynomial growth, *i.e.* if dg is a left invariant Haar measure and V a compact neighborhood of the identity element e of G , then there are constants $c, d > 0$ such that dg-measure$(V^n) \leq cn^d$, $n \in \mathbb{N}$. Notice that the connected nilpotent Lie groups are of polynomial growth.

Let us also identify the elements of the Lie algebra $\mathfrak{g}$ of G with the left invariant vector fields on G and consider $E_1, \ldots, E_p \in \mathfrak{g}$ that satisfy Hörmander's condition *i.e.* together with their successive Lie brackets $\big[E_{i_1}, [E_{i_2}, \ldots, E_{i_s}] \cdots\big]$, they generate $\mathfrak{g}$. To these vector fields it is associated, in a canonical way, a left invariant distance $d_E(.,.)$ on G, called control distance. This distanse has the property that (*cf.* [24]) if $S_E(x,t) = \{y \in G, d_E(x,y) < t\}$, $x \in G$, $t > 0$, then there is $c \in \mathbb{N}$ such that

$$S_E(e,n) \subseteq V^{cn},\ V^n \subseteq S_E(e,cn), \quad n \in \mathbb{N}. \tag{0.1}$$

Moreover the operators $L = -(E_1^2 + \cdots + E_p^2)$ and $\partial/\partial t + L$, according to a classical theorem of L. Hörmander [15], are hypoelliptic.

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The purpose of this paper is to explain how ideas inspired from Homogenization theory can be used to answer questions concerning the Harmonic analysis on G. More precisely, we prove a homogenization formula for the operator L. This formula is similar to the one already known for second order uniformly elliptic differential operators with periodic coefficients on $\mathbb{R}^n$. The novelty here is that we deal with hypoelliptic operators whose coefficients are functions defined on a compact Lie group and not periodic and that the homogenised operator L_H is a left invariant sub-Laplacian (*i.e.* like L, it is a sum of squares of left invariant vector fields that satisfy Hörmander's condition), defined on a homogeneous nilpotent Lie group N_H and invariant with respect to its dilation structure. N_H is uniquely determined from the algebraic structure of G. Then using a rescaling argument inspired from M. Avellaneda and F. H. Lin [2] and [3] and further exploiting the algebraic structure of G, we obtain the following results.

THEOREM 1. *Let $G, E_1, \ldots, E_p$ and L be as above. Then for every integer $k \geq 0$, $1 \leq i \leq p$ and $0 < a < b < 1$ there exists $c > 0$ such that*

$$\left|\frac{\partial^k}{\partial t^k} E_i u(at, x)\right| \leq c t^{-k-\frac{1}{2}} u(bt, x), \quad t \geq 1,\ x \in G$$

for all $u \geq 0$ such that $(\partial/\partial t + L)u = 0$ in $(0, t) \times S_E(x, \sqrt{t})$.

THEOREM 2. *Let $G, E_1, \ldots, E_p$ and L be as above. Then the Riesz transforms $E_i L^{-\frac{1}{2}}$, $L^{-\frac{1}{2}} E_i$, $1 \leq i \leq p$ (cf. [21]), are bounded on L^q, $1 < q < \infty$ and from L^1 to weak-L^1.*

Theorem 2 has been proved, in the case where G is a stratified nilpotent Lie group and L is invariant with respect to its dilation structure by M. Christ and D. Geller [8] and for general nilpotent Lie groups by N. Lohoué and N. Th. Varopoulos [18].

When G is nilpotent then Theorem 1 is a particular case of a more general result of N. Th. Varopoulos [24], namely for all integers $k, \ell \geq 0$ there is $c_{k,\ell} \geq 0$ such that

$$\left|\frac{\partial^k}{\partial t^k} E_{i_1} \cdots E_{i_\ell} u(at, x)\right| \leq c_{k,\ell} t^{-k-\frac{\ell}{2}} u(bt, x), \quad t \geq 1,\ x \in G \tag{0.2}$$

for all $u \geq 0$ such that $(\partial/\partial t + L)u = 0$, in $(0, t) \times S_E(x, \sqrt{t})$.

These inequalities are also true for $0 < t < 1$ (*cf.* N. Th. Varopoulos [23]), but this is a result of the local theory of operators of the type sum of squares of vector fields that satisfy Hörmander's condition.

The motivating example is the universal covering of the group of Euclidean motions on the plane, which is a three dimensional solvable Lie group of polynomial growth. As we shall see in Section 1, every operator L as above, on this group, can be expressed as a second order differential operator on $\mathbb{R}^3$ with periodic coefficients. We shall give in Section 1, a specific example of a sub-Laplacian $L = -(E_1^2 + E_2^2 + E_3^2)$, for which there are families of functions $u_t, v_t, t \geq 1$ and $c > 0$ such that

$$u_t \geq 0,\ Lu_t = 0 \text{ in } S_E(e, t),\ |E_1^2 u_t(e)| \geq \frac{c}{t} u_t(e),\ u_t(e) > c,\ t \geq 1 \tag{0.3}$$

$$(0.4) \qquad v_t \in C_0^\infty(G),\ \|E_1^2 v_t\|_2 \geq t\|Lv_t\|_2,\ t \geq 1.$$

Clearly, (0.3) shows that the inequalities (0.2) are not true, for $\ell \geq 2$, for general non-nilpotent Lie groups of polynomial growth and (0.4) that the higher order Riesz transforms, $E_i^2 L^{-1}$, $L^{-1}E_i^2$, $1 \leq i \leq p$, in general, are not bounded, even on L^2.

In Section 1 we shall discuss the universal covering of the group of the Euclidean motions on the plane and we shall show how (0.3), (0.4) and Theorems 1 and 2 can be proved in this particular case. In the subsequent sections we shall show how these results can be proved in the general case.

To simplify the notation, we shall use the summation convention for repeated indices throughout this paper.

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1. **The motivating example.** Let Q be a simply connected Lie group of dimension three and assume that there is a basis $\{X_1, X_2, X_3\}$ of its Lie algebra $\mathfrak{q}$ (the elements of $\mathfrak{q}$ are identified with the left invariant vector fields on Q) such that

$$[X_1, X_2] = X_3,\ [X_1, X_3] = -X_2,\ [X_2, X_3] = 0.$$

Identifying the analytic subgroups of Q whose Lie algebra is generated by $\{X_2, X_3\}$ and $\{X_1\}$ with $\mathbb{R}^2$ and $\mathbb{R}$ respectively, we can see that Q is isomorphic to the semidirect product $\mathbb{R}^2 \times_\tau \mathbb{R}$ where the action τ of $\mathbb{R}$ on $\mathbb{R}^2$ is given by $\tau\colon \mathbb{R} \longrightarrow L(\mathbb{R}^2)\colon x \longrightarrow \mathrm{rot}_x, \mathrm{rot}_x$ being the counterclockwise rotation by angle x and $L(\mathbb{R}^2)$ the space of linear tranformations of $\mathbb{R}^2$.

Q is isomorphic to the universal covering of the group of Euclidean motions on the plane. It is a (non-nilpotent) solvable Lie group of polynomial growth.

Let

$$E_1 = X_1,\ E_2 = X_1 + X_2,\ E_3 = X_3 \text{ and } L = -(E_1^2 + E_2^2 + E_3^2).$$

We are going to show how Theorems 1 and 2 can be proved in this specific example, construct the families of functions u_t and $v_t, t \geq 1$ mentioned in (0.3) and (0.4) and explain why it is natural to use Homogenization theory.

The fundamental remark is that, if we identify Q with $\mathbb{R}^3$, using the exponential coordinates of the second kind, then L becomes a second order differential operator with periodic coefficients on $\mathbb{R}^3$.

By exponential coordinates of the second kind, we understand the diffeomorphism

$$\phi\colon \mathbb{R}^3 \longrightarrow Q,\ \phi\colon (x_3, x_2, x_1) \longrightarrow \exp x_3X_3 \exp x_2X_2 \exp x_1X_1.$$

If $x = (x_3, x_2, x_1)$ then we have that

$$(1.1) \qquad d\phi^{-1}X_1(x) = \frac{\partial}{\partial x_1} \quad d\phi^{-1}X_2(x) = \cos x_1 \frac{\partial}{\partial x_2} + \sin x_1 \frac{\partial}{\partial x_3}$$
$$d\phi^{-1}X_3(x) = -\sin x_1 \frac{\partial}{\partial x_2} + \cos x_1 \frac{\partial}{\partial x_3}.$$

Let us now identify Q with $\mathbb{R}^3$ (as differential manifolds). Then L becomes a uniformly elliptic differential operator, which can be written in divergence form $L = -\frac{\partial}{\partial x_i}a_{ij}(x)\frac{\partial}{\partial x_j}$, with $a_{11} = 2$, $a_{22} = a_{33} = 1$, $a_{12} = a_{21} = \cos x_1$, $a_{13} = a_{31} = \sin x_1$ and $a_{23} = a_{32} = 0$ and the control distance $d_E(.,.)$ associated to the vector fields E_1, E_2, E_3 equivalent to the Euclidean one *i.e.* $\exists b \geq a > 0$ such that $a|x-y| \leq d_E(x,y) \leq b|x-y|$, $x, y \in \mathbb{R}^3$.

Moreover, if $L_\varepsilon = -\frac{\partial}{\partial x_i}a_{ij}(\frac{x}{\varepsilon})\frac{\partial}{\partial x_j}$ and $E_{\varepsilon,i}(x) = E_i(\frac{x}{\varepsilon})$, $i = 1,2,3$, $0 < \varepsilon \leq 1$ and $B(0,1)$ is the Euclidean unit ball then proving the inequalities (0. 2) is equivalent to proving that for all $k, \ell \in \mathbb{Z}$, $k, \ell \geq 0$ and $0 < a < b < 1$ there is $c > 0$ such that

$$\left|\frac{\partial^k}{\partial t^k}E_{\varepsilon,i_1}\cdots E_{\varepsilon,i_\ell}u_\varepsilon(a,0)\right| \leq cu_\varepsilon(b,0), \quad 0 < \varepsilon \leq 1 \tag{1.2}$$

for all $u_\varepsilon \geq 0$ satisfying $(\partial/\partial t + L_\varepsilon)u_\varepsilon = 0$ in $(0,1) \times B(0,1)$, which is a problem of Homogenization theory.

Results from Homogenization theory (cf. [2], [4]). Let $L = -\frac{\partial}{\partial x_i}a_{ij}(x)\frac{\partial}{\partial x_j}$ be a uniformly elliptic operator in $\mathbb{R}^n$ and assume that its coefficients $a_{ij}(x)$ are periodic (*i.e.* $a_{ij}(x+z) = a_{ij}(x), x \in \mathbb{R}^n, z \in \mathbb{Z}^n$) and Hölder continuous (*i.e.* there is $\alpha \in (0,1)$ and $M > 0$ such that $\|a_{ij}(x)\|_{C^\alpha(\mathbb{R}^n} \leq M$).

We denote by $\chi^j, j = 1, \ldots, n$ the unique solutions of the problem

$$L(x_j - \chi^j) = 0,\ \chi^j(x+z) = \chi^j(x),\ x \in \mathbb{R}^n,\ z \in \mathbb{Z}^n,\ \int_D \chi^j(x)\,dx = 0,\ D = [0,1]^n.$$

The functions χ^j are called correctors.

We denote by $L_0 = -\frac{\partial}{\partial x_i}q_{ij}\frac{\partial}{\partial x_j}$ the homogenised operator whose coefficients q_{ij} are the constants defined by

$$q_{ij} = \int_D \left[a_{ij} - a_{i\ell}\frac{\partial}{\partial x_\ell}\chi^j(x)\right]dx,\ D = [0,1]^n$$

It can be shown that L_0 is an elliptic operator (*cf.* [4]).

Let $L_\varepsilon = -\frac{\partial}{\partial x_i}a_{ij}(\frac{x}{\varepsilon})\frac{\partial}{\partial x_j}$, $0 < \varepsilon \leq 1$. Let also $f \in L^{n+\delta}\big(B(0,1)\big)$, $\delta > 0$, $g \in C^{1,\nu}\big(\partial B(0,1)\big)$, $0 < \nu \leq 1$ and denote by u_ε, $0 \leq \varepsilon \leq 1$, the solutions of the problem

$$L_\varepsilon u_\varepsilon = f \text{ in } B(0,1),\ \ u_\varepsilon = g \text{ on } \partial B(0,1),\ 0 < \varepsilon \leq 1. \tag{1.3}$$

We have the following results.

THEOREM 1.1 (*cf.* [4]). *Let u_ε, $0 \leq \varepsilon \leq 1$ be as above. Then $u_\varepsilon \longrightarrow u_0$, $(\varepsilon \longrightarrow 0)$, uniformly on the compact subsets of $B(0,1)$.*

THEOREM 1.2 (*cf.* M. AVELLANEDA AND F. H. LIN [2]). *Let u_ε, $0 \leq \varepsilon \leq 1$ be as above. Then there is a constant $c > 0$ depending only on α, M, n, ν, δ and independent of ε such that*

$$[u_\varepsilon]_{C^{0,1}(B(0,1))} \leq c\Big([g]_{C^{1,\nu}(\partial B(0,1))} + \|f\|_{L^{n+\delta}(B(0,1))}\Big). \tag{1.4}$$

In our example, we have that

$$\chi^1(x) = 0,\ \chi^2(x) = \frac{1}{2}\sin x_1,\ \chi^3(x) = -\frac{1}{2}\cos x_1$$

and

$$L_0 = -\Big(2\frac{\partial^2}{\partial x_1^2} + \frac{3}{4}\frac{\partial^2}{\partial x_2^2} + \frac{5}{4}\frac{\partial^2}{\partial x_3^2}\Big).$$

L_ε can also be written as

$$\begin{aligned}L_\varepsilon = -2\frac{\partial^2}{\partial x_1^2} - 2\cos\frac{x_1}{\varepsilon}\frac{\partial^2}{\partial x_1\partial x_2} - 2\sin\frac{x_1}{\varepsilon}\frac{\partial^2}{\partial x_1\partial x_3} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}\\ + \frac{1}{\varepsilon}\sin\frac{x_1}{\varepsilon}\frac{\partial}{\partial x_2} - \frac{1}{\varepsilon}\cos\frac{x_1}{\varepsilon}\frac{\partial}{\partial x_3}.\end{aligned} \tag{1.5}$$

The Harnack inequalities. For $\ell = 1$, (1.2) is a parabolic analogue of (1.4) and it can be proved in a similar way (for details, see Section 7).

Let us now see why (1.2) is not true for $\ell \geq 2$. Let us take $f = 0$ and $g = x_3 + 2$ in (1.3). Then $u_0 = x_3 + 2$. Hence $u_0 \geq 0$, $\frac{\partial}{\partial x_3}u = 1$ and $\frac{\partial}{\partial x_1}u_0 = \frac{\partial}{\partial x_2}u_0 = 0$.

Since $L_\varepsilon\frac{\partial}{\partial x_i}u_\varepsilon = \frac{\partial}{\partial x_i}L_\varepsilon u_\varepsilon = 0$, $i = 2, 3$, it follows from Theorem 1.1 that

$$u_\varepsilon \longrightarrow u_0 \text{ and } \frac{\partial}{\partial x_i}u_\varepsilon \longrightarrow \frac{\partial}{\partial x_i}u_0,\ (\varepsilon \longrightarrow 0),\ i = 2, 3 \tag{1.6}$$

uniformly on the compact subsets of $B(0, 1)$.
Moreover , it follows from Theorem 1.2 that there is $c > 0$ such that

$$\Big|\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u_\varepsilon(x)\Big| \leq c,\ x \in B(0, 1)\ \ i = 1, 2, 3,\ j = 2, 3. \tag{1.7}$$

Now, (1.5),(1.6) and (1.7) imply that

$$\frac{\partial^2}{\partial x_1^2}u_\varepsilon(0) \sim \frac{1}{\varepsilon}, \quad (\varepsilon \longrightarrow 0).$$

It follows that the family of functions u_t, $t \geq 1$ defined by $u_t(x) = u_\varepsilon(\varepsilon x)$, $\varepsilon = \frac{1}{t}$ satisfy (0.3).

The Riesz transforms. The construction of the family v_t, $t \geq 1$, mentioned in (0.4) is similar to the construction of the family u_t, $t \geq 1$ above, *i.e.* we can consider, in (1.3), $g = 0$ and $f = L_0\varphi$, where $\varphi \in C_0^\infty\big(B(0, 1)\big)$ is such that $\frac{\partial}{\partial x_3}\varphi \neq 0$ and then proceed in the same way.

Let us now see how we can prove that the Riesz transforms $E_iL^{-\frac{1}{2}}$ and $L^{-\frac{1}{2}}E_i$, $i = 1, 2, 3$ are bounded on L^q, $1 < q < +\infty$ and from L^1 to weak-L^1.

It follows from the observation

$$\sum_{1\leq i\leq 3}\|E_iL^{-\frac{1}{2}}\varphi\|_2^2 = -\sum_{1\leq i\leq 3}(E_i^2L^{-\frac{1}{2}}\varphi, L^{-\frac{1}{2}}\varphi) = (L^{-\frac{1}{2}}\varphi, L^{-\frac{1}{2}}\varphi) = \|\varphi\|_2^2$$

that the tranforms $E_iL^{-\frac{1}{2}}$, as well as their adjoints $L^{-\frac{1}{2}}E_i$, are bounded on L^2. So it is enough to prove that they are bounded from L^1 to weak-L^1. Then by interpolation we can prove that they are bounded on L^q, $1 < q < 2$ and by duality on L^q, $2 < q < \infty$ (*cf.* [20]).

Let us use the notation $E_j^yK(x,y)$ to denote the derivative of the kernel $K(x,y)$ with respect to the variable y with respect to the vector field E_j.

Let $K_i(x,y)$ be the kernel of the transform $E_iL^{-\frac{1}{2}}$.

If $p_t(x,y)$ is the heat kernel (*i.e.* the fundamental solution of the equation $(\partial/\partial t+L)u = 0$) then

$$(1.8) \qquad K_i(x,y) = \int_0^\infty t^{-\frac{1}{2}}E_i^x p_t(x,y)\,dt \text{ and } E_j^yK_i(x,y) = \int_0^\infty t^{-\frac{1}{2}}E_j^yE_i^x p_t(x,y)\,dt.$$

Since the function $u(t,y) = E_i^xp_t(x,y)$ satisfies $(\partial/\partial t+L)u = 0$, Theorem 1 can be applied to both $E_i^xp_t(x,y)$ and $E_j^yE_i^xp_t(x,y)$ and using well known Gaussian estimates of the heat kernel $p_t(x,y)$ (*cf.* [1]) we can deduce from (1.8) that there is $c > 0$ such that

$$(1.9) \qquad |K_i(x,y)| \leq \frac{c}{|x-y|^3}, \text{ and } |E_j^yK_i(x,y)| \leq \frac{c}{|x-y|^4}.$$

It is well known that once we have the estimates (1.9) then it can be proved that $E_iL^{-\frac{1}{2}}$ is bounded from L^1 to weak-L^1 (*cf.* [20]).

Unfortunately, the estimates (1.9) are not availlable for the kernels $K_i^*(x,y)$ of the tranforms $L^{-\frac{1}{2}}E_i$, since in that case we have that

$$(1.10) \qquad K_i^*(x,y) = \int_0^\infty t^{-\frac{1}{2}}E_i^y p_t(x,y)\,dt \text{ and } E_j^yK_i^*(x,y) = \int_0^\infty t^{-\frac{1}{2}}E_j^yE_i^y p_t(x,y)\,dt.$$

and, as we have seen, the Harnack inequalities needed to estimate $E_j^yE_i^yp_t(x,y)$ are not true.

The way to get around this difficulty, is to observe , as it is clear from (1.1), that the natural fields to use are the $\frac{\partial}{\partial x_i}$ and not the E_i, $i = 1,2,3$. Indeed, in that case we can take advantage of the fact that for $i = 2,3$, $\frac{\partial}{\partial x_i}$ and L commute and obtain the desired estimates for $\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}p_t(x,y)$, $i = 2,3, j = 1,2,3$ applying Theorem 1 twice and then prove that the transforms $L^{-\frac{1}{2}}\frac{\partial}{\partial x_2}$ and $L^{-\frac{1}{2}}\frac{\partial}{\partial x_3}$ are bounded from L^1 to weak-L^1 (their L^2 boundedness follows from that of the transforms $L^{-\frac{1}{2}}E_i$, $i = 1,2,3$, proved above).

In order to prove that the transform $L^{-\frac{1}{2}}\frac{\partial}{\partial x_1}$, is bounded from L^1 to weak-L^1, we argue as follows. We consider the transform $L^{-\frac{1}{2}}H$ where the vector field H is defined by

$$H = \frac{\partial}{\partial x_1} + \left(\frac{\partial}{\partial x_1}\chi^2\right)\frac{\partial}{\partial x_2} + \left(\frac{\partial}{\partial x_1}\chi^3\right)\frac{\partial}{\partial x_3} = \frac{\partial}{\partial x_1} + \frac{1}{2}\cos x_1\frac{\partial}{\partial x_2} + \frac{1}{2}\sin x_1\frac{\partial}{\partial x_3}.$$

Observe that

$$\frac{\partial}{\partial x_1}H = -\frac{1}{2}\left(L + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \cos x_1\frac{\partial^2}{\partial x_1\partial x_2} + \sin x_1\frac{\partial^2}{\partial x_1\partial x_3}\right).$$

So, as it has already been explained above, we can estimate $\frac{\partial}{\partial x_i}Hp_t(x,y)$, $i = 1,2,3$ applying Theorem 1 twice and prove the estimates (1.8) for the kernel $K_H(x,y)$ of the transform $L^{-\frac{1}{2}}H$, which in turn implies that $L^{-\frac{1}{2}}H$, hence $L^{-\frac{1}{2}}\frac{\partial}{\partial x_1}$, is bounded from L^1 to weak-L^1.

In the following sections, we shall generalise these ideas in any connected Lie group of polynomial growth.

2. **The structure of the Lie algebra.** Let $\mathfrak{g}$ be a Lie algebra and denote by $\mathfrak{q}, \mathfrak{n}$ and $\mathfrak{m}$ respectively the radical, the nil-radical and a Levi sub-algebra of $\mathfrak{g}$. $\mathfrak{q}$ and $\mathfrak{n}$ are, respectively, solvable and nilpotent ideals and $\mathfrak{m}$ a semisimple subalgebra of $\mathfrak{g}$. Moreover (*cf.* [22])

$$\mathfrak{n} \subseteq \mathfrak{q},\ \mathfrak{g} = \mathfrak{q} + \mathfrak{m},\ \mathfrak{q} \cap \mathfrak{m} = \{0\},\ [\mathfrak{q}, \mathfrak{g}] \subseteq \mathfrak{n}. \tag{2.1}$$

We denote by π the natural map $\pi\colon \mathfrak{q} \longrightarrow \mathfrak{q}/\mathfrak{n}$ and we put $k = \dim(\mathfrak{q}/\mathfrak{n})$.

We denote by $\operatorname{ad} X = S(X) + K(X)$ the Jordan decomposition of the derivation $\operatorname{ad} X(Y) = [X, Y]$, $X \in \mathfrak{g}$. $S(X)$ is the semisimple and $K(Y)$ the nilpotent part. It is well known that

(i) $S(X)$ and $K(X)$ are derivations of $\mathfrak{g}$ (*cf.* [22]).

(ii) There are real polynomials $s(x)$ and $k(x)$ such that

$$S(X) = s(\operatorname{ad} X) \text{ and } K(X) = k(\operatorname{ad} X) \tag{2.2}$$

(*cf.* [16]).

(iii)

$$[S(X), K(X)] = 0. \tag{2.3}$$

Notice that the fact that $\operatorname{ad} X(X) = [X, X] = 0$, $X \in \mathfrak{g}$ implies that the constant coefficients of the polynomials $k(x)$, hence also of the polynomials $s(x)$, are zero.

LEMMA 1.1. *There are vectors $Y_1, \ldots, Y_k \in \mathfrak{q}$ such that*

a) $[S(Y_i), S(Y_j)] = 0, [Z, Y_i] = 0,\ 1 \le i, j \le k,\ Z \in \mathfrak{m}$,

b) $\{\pi(Y_1), \ldots, \pi(Y_k)\}$ *is a basis of* $\mathfrak{q}/\mathfrak{n}$.

PROOF. Let $\{Z_1, \ldots, Z_k\}$ any choice of vectors of $\mathfrak{q}$ such that $\{\pi(Z_1), \ldots, \pi(Z_k)\}$ is a basis of $\mathfrak{q}/\mathfrak{n}$. To prove the lemma it is enough to prove that for every integer $1 \le m \le k$ we can choose vectors $Y_1, \ldots, Y_m \in \mathfrak{q}$ such that

$$\begin{gathered}[S(Y_i), S(Y_j)] = 0,\ [Z, Y_i] = 0,\ 1 \le i, j \le m, Z \in \mathfrak{m} \\ \{\pi(Y_1), \ldots, \pi(Y_m), \pi(Z_{m+1}), \ldots, \pi(Z_k)\} \text{ basis of } \mathfrak{q}/\mathfrak{n}.\end{gathered} \tag{2.4}$$

(2.4) will be proved by induction on m. For $m = 1$ observe that (2.1) together with the fact that $\mathfrak{m}$ is semisimple imply that $\mathfrak{q}$ has a subspace $\mathfrak{b}$ which is complementary to $\mathfrak{n}$, *i.e.* such that $\mathfrak{q} = \mathfrak{b} \oplus \mathfrak{n}$ and $\operatorname{ad} Z(\mathfrak{b}) = \{0\}, Z \in \mathfrak{m}$ (*cf.* [16]). For Y_1 we choose any nonzero element of $\mathfrak{b}$ such that $\pi(Y_1)$ is linearly independent from the vectors $\pi(Z_2), \ldots, \pi(Z_k)$. Assume now that (2.4) is true for $m = j, 1 \leq j < k$. To prove that it is also true for $m = j + 1$ assume that the vectors $Y_1, \dots, Y_j$ have been chosen and consider the linear space $\mathfrak{d}$ that is generated by $\mathfrak{n}$ and the vectors $Z_{j+1}, \dots, Z_k$. It follows from the fact that $[\mathfrak{q}, \mathfrak{g}] \subseteq \mathfrak{n}$ that $\mathfrak{d}$ is actually an ideal of $\mathfrak{g}$. Furthermore $\mathfrak{n}$ is an invariant subspace of $\mathfrak{d}$ with respect to the Lie algebra of linear tranformations generated by the $\{\operatorname{ad} Z, S(Y_i), Z \in \mathfrak{m}, 1 \leq i \leq j\}$. Hence, $\mathfrak{d}$ has a subspace $\mathfrak{b}$ that is complementary to $\mathfrak{n}$, *i.e.* such that $\mathfrak{d} = \mathfrak{b} \oplus \mathfrak{n}$ and $\operatorname{ad} Z(\mathfrak{b}) = \{0\}, Z \in \mathfrak{m}, S(Y_i)(\mathfrak{b}) = \{0\}, 1 \leq i \leq j$. For Y_{j+1} we choose any nonzero element of $\mathfrak{b}$ such that $\pi(Y_{j+1})$ is linearly independent from the vectors $\pi(Z_{j+2}), \dots, \pi(Z_k)$. $S(Y_{j+1})$ will commute with the derivations $S(Y_1), .., S(Y_j)$ because of (2.2) and the fact that $S(Y_i)Y_{j+1} = 0, 1 \leq i \leq j$. This proves (2.4) and the lemma follows.

PROPOSITION 2.2. *There are vectors* $X_1, \dots, X_k \in \mathfrak{q}$, $k = \dim \mathfrak{q} / \mathfrak{n}$, $X_1, \dots, X_k \in \mathfrak{q}$, $k = \dim \mathfrak{q} / \mathfrak{n}$, *such that*

a) $S(X_i)X_j = 0$, $\operatorname{ad} Z(X_i) = 0$, $Z \in \mathfrak{m}$, $1 \leq i, j \leq k$.

b) $\{\pi(X_1), \dots, \pi(X_k)\}$ *is a basis of* $\mathfrak{q} / \mathfrak{n}$.

PROOF. Let $Y_1, \dots, Y_k \in \mathfrak{q}$ as in the Lemma 2.1 above and denote by $\mathfrak{s}$ be the Lie algebra of linear transformations of $\mathfrak{q}$ generated by $\operatorname{ad} Z$, $S(Y_i)$, $Z \in \mathfrak{m}$, $1 \leq i \leq k$. $\mathfrak{n}$ is invariant with respect to $\mathfrak{s}$. Hence, there is a complementary subspace $\mathfrak{b}$ to $\mathfrak{n}$ such that $\mathfrak{q} = \mathfrak{n} \oplus \mathfrak{b}$ and $A(\mathfrak{b}) \subseteq \mathfrak{b}, A \in \mathfrak{s}$ (*cf.* [16]). (2.1) and (2.3) imply that $\operatorname{ad} Z(\mathfrak{b}) = S(Y_i)\mathfrak{b} = \{0\}, Z \in \mathfrak{m}, 1 \leq i \leq k$.

Let $N_1, \dots, N_k \in \mathfrak{n}$ such that $X_i = Y_i - N_i \in \mathfrak{b}$, $i = 1, \dots, k$. The vectors $X_1, \dots, X_k$ have all the properties required by the proposition: they satisfy b) since they form a basis of $\mathfrak{b}$. It is also clear that $\operatorname{ad} Z(X_i) = 0, Z \in \mathfrak{m}$. So it only remains to verify that $S(X_i)X_j = 0$. This will follow from the fact that $S(X_i) = S(Y_i)$. This assertion is proved as follows. First we observe that $S(Y_i)X_i = 0$ and that $S(Y_i)N_i = S(Y_i)Y_i - S(Y_i)X_i = 0$. Next we observe that since $K(Y_i)$ is a derivation we have that $[K(Y_i), \operatorname{ad} N_i] = \operatorname{ad}\big(K(Y_i)N_i\big)$, which combined with the fact that $K(Y_i)N_i \in \mathfrak{n}$ implies that the linear transformation $[K(Y_i), \operatorname{ad} N_i]$ is nilpotent. This in turn implies that although the $K(Y_i)$ and $\operatorname{ad} N_i$ do not commute, we can nevertheless find $m \in \mathbf{N}$ such that $\big(K(Y_i) + \operatorname{ad} N_i\big)^m = 0$, *i.e.* $K(Y_i) + \operatorname{ad} N_i$ is a nilpotent transformation. In other words we have proved that $S(Y_i)$ and $K(Y_i) + \operatorname{ad} N_i$ are semisimple and nilpotent transformations respectively and that they commute. Since $\operatorname{ad} X_i = S(Y_i) + K(Y_i) + \operatorname{ad} N_i$, it follows from the uniqueness of the Jordan decomposition that $S(Y_i)$ is the semisimple part of $\operatorname{ad} X_i$ and the proposition follows.

In what follows we shall consider and fix, once and for all, vectors $X_1, \dots, X_k \in \mathfrak{q}$ having the properties described in the above proposition.

The nil-shadow $\mathfrak{q}_N$ *of* $\mathfrak{q}$. We can easily see that the conditions

$$[X_i, X_j]_N = [X_i, X_j],\ [X_i, Y]_N = K(X_i)Y,\ [Y, Z]_N = [Y, Z],\ 1 \leq i, j \leq k,\ Y, Z \in \mathfrak{n}$$

define a unique product $[.,.]_N$ on the linear space $\mathfrak{q}$. We can verify directly (writing the elements X of $\mathfrak{q}$ as a sum $X = X' + Y$ with X' a linear combination of the vectors $X_1, \dots, X_k$ and $Y \in \mathfrak{n}$) that $[.,.]_N$ satisfies the Jacobi identity. So, $\mathfrak{q}_N = (\mathfrak{q}, [.,.]_N)$ is a Lie algebra, which is also nilpotent. $\mathfrak{q}_N$ is called the nil-shadow of $\mathfrak{q}$.

The filtration of $\mathfrak{q}$. We put $\mathfrak{r}_1 = \mathfrak{q}$ and $\mathfrak{r}_{i+1} = [\mathfrak{r}_1, \mathfrak{r}_i]_N$, $i \geq 1$. Then, since $\mathfrak{q}_N$ is nilpotent, we have the following filtration of $\mathfrak{q}$:

$$\mathfrak{q} = \mathfrak{r}_1 \supseteq \mathfrak{n} \supseteq \mathfrak{r}_2 \supseteq \cdots \supseteq \mathfrak{r}_m \supseteq \mathfrak{r}_{m+1} = \{0\}, \quad \mathfrak{r}_m \neq \{0\}.$$

PROPOSITION 2.3. *1)* $\mathfrak{r}_1 \supseteq \mathfrak{n} \supseteq \mathfrak{r}_2$.

2) $\mathfrak{r}_i$ *is an ideal of* $\mathfrak{g}$, $i = 1, 2, \ldots$

3) There are subspaces $\mathfrak{a}_1, \ldots, \mathfrak{a}_m$ *of* $\mathfrak{q}$ *such that*

a) $\operatorname{ad} Z(\mathfrak{a}_i) \subseteq \mathfrak{a}_i$, $S(X_j)\mathfrak{a}_i \subseteq \mathfrak{a}_i$, $Z \in \mathfrak{m}$, $j = 1, \ldots, k$, $i = 1, \ldots, m$

b) $\mathfrak{r}_i = \mathfrak{a}_i \oplus \cdots \oplus \mathfrak{a}_m$ *and*

c) $\mathfrak{a}_i = \mathfrak{a}_{0i} \oplus \mathfrak{a}_{1i} \oplus \mathfrak{a}_{2i}$, *where* $\mathfrak{a}_{0i} = \{Y \in \mathfrak{a}_i, S(X_j)Y = 0, 1 \leq j \leq k, \operatorname{ad} Z(Y) = 0, Z \in \mathfrak{m}\}$, $\mathfrak{a}_{0i} \oplus \mathfrak{a}_{1i} = \{Y \in \mathfrak{a}_i, S(X_j)Y = 0, 1 \leq j \leq k\}$, $\operatorname{ad} Z(\mathfrak{a}_{1i}) \subseteq \mathfrak{a}_{1i}$, $Z \in \mathfrak{m}$, $S(X_j)\mathfrak{a}_{2i} \subseteq \mathfrak{a}_{2i}$, $1 \leq j \leq k$, $\operatorname{ad} Z(\mathfrak{a}_{2i}) \subseteq \mathfrak{a}_{2i}$, $Z \in \mathfrak{m}$.

PROOF. 1) follows from (2.1) and the way $[.,.]_N$ was defined. 2) can be proved by induction. It is trivially true for $i = 1$. So, assume that it is true for $i = n$. We are going to verify that it is also true for $i = n + 1$.

Let $Y \in \mathfrak{r}_1$, $Z \in \mathfrak{r}_i$. If $X \in \mathfrak{n}$, then $\operatorname{ad} X([Y, Z]_N) = \big[X, [Y, Z]_N\big]_N \in \mathfrak{r}_{n+2} \subseteq \mathfrak{r}_{n+1}$. If $X \in \mathfrak{m}$, $Z \in \mathfrak{n}$ and $Y = X_j$ for some $1 \leq j \leq k$, then $\operatorname{ad} X([X_j, Z]_N) = \operatorname{ad} X K(X_j)Z = K(X_j)\operatorname{ad} X(Z) = [X_j, \operatorname{ad} X(Z)]_N \in \mathfrak{r}_{n+1}$, since $\operatorname{ad} X(X_j) = 0$ and $K(X_j)$ is a polynomial in $\operatorname{ad} X_j$. If $X \in \mathfrak{m}$, $Z = X_\ell$ and $Y = X_j$ for some $1 \leq j, \ell \leq k$, then $\operatorname{ad} X[X_\ell, X_j]_N = \operatorname{ad} X[X_\ell, X_j] = 0$. If $Z \in \mathfrak{n}$, $Y = X_j$ and $X = X_\ell$ for some $1 \leq j, \ell \leq k$, then $\operatorname{ad} X_\ell([X_j, Z]_N) = \operatorname{ad} X_\ell K(X_j)Z = K(X_\ell)K(X_j)Z + S(X_\ell)K(X_j)Z = K(X_\ell)K(X_j)Z + K(X_j)S(X_\ell)Z$, since $S(X_\ell)X_j = 0$ and $K(X_j)$ is a polynomial in $\operatorname{ad} X_j$. Hence $\operatorname{ad} X_\ell([X_j, Z]_N) = \big[X_\ell, [X_j, Z]_N\big]_N + [X_j, S(X_\ell)Z]_N \in \mathfrak{r}_{n+1}$. Finally, if $X = X_h$, $Y = X_\ell$ and $Z = X_j$ for some $1 \leq h, \ell, j \leq k$, then $\operatorname{ad} X_h([X_\ell, X_j]_N) = \big[X_h, [X_\ell, X_j]_N\big]_N \in \mathfrak{r}_{n+2} \subseteq \mathfrak{r}_{n+1}$. Since the general case is a linear combination of the cases examined above, we conclude that $\mathfrak{r}_{n+1}$ is also an ideal of $\mathfrak{g}$. This proves the inductive step and 2) follows.

3a) and 3b) follow from the observation that, according to 2), the spaces $\mathfrak{r}_1, \ldots, \mathfrak{r}_m$ are invariant with respect to the Lie algebra of linear transformations of $\mathfrak{q}$ generated by the tranformations $\operatorname{ad} Z, S(X_i)$, $Z \in \mathfrak{m}, i = 1, \ldots, k$ (*cf.* [16]). Given 3a) and 3b), 3c) follows again from the observation that $\mathfrak{a}_{0i}$ and $\mathfrak{a}_{0i} \oplus \mathfrak{a}_{1i}$ are invariant with respect to the Lie algebra of linear transformations of $\mathfrak{q}$ generated by the tranformations $\operatorname{ad} Z$, $S(X_i)$, $Z \in \mathfrak{m}, i = 1, \ldots, k$.

We put $n = \dim \mathfrak{q}$, $n_0 = 0$ and $n_i = \dim(\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_i)$, $i = 1, \ldots, m$. Then

$$1 \leq k \leq n_1 < \cdots < n_m = n.$$

The choice of the basis of $\mathfrak{q}$. Assume now that $\mathfrak{g}$, hence $\mathfrak{q}$ is of type R, *i.e.* that all the eigenvalues of the derivations $\operatorname{ad} X$, $X \in \mathfrak{g}$ are purely imaginary (*i.e.* of the type $ia, a \in \mathbb{R}$).

PROPOSITION 2.4. *If* $\mathfrak{g}$ *is of type R, then there is a basis* $\{X_1, \ldots, X_n\}$ *of* $\mathfrak{q}$ *such that*

1) $X_1, \ldots, X_k$ *are as above and* $X_{k+1}, \ldots, X_n \in \mathfrak{n}$, *2)* $\{X_{n_{i-1}+1},\dots,X_{n_i}\}$ *is a basis of* $\mathfrak{a}_i$, $i=1,\dots,m$

3) $\{X_{n_{i-1}+1},\dots,X_{n_{0i}}\},\{X_{n_{0i}+1},\dots,X_{n_{1i}}\}$ *and* $\{X_{n_{1i}+1},\dots,X_{n_i}\}$ *are basis of* $\mathfrak{a}_{0i}$, $\mathfrak{a}_{1i}$ *and* $\mathfrak{a}_{2i}$ *respectively,* $i=0,\dots,m$ *and*

4) *the number of the vectors* $\{X_{n_{1i}+1},\dots,X_{n_i}\}$ *is even and they can be combined in pairs* $\{X_{n_{1i}+1},X_{n_{1i}+2}\},\dots,\{X_j,X_{j+1}\},\dots,\{X_{n_i-1},X_{n_i}\}$ *so that for every pair* $\{X_j,X_{j+1}\}$ *and every* $\ell=1,\dots,k$ *there is* $a_\ell\in\mathbb{R}$ *such that*

$$
\begin{aligned}
e^{S(X_\ell)}X_j &= \cos a_\ell X_j+\sin a_\ell X_{j+1}\\
e^{S(X_\ell)}X_{j+1} &= -\sin a_\ell X_j+\cos a_\ell X_{j+1}.
\end{aligned}
\tag{2.5}
$$

PROOF. For $\{X_{n_{i-1}+1},\dots,X_{n_{0i}}\}$ and $\{X_{n_{0i}+1},\dots,X_{n_{1i}}\}$ we choose any basis of $\mathfrak{a}_{0i}$ and $\mathfrak{a}_{1i}$ respectively, so that 1) is satisfied. In order to choose $\{X_{n_{1i}+1},\dots,X_{n_i}\}$ let us denote by $\mathfrak{a}_{2i,\mathbb{C}}$ the complexification of $\mathfrak{a}_{2i}$ and denote by $S(X_j)_\mathbb{C}$ the extension of $S(X_j)$ to $\mathfrak{a}_{2i,\mathbb{C}}$, $i=1,\dots,k$. Since $S(X_j)_\mathbb{C}$ is also semisimple, we can decompose $\mathfrak{a}_{2i,\mathbb{C}}$ as $\mathfrak{a}_{2i,b_1}\oplus\cdots\oplus\mathfrak{a}_{2i,b_h}$ where $\mathfrak{a}_{2i,b_\ell}=\{Y\in\mathfrak{a}_{2i,\mathbb{C}},S(X_j)_\mathbb{C}(Y)=ib_\ell Y\}$ and $ib_1,\dots,ib_h\in i\mathbb{R}$ are the different eigenvalues of $S(X_j)_\mathbb{C}$. Since $S(X_\ell)_\mathbb{C}S(X_j)_\mathbb{C}=S(X_j)_\mathbb{C}S(X_\ell)_\mathbb{C}$, $\ell=1,\dots,k$, $S(X_\ell)_\mathbb{C}\mathfrak{a}_{2i,b_s}\subseteq\mathfrak{a}_{2i,b_s}$, $s=1,\dots,h$. We can apply the same procedure to $\mathfrak{a}_{2i,b_s}$ relative to any other $S(X_\ell)_\mathbb{C}$. This leads to a decomposition

$$
\mathfrak{a}_{2i,\mathbb{C}}=\mathfrak{b}_1\oplus\cdots\oplus\mathfrak{b}_s \tag{2.6}
$$

of $\mathfrak{a}_{2i,\mathbb{C}}$ into $\{S(X_j)_\mathbb{C},j=1,\dots,k\}$-invariant subspaces, such that the linear tranformations induced in the $\mathfrak{b}_\ell$ by every $S(X_j)_\mathbb{C}$ are scalar multiplications by some ia, $a\in\mathbb{R}$. Moreover the subspaces $\mathfrak{b}_\ell$ can be taken to be one-dimensional. Let us identify $\mathfrak{a}_{2i,\mathbb{C}}$ with $\{Z+iE,Z,E\in\mathfrak{a}_{2i}\}$ and put $\bar{Y}=Z-iE$, $\operatorname{Re}Y=Z$, $\operatorname{Im}Y=E$ for $Y=Z+iE\in\mathfrak{a}_{2i,\mathbb{C}}$, $Z,E\in\mathfrak{a}_{2i}$ and $\bar{A}=\{\bar{Y},\,Y\in A\}$ for $A\subseteq\mathfrak{a}_{i,\mathbb{C}}$. We observe that if ia, $a\in\mathbb{R},a\neq0$ is an eigenvalue of $S(X_j)_\mathbb{C}$ then $-ia$ is also an eigenvalue of the same multiplicity and that if Y is an eigenvector for ia, $Y\neq0$ then $\operatorname{Re}Y\neq0$, $\operatorname{Im}Y\neq0$, $\operatorname{Re}Y\neq\operatorname{Im}Y$ and $\bar{Y}$ is an eigenvector for the eigenvalue $-ia$. Using this observation we can easily see that the subspaces $\mathfrak{b}_\ell$ can be chosen in such a way, that the decomposition (2.6) can be written as

$$
\mathfrak{a}_{2i,\mathbb{C}}=\mathfrak{b}_{i_1}\oplus\bar{\mathfrak{b}}_{i_1}\oplus\cdots\oplus\mathfrak{b}_{i_r}\oplus\bar{\mathfrak{b}}_{i_r}
$$

where $\mathfrak{b}_\ell=\{zY_\ell,\,z\in\mathbb{C}\}$ for some $Y_\ell\in\mathfrak{a}_{2i,\mathbb{C}}$, $Y_\ell=Z+iE$, $Z,E\in\mathfrak{a}_{2i}$, $Z\neq E$, $Z,E\neq0$.

We take $X_{n_{1i}+1}=\operatorname{Re}Y_{i_1}$, $X_{n_{1i}+2}=\operatorname{Im}Y_{i_1},\dots,X_{n_i-1}=\operatorname{Re}Y_{i_r}$, $X_{n_i}=\operatorname{Im}Y_{i_r}$. We can easily see that the basis of $\mathfrak{q}$, constructed in this way, satisfies the requirements of the proposition.

CONVENTION. In the rest of this article we shall fix a basis $\{X_1,\dots,X_n\}$ of $\mathfrak{q}$ having the properties described in the above proposition.

3. **The exponential coordinates of the second kind.** Let G be a connected Lie group of polynomial growth and $\mathfrak{g}$ its Lie algebra. According to a wellknown theorem of Y. Guivarc'h [14], $\mathfrak{g}$ is of type R, *i.e.* all the eigenvalues of the derivations $\operatorname{ad} X(Y) = [X, Y], X, Y \in \mathfrak{g}$ are imaginary. We identify the elements of $\mathfrak{g}$ with the left invariant vector fields on G. We denote by $\operatorname{Ad} z$ the differential of the inner automorphim $g \longrightarrow zgz^{-1}$ of G.

Let $\mathfrak{q}$,$\mathfrak{n}$ and $\mathfrak{m}$ be as in Section 2. Let $S(X)$, $K(X)$, $X \in \mathfrak{g}$ be as in Section 2 and the basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{q}$ as in Proposition 2.3.

Denote by Q, N and M the analytic subgroups of G having Lie algebras $\mathfrak{q}$,$\mathfrak{n}$ and $\mathfrak{m}$ respectively. Q and N are closed normal analytic subgroups of G, solvable and nilpotent, respectively and M a maximal semisimple analytic subgroup of G. The assumption that G is of polynomial growth implies that M is compact and therefore closed. Moreover

$$G = QM, \; N \subseteq Q \text{ and } Q/N \text{ is abelian.} \tag{3.1}$$

When G is simply connected then Q, N and M are also simply connected and $Q \cap M = \{e\}$. Hence G is isomorphic to the semidirect product $Q \times_\tau M$, where $\tau: z \longrightarrow \operatorname{Aut}(Q): z \longrightarrow \tau_z$, $\operatorname{Aut}(Q)$ the set of automorphisms of Q and $\tau_z(g) = zgz^{-1}$, $g \in Q$, $z \in M$. When G is not simply connected, then $Q \cap M \subseteq$ center (M) and the mapping

$$Q \times_\tau M \longrightarrow G = QM, \; (x, z) \longrightarrow xz$$

is a covering of G (*cf.* [22], p. 255, exercise 41). Hence there is a central discrete subgroup Γ of $Q \times_\tau M$, isomorphic with $Q \cap M$, such that G is isomorphic with $Q \times_\tau M/\Gamma$. The fact that M is compact implies that Γ is finite.

PROPOSITION 3.1. *Let X and Z be left invariant vector fields on $Q \times_\tau M$, $X \in \mathfrak{q}$, $Z \in \mathfrak{m}$. Then*

$$(X + Z)(x, z) = \big(\operatorname{Ad} z(X)(x), Z(z)\big), \quad x \in Q, \; z \in M. \tag{3.2}$$

PROOF. Let $x \in Q$, $z \in M$, $g = (x, z)$ and $t \in \mathbb{R}$. Then the proposition follows from the observation that

$$g \exp tZ = (x, z) \exp tZ = (x, z \exp tZ)$$
$$g \exp tX = (x, z) \exp tX = (xz \exp tXz^{-1}, z) = \big(x \exp t \operatorname{Ad} z(X), z\big).$$

3.1 *The simply connected case.* Let $\bar{G}$ be the universal covering of G. Then $\bar{G}$ is simply connected and we denote by $\bar{Q}$, $\bar{N}$ and $\bar{M}$ the analytic subgroups of $\bar{G}$ whose Lie algebras are $\mathfrak{q}$, $\mathfrak{n}$ and $\mathfrak{m}$ respectively.

It is well known (*cf.* [22]) that the map

$$\phi: \mathbb{R}^n \longrightarrow \bar{Q}, \; \phi : x = (x_n, \ldots, x_1) \longrightarrow \exp x_n X_n \cdots \exp x_1 X_1$$

is a diffeomorphism, called exponential coordinates of the second kind.

We want to give an expression for $d\phi^{-1}$. To this end, we shall need some notations. We denote by $\overline{\mathrm{ad}}X_i$ and $\bar{K}(X_i)$ the linear transformations of $\mathfrak{q}$ defined by

$$\overline{\mathrm{ad}}(X_i)X_j = 0, \text{ for } i \geq j \text{ and } \overline{\mathrm{ad}}(X_i)X_j = \mathrm{ad}(X_i)X_j, \text{ for } i < j$$
$$\bar{K}(X_i)X_j = 0, \text{ for } i \geq j \text{ and } \bar{K}(X_i)X_j = K(X_i)X_j, \text{ for } i < j.$$

It follows from (2.2) and the fact that $S(X_i)X_j = 0$, $1 \leq i, j \leq k$ that

$$S(X_i)\bar{K}(X_j) = \bar{K}(X_j)S(X_i), \quad 1 \leq i, j \leq k. \tag{3.3}$$

If $B(x) = b_n(x)\frac{\partial}{\partial x_n} + \cdots + b_1(x)\frac{\partial}{\partial x_1}$ is a vector field on $\mathbb{R}^n$, then we put $\mathrm{pr}_i B(x) = b_i(x)$. We also use the same notation for the left invariant vector fields on Q, *i.e.* if $E = c_nX_n + \cdots + c_1X_1$, then we put $\mathrm{pr}_i E = c_i$.

We also put $\sigma(i) = j$, if $u_{j-1} < i \leq n_j$ ($n_0, \ldots, n_m$ are as in Section 2).

PROPOSITION 3.2. *With the above notations we have*

$$\begin{aligned}
\mathrm{pr}_i\, d\phi^{-1}E(x) &= \mathrm{pr}_i\Big[e^{x_n \overline{\mathrm{ad}}X_n} \cdots e^{x_1\overline{\mathrm{ad}}X_1}\Big](E) \\
&= \mathrm{pr}_i\Big[e^{x_n\bar{K}(X_n)} \cdots e^{x_1\bar{K}(X_1)} e^{x_kS(X_k)} \cdots e^{x_1S(X_1)}\Big](E) \\
&= \mathrm{pr}_i\bigg\{\bigg[\sum_{\lambda_1\sigma(1)+\cdots+\lambda_{i-1}\sigma(i-1)\leq\sigma(i)-1} x_1^{\lambda_1}\cdots x_{i-1}^{\lambda_{i-1}} \\
&\qquad \bar{K}^{\lambda_{i-1}}(X_{i-1})\cdots\bar{K}^{\lambda_1}(X_1)\bigg] e^{x_kS(X_k)}\cdots e^{x_1S(X_1)}\bigg\}(E).
\end{aligned} \tag{3.4}$$

PROOF. Clearly, the third equality in (3.4) is a more explicit version of the second one and the second equality follows immediately from the first one using (3.3). So it is enough to prove the first equality in (3.4).

Let $g = \exp x_nX_n \cdots \exp x_1X_1 \in \bar{Q}$ and $\gamma(t) = g \exp tE$, $t > 0$ an integral curve of E. Then to prove the proposition it is enough to prove that

$$\begin{aligned}
\gamma(t) &= \exp\Big(x_n + t\,\mathrm{pr}_n\, e^{x_{n-1}\overline{\mathrm{ad}}X_{n-1}} \cdots e^{x_1\overline{\mathrm{ad}}X_1}E + O(t^2)\Big)X_n \\
&\qquad \exp\Big(x_2 + t\,\mathrm{pr}_2\, e^{x_1\overline{\mathrm{ad}}X_1}E + O(t^2)\Big)X_2 \exp(x_1 + t\,\mathrm{pr}_1\, E)X_1.
\end{aligned} \tag{3.5}$$

(3.5) can be proved by induction on n: It is trivially true for $n = 1$. So assume that it is true for $n \leq \ell$. To prove that it is also true for $n = \ell + 1$, observe that it follows from the Campell-Hausdorff formula that

$$\begin{aligned}
\exp tE &= \exp t(c_{\ell+1}X_{\ell+1} + \cdots + c_1X_1) \\
&= \exp\Big[\big(tc_{\ell+1} + O(t^2)\big)X_{\ell+1} + \cdots + \big(tc_2 + O(t^2)\big)X_2\Big] \exp c_1tX_1.
\end{aligned}$$

Hence

$$\begin{aligned}
&\gamma(t) \\
&= \exp x_{\ell+1}X_{\ell+1} \cdots \exp x_1X_1 \exp\Big[\big(tc_{\ell+1} + O(t^2)\big)X_{\ell+1} + \cdots + \big(tc_2 + O(t^2)\big)X_2\Big] \\
&\qquad \exp -x_1X_1 \exp x_1X_1 \exp(x_1 + tc_1)X_1 \\
&= \exp x_{\ell+1}X_{\ell+1} \cdots \exp x_2X_2 \exp e^{x_1\overline{\mathrm{ad}}X_1}\Big[\big(tc_{\ell+1} + O(t^2)\big)X_{\ell+1} + \cdots + \big(tc_2 + O(t^2)\big)X_2\Big] \\
&\qquad \exp(x_1 + tc_1)X_1.
\end{aligned} \tag{3.6}$$

Observing that the linear subspace of $\mathfrak{q}$ generated by the vectors $X_{\ell+1},\ldots,X_2$ is in fact an ideal of the Lie algebra $\mathfrak{q}$ we can see that it follows from (3.6) and the inductive hypothesis that (3.5) is also true for $n = \ell + 1$. This proves the inductive step and the proposition follows.

Let $\bar{Q}_N$ be a simply connected nilpotent Lie group that admits as Lie algebra the nil-shadow $\mathfrak{q}_N$ of $\mathfrak{q}$. $\bar{Q}_N$ is also called the nil-shadow of $\bar{Q}$.

We identify the elements of $\mathfrak{q}_N$ with the left invariant vector fields on Q_N and if $X \in \mathfrak{q}$ then we denote by ${}_NX$ the element of $\mathfrak{q}_N$ satisfying ${}_NX(0) = X(0)$. We extend the transformations $S(X)$, $X \in \mathfrak{q}$ and $\operatorname{Ad} z$, $z \in M$ to $\mathfrak{q}_N$ by putting $S(X)_NY = {}_N\big(S(X)Y\big)$ and $\operatorname{Ad} z({}_NY) = {}_N\big(\operatorname{Ad} z(Y)\big)$.

Using again the exponential coordinates of the second kind

$$\phi_N\colon \mathbb{R}^n \longrightarrow \bar{Q}_N,\ \phi\colon (x_n,\ldots,x_1) \longrightarrow \exp x_{nN}X_n \cdots \exp x_{1N}X_1$$

we can see that $\bar{Q}_N$ is diffeomorphic with $\mathbb{R}^n$.

From now on, using the exponential coordinates of the second kind ϕ and ϕ_N, we shall identify $\bar{Q}$ and $\bar{Q}_N$ as differential manifolds with $\mathbb{R}^n$.

It follows from (3.2) that if $x = (x_n,\ldots,x_1) \in \mathbb{R}^n$ and $E \in \mathfrak{q}$ then

$$E(x) = e^{x_kS(X_k)} \cdots e^{x_1S(X_1)}{}_NE(x). \tag{3.7}$$

Using the diffeomorphism

$$\Phi\colon \mathbb{R}^n \times \bar{M} \longrightarrow \bar{G} = \bar{Q}\bar{M},\ \Phi\colon (x,z) \longrightarrow \phi(x)z$$

we identify the groups $\bar{G}$ and $\bar{Q}_N \times \bar{M}$ as differential manifolds with $\mathbb{R}^n \times \bar{M}$. Also, if $E = (X,Z)$ is a vector field on $\mathbb{R}^n \times \bar{M}$ then we write $E = X + Z$.

Putting (3.2) and (3.7) together we have that

$$\begin{gathered}(X+Z)(x,z) = e^{x_kS(X_k)} \cdots e^{x_1S(X_1)} \operatorname{Ad} z({}_NX)(x,z) + Z(x,z),\\ X \in \mathfrak{q},\ Z \in \mathfrak{m},\ x = (x_n,\ldots,x_1) \in \mathbb{R}^n,\ z \in \bar{M}.\end{gathered} \tag{3.8}$$

3.2 *The fundamental group of G.* As we have seen the universal cover $\bar{G}$ of G is isomorphic with the group $\bar{Q} \times_\tau \bar{M}$. $\bar{Q} \times_\tau \bar{M}$, being a simply connected space is the universal covering of the group $Q \times_\tau M$ which in turn as we have seen is a finite cover of G (*cf.* [22], p. 255, exercise 41).

Let Γ' be the fundamental group of G. Then Γ' is isomorphic to a finitely generated discrete normal subgroup of $\bar{G}$. Let $\Gamma_1 = \{g \in \Gamma' : g \in \bar{Q}\}$ and $A = \{g \in \Gamma' : g \in \bar{M}\}$. Then the group Γ of the finite covering $Q \times_\tau M \longrightarrow G = QM\colon (x,z) \longrightarrow xz$, is isomorphic with Γ'/Γ_1A. Moreover M is isomorphic with $\bar{M}/A$, Q with $\bar{Q}/\Gamma_1$ and $Q \times_\tau M$ with $\bar{G}/\Gamma_1A$. We shall identify these groups using the corresponding isomorphisms. Observe that Γ_1 is isomorphic with $\mathbb{Z}^d$ for some $d \leq n$.

We are going to prove the following:

PROPOSITION 3.3. *Let Γ', Γ_1 and A be as above. Then*

(i) Γ' is also a subgroup of $\bar{Q}_N \times \bar{M}$ (recall that $\bar{G}$ and $\bar{Q}_N \times \bar{M}$ have been identified as differential manifolds, hence Γ' is a subset of $\bar{Q}_N \times \bar{M}$).

(ii) The basis $\{X_n, \ldots, X_1\}$ of $\mathfrak{q}$ can be chosen in such a way that it has the additional property that

$$\exp X_{i_1}, \ldots, \exp X_{i_d} \text{ generate } \Gamma_1 \text{ for some integers } 1 \leq i_1 < \cdots < i_d \leq n.$$

To prove the above proposition we shall need some lemmas.

We denote by $Z(\bar{G})$, $Z(\bar{Q})$, $Z(\bar{M})$, $Z(\bar{Q}_N)$ and $Z(\bar{Q}_N \times \bar{M})$ the centers of the groups $\bar{G}$, $\bar{Q}$, $\bar{M}$, $\bar{Q}_N$ and $\bar{Q}_N \times \bar{M}$ respectively.

LEMMA 3.4. *If $g = xz$, $g \in Z(\bar{G})$, $x \in \bar{Q}$, $z \in \bar{M}$, then $z \in Z(\bar{M})$ and $xy = yx$, $y \in \bar{M}$.*

PROOF. Since $\mathfrak{g}$ can be written in a unique way as a product $g = xz$, $x \in \bar{Q}$, $z \in \bar{M}$ and $\forall y \in \bar{M}$, $ygy^{-1} = yxy^{-1}yzy^{-1}$, and $yxy^{-1} \in \bar{Q}$, $yzy^{-1} \in \bar{M}$ we have that $yxy^{-1} = x$, $yzy^{-1} = z$, $\forall y \in \bar{M}$. Hence the lemma.

LEMMA 3.5. *If $g = xz$, $x = \exp x_n X_n \cdots \exp x_1 X_1$, $g \in Z(\bar{G})$, $x \in \bar{Q}$, $z \in \bar{M}$ then $e^{x_k S(X_k)} \cdots e^{x_1 S(X_1)} \operatorname{Ad} z(X) = X$, $X \in \mathfrak{q}$.*

PROOF. It is enough to prove that $e^{x_k S(X_k)} \cdots e^{x_1 S(X_1)} \operatorname{Ad} z(X_\ell) = X_\ell$, $1 \leq \ell \leq n$. We have that $X_\ell \in \mathfrak{a}_i$ for some $1 \leq i \leq m$ (*cf.* Proposition 2.2). Let R_{i+1} be the analytic subgroup of G that has $\mathfrak{r}_{i+1}$ as its Lie algebra. The fact that $\mathfrak{r}_{i+1}$ is an ideal of $\mathfrak{g}$ implies that R_{i+1} is a normal subgroup of G. Now the lemma follows from the observation that if $y_t = \exp tX_\ell$, $t \in \mathbb{R}$, then $gy_tg^{-1} = y \exp tY_\ell$ where $y \in R_{i+1}$ and $Y_\ell = e^{x_k S(X_k)} \cdots e^{x_1 S(X_1)} \operatorname{Ad} z(X_\ell) \in \mathfrak{a}_i$.

LEMMA 3.6. *If $g = \exp x_n X_n \cdots \exp x_1 X_1 z \in Z(\bar{G})$, $z \in \bar{M}$, then for all ℓ such that $x_\ell \neq 0$ we have $S(X_i)X_\ell = 0$, $i = 1, \ldots, k$ (i.e. $x_\ell = 0$, $n_{1j} < \ell \leq n_j, 1 \leq j \leq m$, cf. Proposition 2.4).*

PROOF. Assume that there is $1 \leq i \leq k$ for which the lemma is not true and put $j = \inf\{\ell : x_\ell \neq 0, S(X_i)X_\ell \neq 0\}$. Then because of the way the basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{q}$ was constructed (*cf.* Propositions 2.2 and 2.4) there is h such that either $h = j-1$ or $h = j+1$ and $a \neq 0$ for which we have $e^{S(X_i)}X_j = \cos aX_j + \sin aX_h$. Let $\mathfrak{r}$ be the linear subspace of $\mathfrak{q}$ generated by the vectors $X_{j+1}, \ldots, X_n$, in the case $h = j-1$, or by the vectors $X_{j+2}, \ldots, X_n$ in the case $h = j+1$. Then $\mathfrak{r}$ is an ideal of $\mathfrak{q}$. Let R the analytic subgroup of $\bar{G}$ having $\mathfrak{r}$ as its Lie algebra. Let also $b = \inf(j, h)$. Then $g = y \exp(x_j X_j + x_h X_h) \exp x_{b+1} X_{b+1} \cdots \exp x_1 X_1 z$ for some $y \in R$. Let $z_t = \exp tX_i$, $t \in \mathbb{R}$. Since $z_t^{-1} R z_t = R$ and $z_t^{-1} z z_t = z$, there is $y_t \in R$ such that $z_t^{-1} g z_t = y_t \exp e^{tS(X_i)}(x_j X_j + x_h X_h) \exp x_{b+1} X_{b+1} \cdots \exp x_1 X_1 z$, which contradicts the hypothesis that $g \in Z(\bar{G})$ since there is $t \in \mathbb{R}$ such that $e^{tS(X_i)}(x_j X_j + x_h X_h) \neq x_j X_j + x_h X_h$. The lemma follows.

LEMMA 3.7. *If $g = \exp x_n X_n \cdots \exp x_1 X_1 z \in Z(\bar{G})$, $z \in \bar{M}$, then* $\operatorname{ad} Z(X_\ell) = 0$ *for all X_ℓ such that $x_\ell \neq 0$ and $Z \in \mathfrak{m}$ (i.e. $x_\ell, n_{0j} < \ell \leq n_j$, $1 \leq j \leq m$, cf. Proposition 2.4).*

PROOF. Let $j = \inf\{\ell : x_\ell \neq 0$ and $\exists Z \in \mathfrak{m}$ such that $\operatorname{ad} Z(X_\ell) \neq 0\}$. Then, in view of Lemma 5, $n_{0h} < j \leq n_h$ for some $1 \leq h \leq m$. Let R denote the analytic subgroup of $\bar{G}$ having as Lie algebra the ideal of $\mathfrak{g}$ generated by the vectors $X_{n_h+1}, \ldots, X_n$ (*cf.* Propositions 2.2 and 2.4). Then there is $y \in R$ such that $g = y \exp(x_{n_h} X_{n_h} + \cdots + x_{n_{0h}+1} X_{n_{0h}+1}) \exp x_{n_{0h}} X_{n_{0h}} \cdots \exp x_1 X_1 z$. Let $z_t = \exp tZ$, $t \in \mathbb{R}$. Since $z_t R z_t^{-1} = R$, $z_t z z_t^{-1} = z$, there is $y_t \in R$, $y_t = z_t y z_t^{-1}$ such that $z_t g z_t^{-1} = y_t \exp e^{t\operatorname{ad} Z}(x_{n_h} X_{n_h} + \cdots + x_{n_{0h}+1} X_{n_{0h}+1}) \cdot \exp x_{n_{0h}} X_{n_{0h}} \cdots \exp x_1 X_1 z$. By definition of the $\mathfrak{a}_{1h}$, $\mathfrak{a}_{2h}$ there is $t \in \mathbb{R}$ such that $e^{t\operatorname{ad} Z}(x_{n_h} X_{n_h} + \cdots + x_{n_{0h}+1} X_{n_{0h}+1}) \neq x_{n_h} X_{n_h} + \cdots + x_{n_{0h}+1} X_{n_{0h}+1}$. Hence the lemma.

From the above lemmas we have the following:

COROLLARY 3.8. *Let $*$ denote the product with respect to the group $\bar{Q}_N \times \bar{M}$. Then $y * g = yg$, $g * y = gy$, $g \in Z(\bar{G})$, $y \in \bar{G}$. In particular, Γ' is also a subgroup of $\bar{Q}_N \times \bar{M}$.*

Let $\mathfrak{q}_0 = \{X \in \mathfrak{q} : S(X_i)X = 0, \operatorname{ad} Z(X) = 0, i = 1, \ldots k, Z \in \mathfrak{m}\}$. We can easily see that $\mathfrak{q}_0$ is generated by the vectors $\{X_j : S(X_i)X_j = 0, \operatorname{ad} Z(X_j) = 0, i = 1, \ldots, k, Z \in \mathfrak{m}, j = 1, \ldots, n\}$ and that it is a nilpotent subalgebra of $\mathfrak{q}$. We denote by Q_0 the analytic subgroup of $\bar{Q}$ having $\mathfrak{q}_0$ as its Lie algebra.

We have the following corollary to Lemmas 3.6 and 3.7.

COROLLARY 3.9. *If $g \in Z(\bar{G})$, $g = xz$, $x \in \bar{Q}$, $z \in \bar{M}$, then $x \in Q_0$. Hence there is $X \in \mathfrak{q}_0$ such that $x = \exp X$ (cf. [22]).*

PROOF OF PROPOSITION 3.3. (i) follows from Corollary 3.8. So, we only have to prove (ii). To this end, let us consider the filtration

$$\mathfrak{q} = \mathfrak{r}_1 \supseteq \mathfrak{n} \supseteq \mathfrak{r}_2 \supseteq \cdots \supseteq \mathfrak{r}_m \supseteq \mathfrak{r}_{m+1} = \{0\}$$

of $\mathfrak{q}$ constructed in Proposition 2.2. and denote by $R_1, \ldots, R_m$ the analytic subgroups of G having as Lie algebras $\mathfrak{r}_1, \ldots, \mathfrak{r}_m$ respectively. Let D_0 be the image of Γ_1 by the map $\bar{Q} \longrightarrow \bar{Q}/\bar{N}$. Then D_0 is isomorphic to $\mathbb{Z}^{b_0}$ for some $b_0 \leq k$. It follows from Corollary 3.9, that there are vectors $Y_1, \ldots, Y_{b_0}$ such that

a) $Y_1, \ldots, Y_{b_0} \in \mathfrak{q}_0$, $\exp Y_1, \ldots, \exp Y_{b_0} \in \Gamma_1$ and
b) the images of $\exp Y_1, \ldots, \exp Y_{b_0}$ by the map $\bar{Q} \longrightarrow \bar{Q}/\bar{N}$ generate D_0.

Let B_0 the subgroup of Γ_1 generated by $\exp Y_1, \ldots, \exp Y_{b_0}$. Then B_0 is isomorphic with $\mathbb{Z}^{b_0}$ (recall that Γ_1 is abelian). Moreover there is a subgroup B_0' of Γ_1 such that

$$\Gamma_1 = B_0 \times B_0',\ B_0' \subseteq \bar{N},\ B_1' \text{ isomorphic with } \mathbb{Z}^{b_0'},\ b_0 + b_0' = d.$$

Let D_1 be the image of B_0' by the map $\bar{N} \longrightarrow \bar{N}/R_2$. Then D_1 is isomorphic to $\mathbb{Z}^{b_1}$ for some $b_1 \leq n_1 - k$. Again, it follows from Corollary 8.9 that there are vectors $Y_{b_0+1}, \ldots, Y_{b_0+b_1} \in \mathfrak{q}$ such that

a) $Y_{b_0+1}, \ldots, Y_{b_0+b_1} \in \mathfrak{q}_0 \cap \mathfrak{n}$, $\exp Y_{b_0+1}, \ldots, \exp Y_{b_0+b_1} \in \Gamma_1$ and
b) the images of $\exp Y_{b_0+1}, \ldots, \exp Y_{b_0+b_1}$ by the map $\bar{N} \longrightarrow \bar{N}/R_2$ generate D_1.

Let B_1 the subgroup of B'_0 generated by $\exp Y_{b_0+1}, \ldots, \exp Y_{b_0+b_1}$. Then B_1 is isomorphic with $\mathbb{Z}^{b_1}$ and there is a subgroup B'_1 of B'_0 such that

$$B'_0 = B_1 \times B'_1,\ B'_1 \subseteq R_2,\ B'_1 \text{ isomorphic with } \mathbb{Z}^{b'_1},\ b_1 + b'_1 = b'_0.$$

Repeating the same argument we can construct for all $i = 2, \ldots, m$ subgroups B_i, B'_i of Γ_1 and vectors $Y_{b_{i-1}+1}, \ldots, Y_{b_{i-1}+b_i} \in \mathfrak{q}_0$ such that

a) $B'_{i-1} = B_i \times B'_i, B'_i \subseteq R_{i+1}, \Gamma_1 = B_0 \times \cdots \times B_m, d = b_0 + \cdots + b_m$,

b) $Y_{b_{i-1}+1}, \ldots, Y_{b_{i-1}+b_i} \in \mathfrak{q}_0 \cap \mathfrak{r}_i$, $\exp Y_{b_0+1}, \ldots, \exp Y_{b_0+b_1}$ generate B_i and

c) the images of $\exp Y_{b_{i-1}+1}, \ldots, \exp Y_{b_{i-1}+b_i}$ by the map $R_i \longrightarrow R_i/R_{i+1}$ generate the image of B'_{i-1} by the same map.

Now we choose vectors $X_{i_1}, \ldots, X_{i_d} \in \mathfrak{q}_0$ from the basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{q}$ so that $0 < i_j \leq n_{01}$ for $0 < j \leq b_0$, $n_{h-1} < i_j \leq n_{0h}$ for $b_{h-1} < j \leq b_h$ and $h \geq 1$ and so that $\{X_i, \leq i \leq n,\ i \neq i_j, 1 \leq j \leq d,\ Y_1, \ldots, Y_d\}$ continues to be a basis of $\mathfrak{q}$. The new basis of $\mathfrak{q}$ obtained by replacing the vectors $X_{i_1}, \ldots, X_{i_d}$ by the vectors $Y_1, \ldots, Y_d$ respectively satisfies (ii).

3.3 *The non-simply connected case.* We call nil-shadow Q_N of Q the group $Q_N = \bar{Q}_N/\Gamma_1$ and we put $G_N = \bar{Q}_N \times \bar{M}/\Gamma'$. It follows from Corollary 3.8 that G and G_N are identical as differential manifolds . We fix a basis $\{X_n, \ldots, X_1\}$ of $\mathfrak{q}$ as in Proposition 3.3.

We define $\mathbb{O}_i = \mathbb{T}(= \mathbb{R}/\mathbb{Z})$, if $\exp X_i \in \Gamma_1$ and $\mathbb{O}_i = \mathbb{R}$, if not, $i = 1, \ldots, n$ and we put $\mathbb{O} = \mathbb{O}_n \times \cdots \times \mathbb{O}_1$.

As in the simply connected case, we have the diffeomorphisms, which we shall also denote by ϕ, ϕ_N :

$$\phi\colon \mathbb{O} \longrightarrow Q,\ \phi : x = (x_n, \ldots, x_1) \longrightarrow \exp x_n X_n \cdots \exp x_1 X_1$$
$$\phi_N\colon \mathbb{O} \longrightarrow Q_N,\ \phi_N : x = (x_n, \ldots, x_1) \longrightarrow \exp x_{nN} X_n \cdots \exp x_{1N} X_1.$$

Using these difeomorphisms , we identify Q, Q_N with $\mathbb{O}$ and $Q \times_\tau M$, $Q_N \times M$ with $\mathbb{O} \times M$ as differential manifolds. The map, which we shall also denote by Φ

$$\Phi\colon \mathbb{O} \times M \longrightarrow G = QM,\ \Phi\colon (x, z) \longrightarrow \phi(x)z$$

becomes a finite covering map for G. Using this map we identify, as differential manifolds, the groups G and G_N with $\mathbb{O} \times M/\Gamma$, where $\Gamma = \Gamma'/\Gamma_1 A$.

From what has been proved in Section 3.2 , we have the following

COROLLARY 3.10. *Let $g \in G$. Then, there are $x = \exp x_n X_n \cdots \exp x_1 X_1 \in Q$ and $z \in M$ such that $g = xz$. If we also have $g = x'z'$ for $x' = \exp x'_n X_n \cdots \exp x'_1 X_1 \in Q, z' \in M$, then $x'_i = x_i$, whenever $\mathbb{O}_i = \mathbb{R}$.*

We denote by $\mathfrak{g}_N$ the Lie algebra of G_N and by $[.,.]_N$ the Lie product in $\mathfrak{g}_N$. Notice that $\mathfrak{g}_N = \mathfrak{q}_N + \mathfrak{m}$ and that $[\mathfrak{q}_N, \mathfrak{m}]_N = 0$. We identify the elements of $\mathfrak{g}_N$ with the left invariant vector fields on G_N.

If $X \in \mathfrak{q}$ is a left invariant vector field on G, $X \in \mathfrak{q}$, then we denote by ${}_N X \in \mathfrak{q}_N$ the left invariant vector field on G_N that satisfies ${}_N X(0) = X(0)$. If $E = (X, Z)$ is a vector field on $\mathbb{O} \times M$, then we write $E = X + Z$. With these changes in the notations (3.8) remains true, *i.e.*

$$(X+Z)(x,z) = e^{x_k S(X_k)} \cdots e^{x_1 S(X_1)} \operatorname{Ad} z({}_N X)(x,z) + Z(x,z), \tag{3.9}$$
$$X \in \mathfrak{q},\ Z \in \mathfrak{m},\ x = (x_n, \ldots, x_1) \in \mathbb{O}, z \in M.$$

4. **The volume growth.** Let G be a connected Lie group of polynomial growth, dg a left invariant Haar measure on G.

We shall use the notations of Section 3. As explained in that section we identify $Q \times_\tau M$ and $Q_N \times M$ with $\mathbb{O} \times M$ as differential manifolds and G and G_N with $Q \times_\tau M/\Gamma$.

$n_0, n_1, \ldots, n_m$ are as in Section 2 and we put

$$\sigma(i) = 0,\ \text{if } \mathbb{O}_i = \mathbb{T},\ \sigma(i) = j,\ \text{if } \mathbb{O}_i = \mathbb{R} \text{ and } n_{j-1} < i \leq n_j,\ i = 1, \ldots, n$$
$$d = \sum_{1 \leq i \leq n} \sigma(i).$$

Let $E_1, \ldots, E_p$ be as in Theorem 1, *i.e.* left invariant vector fields on G that satisfy Hörmander's condition. The control distance $d_E(.,.)$ associated to these vector fields is defined as follows (*cf.* [6], [24]):

We call an absolutely continuous path $\dot{\gamma}\colon [0,1] \longrightarrow G$ admissible if and only if $\dot{\gamma}(t) = a_1(t)E_1 + \cdots + a_p(t)E_p$ for almost all $t \in [0,1]$ and we put $|\dot{\gamma}(t)|^2 = a_1^2(t) + \cdots + a_p^2(t)$. Then we define

$$d_E(x,y) = \inf\Big\{ \int_0^1 |\dot{\gamma}(t)|\, dt, \gamma \text{ admissible path such that } \gamma(0) = x, \gamma(1) = y \Big\}.$$

We put $S_E(x,t) = \{y \in G : d_E(x,y) < t\}, x \in G, t > 0$.

We want to describe the shape of the balls $S_E(e,t)$, $t \geq 1$ and to estimate the dg-measure$(S_E(e,t))$. To this end we shall need some notations. If $g \in G$, $g = xz$, $z \in M$, $x \in Q$, $x = (x_n, \ldots, x_1)$, then we put

$$g_t = x_t z,\ x_t = (t^{\sigma(n)} x_n, \ldots, t^{\sigma(1)} x_1), \quad t > 0.$$

$$D(g,t) = \{h \in G : h = yw, w \in M, y \in Q, y = (y_n, \ldots, y_1),$$
$$x_i - t^{\sigma(i)} < y_i < x_i + t^{\sigma(i)} \text{ for } \sigma(i) \neq 0, 1 \leq i \leq n\},\ t > 0.$$

We also put $D_t = D(e,t)$ and $D = D(e,1)$.

PROPOSITION 4.1. *Let $S_E(x,t)$ and D_t be as above. Then there is $c > 0$ such that*

$$S_E(e, c^{-1}t) \subseteq D_t \subseteq S_E(e, ct), \quad t \geq 1$$
$$c^{-1} t^d \leq\ dg\text{-measure}\big(S_E(e,t)\big) \leq c t^d, \quad t \geq 1.$$

PROOF. As we see from (0.1), the balls $S_E(e,t)$, $t \geq 0$, behave for large t in the same way as the powers V^n, $n \in \mathbb{N}$ of a compact neighborhood V of e. Hence the vector fields $\{E_1, \ldots, E_p\}$ can be replaced with a basis $\{X_n, \ldots, X_1, Z_0, \ldots, Z_{-r}\}$ of the Lie algebra $\mathfrak{g}$ of G, $Z_0, \ldots, Z_{-r} \in \mathfrak{m}$. Also it follows from Corollary 3.10, that it is enough to prove the proposition in the case $\Gamma = \{e\}$. Furthermore, it follows from (3.9), that $\{X_n, \ldots, X_1, Z_0, \ldots, Z_{-r}\}$ can be replaced by $\{{}_NX_n, \ldots, {}_NX_1, Z_0, \ldots, Z_{-r}\}$ and then the proposition becomes a wellknown result (*cf.* [12], [14], [25]).

Arguing in the same way as in the above proposition, we can prove the following lemma which we shall need later on.

LEMMA 4.2. *Let $S_E(g,t), D(g,t)$ and D be as above. Then there is $A > 0$ and $\mu \in \mathbb{N}$ such that for all $g \in D$, $R \in (0,1]$ and $t > t_0 = t_0(R)$, we have*

$$S_E(g_t, tR) \subseteq D(g_t, AtR^{\frac{1}{\mu}}),\ D(g_t, tR) \subseteq S_E(g_t, AtR^{\frac{1}{\mu}}).$$

5. **Generalisations of some classical results of Homogenization theory.** Let G be a connected Lie group of polynomial growth.

Let $E_1, \ldots, E_p$ and L be as in Theorem 1, *i.e.* $E_1, \ldots, E_p$ are left invariant vector fields on G that satisfy Hörmander's condition and $L = -(E_1^2 + \cdots + E_p^2)$.

The purpose of this section is to show how some classical results of Homogenization Theory (*cf.* [4]) can be generalised in our context. In particular, we shall prove a homogenization formula for the operator L. The homogenized operator L_H will be a left invariant sub-Laplacian defined on a limit group N_H. N_H is a homogeneous nilpotent Lie group, determined uniquely from the algebraic structure of G. L_H is invariant with respect to the dilation structure of N_H and depends on both G and L. The importance of N_H and L_H lies in the information they provide about the geometry of G and the behavior of L at infinity.

Let Q, M, Γ, $\mathbb{O}$, Q_N, G_N and $Q \times_\tau M$ be as in Section 3. As explained in that section, we identify, as differential manifolds, Q and Q_N with $\mathbb{O}$, $Q \times_\tau M$ and $Q_N \times M$ with $\mathbb{O} \times M$ and $G_N, Q_N \times M/\Gamma$ and $Q \times_\tau M/\Gamma$ with G.

We fix a basis $\{X_n, \ldots, X_1, Z_0, \ldots, Z_{-r}\}$ of $\mathfrak{g}$, with $\{X_n, \ldots, X_1\}$ a basis of $\mathfrak{q}$ as in Proposition 3.3 and $\{Z_0, \ldots, Z_{-r}\}$ a basis of $\mathfrak{m}$.

$n_0, n_1, \ldots, n_m$ are as in Section 2, $D(g,t), D_t, D$ as in Section 4 and $\sigma(i)$, $i = 1, \ldots, n$ as in (4.1).

5.1 *The dilation.* We denote by τ_ε, $0 < \varepsilon \leq 1$ the dilation of $\mathbb{O} \times M$ defined by

$$\tau_\varepsilon \colon \mathbb{O} \times M \longrightarrow \mathbb{O} \times M,\ \tau_\varepsilon \colon \big((x_n, \ldots, x_1), z\big) \longrightarrow \big((\varepsilon^{\sigma(n)} x_n, \ldots, \varepsilon^{\sigma(1)} x_1), z\big).$$

As we can see from Corollary 3.10, τ_ε induces a dilation on G, which we shall also denote by τ_ε, by putting $\tau_\varepsilon(xz) = \tau_\varepsilon(x)z$, $x \in Q$, $z \in M$.

We put

$$E_{\varepsilon,i} = \frac{1}{\varepsilon}\, d\tau_\varepsilon(E_i),\ i = 1, \ldots, p \text{ and } L_\varepsilon = -(E_{\varepsilon,1}^2 + \cdots + E_{\varepsilon,p}^2), \quad 0 < \varepsilon \leq 1.$$

5.2 *The compactness.* If $(s,x) \in \mathbb{R} \times G$ and $u \in C^\infty([s-\rho^2, s]) \times S_E(x,\rho))$, then we write

$$\operatorname{Osc}(u,s,x,\rho) = \sup\{|u(t,y) - u(t',y')|,\ (t,y),(t',y') \in [s-\rho^2, s] \times S_E(x,\rho)\}.$$

THEOREM 5.1 (*cf.* [19]). *For every* $0 < \delta < 1$, *there is* $0 < a < 0$ *such that*

$$\operatorname{Osc}(u, s, x, \delta\rho) \leq a \operatorname{Osc}(u, s, x, \rho), \quad (s, x) \in \mathbb{R} \times G$$

for all $u \in C^\infty\big([s-\rho^2, s] \times S_E(x, \rho)\big)$ *such that* $(\partial/\partial t + L)u = 0$ *in* $[s-\rho^2, s] \times S_E(x, \rho)$, $\rho > 0$.

The above theorem provides a compactness on the space of functions u_ε, satisfying

$$\|u_\varepsilon\|_\infty \leq 1, \ (\partial/\partial t + L_\varepsilon)u_\varepsilon = 0 \text{ in } (-1, 1) \times D, \quad 0 < \varepsilon \leq 1. \tag{5.1}$$

In particular we have the following:

PROPOSITION 5.2. *Let* $u_\varepsilon, 0 < \varepsilon \leq 1$ *be a family of functions satisfying (5.1). Then there is a subsequence, also denoted by* u_ε, *such that*

$$u_\varepsilon \longrightarrow u_0, \quad (\varepsilon \longrightarrow 0)$$

uniformly on the compact subsets of $(-1, 1) \times D$. *Moreover,* $u_0(t, g) = u_0(t, g')$, *for all* $g, g' \in D$, $g = xz$, $g' = x'z$, $x, x' \in Q$, $z, z' \in M$, $x = (x_n, \ldots, x_1)$, $x' = (x'_n, \ldots, x'_1)$ *such that* $x_i = x'_i$ *if* $\mathfrak{O}_i = \mathbb{R}$, $i = 1, \ldots, n$.

PROOF. From Lemma 4.2 and with the same notations we have that there are constants $0 < r \leq 1$, $1 < C < B < A$ and $\mu, \nu \in \mathbb{N}$, $\mu < \nu$ such that for all $g \in D$, $R \in (0, r)$ and t large enough, we have

$$D(g_t, tR) \subseteq S_E(g_t, CtR^{\frac{1}{\mu}}) \subseteq S_E(g_t, BtR^{\frac{1}{\mu}}) \subseteq D(g_t, AtR^{\frac{1}{\nu}}).$$

On the other hand $(\partial/\partial t + L_\varepsilon)u_\varepsilon = 0$ in $(-1, 1) \times D$ if and only if $(\partial/\partial t + L)v_t = 0$ in $(-t^2, t^2) \times D_t$ where $t = 1/\varepsilon$ and $v_t(s, g) = u_\varepsilon\big(\varepsilon^2 s, \tau_\varepsilon(g)\big)$.

So, applying Theorem 5.1 above we have that, for all $\delta > 0$ and $(t, g) \in (-1, 1) \times D$ there is a neighborhood $(t - s, t + s) \times D(g, r) \subseteq (-1, 1) \times D$, $r, s > 0$ of (t, g) and $\varepsilon_1 \in (0, 1)$ such that $|u_\varepsilon(b, h) - u_\varepsilon(b', h')| < \delta$, $(b, h), (b', h') \in (t - s, t + s) \times D(g, r)$, $\varepsilon < \varepsilon_1$ and the proposition follows.

Let $\mathfrak{O}_c = \{x = (x_n, \ldots, x_1) \in \mathfrak{O} : x_i = 0 \text{ if } \mathfrak{O}_i = \mathbb{R}, i = 1, \ldots, n\}$, $\mathfrak{O}_H = \mathfrak{O}/\mathfrak{O}_c$ and denote by D_H the image of D by the map π defined by $\pi(g) = x + \mathfrak{O}_c$, $g = xz$, $x \in Q$, $z \in M$ (it follows from Corollary 3.10, that π is well defined). Then we have the following:

COROLLARY 5.3. *The limit function* u_0 *of the Proposition 5.2 can be viewed as a function defined on* D_H.

5.3 *The limit group* N_H. Let $K(X)$ and $\mathfrak{a}_1, \ldots, \mathfrak{a}_m$ be as in Section 2. Then we have the direct sum decomposition

$$\mathfrak{q} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_m.$$

We denote by $[.,.]_H$ the unique product on the linear space $\mathfrak{q}$ satisfying for $X \in \mathfrak{a}_i$ and $Y \in \mathfrak{a}_j$

$$[X, Y]_H = \operatorname{pr}_{\mathfrak{a}_{i+j}}[X, Y]_N, \text{ if } i + j \leq m \text{ and } [X, Y]_H = 0, \text{ if } i + j > m.$$

It is easy to see that $[.,.]_H$ satisfies the Jacobi identity (observe that if $Z \in \mathfrak{a}_h$ and X, Y are as above then it follows from the way the spaces $\mathfrak{r}_i, \mathfrak{a}_i, i = 1, \ldots, m$ were defined that $\big[X, [Y, Z]_H\big]_H = \mathrm{pr}_{\mathfrak{a}_{i+j+h}}\big[X, [Y, Z]_N\big]_N$). So, $\mathfrak{q}_H = (\mathfrak{q}, [.,.]_H)$ is a nilpotent Lie algebra which is also stratified.

Let Q_H be a simply connected Lie group that admits $\mathfrak{q}_H$ as its Lie algebra. If $X \in \mathfrak{q}_H$ then we denote by ${}_HX$ the left invariant vector field on Q_H satisfying ${}_HX(e) = X$ (e is the identity element of Q_H). Using the exponential coordinates of the second kind

$$\phi_H : \mathbb{R}^n \longrightarrow Q_H, \ \phi : (x_n, \ldots, x_1) \longrightarrow \exp x_{nH}X_n \cdots \exp x_{1H}X_1$$

we identify Q_H with $\mathbb{R}^n$.

Let $\mathfrak{b}$ be the subalgebra of $\mathfrak{q}_H$ generated by the vectors $\{X_i : \mathbb{O}_i = \mathbb{T}, i = 1, \ldots, n\}$ and denote by C the analytic subgroup of Q_H having $\mathfrak{b}$ as its Lie algebra.

The limit group N_H is defined to be the quotient $N_H = Q_H / C$. It is a stratified nilpotent Lie group.

Observe that if we identify N_H, as a differential manifold, with $\mathbb{O}_H$ (using the exponential coordinates of the second kind) then Corollary 5.3 implies the following

COROLLARY 5.4. *The limit function u_0 in the Proposition 5.2 can be viewed as a function defined on N_H.*

CONVENTION. For simplicity, in what follows, we shall assume that Γ is trivial and hence that $G = Q \times_\tau M$. So the elements of G will be the pairs (x, z), $x \in Q$, $z \in M$. Because of Corollary 3.10, as we have seen so far and as it can be easily verified this presents no loss of generality.

5.4 *The coefficients of the operator L.* To simplify notation we shall denote by $\partial_n, \ldots, \partial_1, \partial_0, \ldots, \partial_{-r}$ respectively the vector fields $\frac{\partial}{\partial x_n}, \ldots, \frac{\partial}{\partial x_1}, Z_0, \ldots, Z_{-r}$.

Let us fix a vector field E_h, $1 \leq h \leq p$. Then from (3.4) and (3.9) and with the same notations we have that

$$E_h = (a_n^h + b_n^h)\partial_n + \cdots . +(a_{-r}^h + b_{-r}^h)\partial_{-r}$$

where

$$a_i^h(x, z) = \alpha_i^h(x, x, z), \ b_i^h(x, z) = \beta_i^h(x, x, z),$$

$$\begin{aligned}\alpha_i^h(x, y, z) = \mathrm{pr}_i\Big\{\Big[&\sum_{\lambda_1\sigma(1)+\cdots+\lambda_{i-1}\sigma(i-1)=\sigma(i)-1} x_1^{\lambda_1}\cdots x_{i-1}^{\lambda_{i-1}} \\ &\bar{K}^{\lambda_{i-1}}(X_{i-1})\cdots\bar{K}^{\lambda_1}(X_1)\Big]e^{y_kS(X_k)}\cdots e^{y_1S(X_1)}\,\mathrm{Ad}\,z\Big\}(E_h)\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}\beta_i^h(x, y, z) = \mathrm{pr}_i\Big\{\Big[&\sum_{\lambda_1\sigma(1)+\cdots+\lambda_{i-1}\sigma(i-1)<\sigma(i)-1} x_1^{\lambda_1}\cdots x_{i-1}^{\lambda_{i-1}} \\ &\bar{K}^{\lambda_{i-1}}(X_{i-1})\cdots\bar{K}^{\lambda_1}(X_1)\Big]e^{y_kS(X_k)}\cdots e^{y_1S(X_1)}\,\mathrm{Ad}\,z\Big\}(E_h),\end{aligned} \tag{5.3}$$

$x = (x_n, \ldots, x_1)$, $y = (y_n, \ldots, y_1)$, $x, y \in \mathbb{O}$, $z \in M$, $-r \leq i \leq n$.

We have the following proposition which is a direct consequence of the above definitions and the way the vectors $X_1, \ldots, X_n$ were chosen (*cf.* Propositions 2.4 and 3.3).

PROPOSITION 5.5. *The coefficients $\alpha_i^h(x,y,z)$ and $\beta_i^h(x,y,z)$ have the following properties:*

1) $\alpha_i^h(x,y,z)$ *=constant, for* $-r \le i \le k$,
2) *if* $k < i \le n_1$, *then* $\alpha_i^h(x,y,z) = \alpha_i^h(y,z)$ *and it is periodic with respect to* y,
3) *if* $n_1 < i \le n$, *then* $\alpha_i^h(x,y,z)$ *and* $\beta_i^h(x,y,z)$ *can be written as finite sums of terms of the form* $p(x)\varphi(y)f(z)$, *where* $p(x) = cx_{i_1}\cdots x_{i_\ell}$, $c \in \mathbb{R}$, $1 \le i_j < i$, $1 \le j \le \ell$, $\varphi(y) = \cos ay_j$ *or* $\sin ay_j$ *for some* $1 \le j \le k$, *hence a periodic function and* $f(z)$ *a* C^∞ *function defined on* M *and*
4) $\beta_i^h(x,y,z) = 0$, $-r \le i \le n_1$.

Let $[.,.]_H$ be as in Section 5.3 and denote by $\bar{K}_H(X_i)$, $1 \le i \le n$ the linear transformations of $\mathfrak{g}$ defined by

$$\bar{K}_H(X_i)Z = 0,\ Z \in \mathfrak{m},\ \bar{K}_H(X_i)X_j = 0,\ j \le i \text{ and } \bar{K}_H(X_i)X_j = [X_i, X_j]_H,\ i \le j.$$

Then (5.2) becomes

$$(5.4)\qquad \alpha_i^h(x,y,z) = \mathrm{pr}_i\Big[e^{x_{i-1}\bar{K}_H(X_{i-1})}\cdots e^{x_1\bar{K}_H(X_1)}e^{y_kS(X_k)}\cdots e^{y_1S(X_1)}\,\mathrm{Ad}\,z\Big](E_h).$$

and from this we have

$$(5.5)\qquad \alpha_i^h(x,y,z) = \sum_{1\le j\le n_1}\alpha_j^h(y,z)\,\mathrm{pr}_i\Big[e^{x_{i-1}\bar{K}_H(X_{i-1})}\cdots e^{x_1\bar{K}_H(X_1)}\Big](X_j).$$

Let us put, for $-r \le i,j \le n$

$$\alpha_{ij}(x,y,z) = \sum_{1\le h\le p}\alpha_i^h(x,y,z)\alpha_j^h(x,y,z)$$

$$\beta_{ij}(x,y,z) = \sum_{1\le h\le p}\Big[\alpha_i^h(x,y,z)\beta_j^h(x,y,z) + \beta_i^h(x,y,z)\beta_j^h(x,y,z) + \beta_i^h(x,y,z)\alpha_j^h(x,y,z)\Big]$$

$$a_{ij}(x,z) = \alpha_{ij}(x,x,z),\ b_{ij}(x,z) = \beta_{ij}(x,x,z).$$

Then we have (we use the summation convention for repeated indices)

$$L = A + B,\ \text{where } A = -\partial_i a_{ij}(x,z)\partial_j \text{ and } B = -\partial_i b_{ij}(x,z)\partial_j.$$

In the following proposition we have gathered some properties of the coefficients $\alpha_{ij}(x,y,z)$ and $\beta_{ij}(x,y,z)$ which are immediate consequences of the definitions.

PROPOSITION 5.6. *1) The coefficients $\alpha_{ij}(x,y,z)$ and $\beta_{ij}(x,y,z)$ are finite sums of terms of the form $p(x)\varphi(y)f(z)$, where* $p(x) = cx_{i_1}\cdots x_{i_\ell}$, $c \in \mathbb{R}$, $1 \le i_h < \max(i,j)$, $1 \le h \le \ell$, $\varphi(y) = \cos ay_j$ *or* $\sin ay_j$ *for some* $1 \le j \le k$, *hence a periodic function and* $f(z)$ *is a* C^∞ *function defined on* M.

2) $\alpha_{ij}(x,y,z) = \alpha_{ij}(y,z)$, $-r \le i,j \le n_1$.
3) $\alpha_{ij}(x,y,z) = $ *constant*, $-r \le i,j \le k$.
4) $\beta_{ij}(x,y,z) = 0$, $-r \le i,j \le n_1$.

5.5 *The correctors.* The variables x,y,z used below are such that $x,y \in \mathbb{O}$, $x = (x_n,\ldots,x_1)$, $y = (y_n,\ldots,y_1)$, $z \in M$.

To simplify notations we shall denote by $D_n, \ldots, D_1, D_0, \ldots, D_{-r}$ respectively the vector fields $\frac{\partial}{\partial y_n}, \ldots, \frac{\partial}{\partial y_1}, Z_0, \ldots, Z_{-r}$.

We put

$$A(x) = -D_i \alpha_{ij}(x, y, z) D_j.$$

If $f(x, y, z)$ is a finite sum of functions periodic with respect to the variable y then we denote by $\mathfrak{M}(f)(x)$ the mean of f, defined by

$$\mathfrak{M}(f)(x) = \lim_{t \to \infty} \frac{1}{|D_t|} \int_{D_t} f(x, y, z)\, dy\, dz$$

where $|D_t|$ denotes the volume of D_t.

The correctors $\chi^j(x, y, z)$, $1 \leq j \leq n$ are defined to be C^∞ functions satisfying

$$A(x)\chi^j(x, y, z) = -D_i \alpha_{ij}(x, y, z), \quad \mathfrak{M}(\chi^j) = 0. \tag{5.6}$$

They are defined as follows:

For $1 \leq j \leq n_1$ they are defined to be the unique solutions of the problem

$$A(x)\chi^j(x, y, z) = -D_i \alpha_{ij}(x, y, z), \quad \mathfrak{M}(\chi^j) = 0.$$

Notice that, in view of Proposition 5.6,

$$\sum_{-r \leq i \leq n} D_i \alpha_{ij}(x, y, z) = \sum_{-r \leq i \leq k} D_i \alpha_{ij}\big((y_k, \ldots, y_1), z\big), \quad 1 \leq j \leq n_1$$

which is a periodic function of mean zero and therefore the correctors χ^j, $1 \leq j \leq n_1$ are well defined.

For $n_1 < j \leq n$ the correctors χ^j are defined by

$$\chi^j(x, y, z) = \sum_{1 \leq \ell \leq n_1} \chi^\ell(y, z)\, \mathrm{pr}_j\Big[e^{x_{j-1}\bar{K}_H(X_{j-1})} \cdots e^{x_1 \bar{K}_H(X_1)}\Big](X_\ell).$$

An immediate consequence of the definition is the following:

PROPOSITION 5.7. *1)* $A(x)\big(\chi^j(x, y, z) - y_j\big) = 0$, $1 \leq j \leq n$.

2) $\chi^j(x, y, z) = \chi^j\big(x, (y_k, \ldots, y_1), z\big)$, $1 \leq j \leq n$.

3) $\chi^j = 0$, $1 \leq j \leq k$.

4) If $k < j \leq n_1$, *then* $\chi^j(x, y, z) = \chi^j(y, z)$ *and is periodic with respect to* y.

5.6 *The homogenised operator* L_H. We put

$$q_{ij}(x) = \mathfrak{M}\{\alpha_{ij}(x, y, z) - \alpha_{i\ell}(x, y, z) D_\ell \chi^j(x, y, z)\}, \quad 1 \leq i, j \leq n$$

and we denote by L_0 the operator (defined in $\mathbb{R}^n$)

$$L_0 = -\partial_i q_{ij}(x) \partial_j$$

PROPOSITION 5.8. *1)* $q_{ij}(x) = q_{ji}(x)$, $1 \leq i, j \leq n$.

2) $q_{ij}(x) =$*constant*, $1 \leq i, j \leq n_1$.

3)

$$\sum_{1\le \ell,\mu\le n_1}\Big\{\mathrm{pr}_i\Big[e^{x_{i-1}\bar K_H(X_{i-1})}\cdots e^{x_1\bar K_H(X_1)}\Big](X_\ell)\Big\}$$
$$q_{\ell\mu}\Big\{\mathrm{pr}_j\Big[e^{x_{j-1}\bar K_H(X_{j-1})}\cdots e^{x_1\bar K_H(X_1)}\Big](X_\mu)\Big\},\quad 1\le i,j\le n.$$

PROOF. 2) and 3) follow from the definitions and Propositions 5.6 and 5.7. To prove 1) let us observe that

$$q_{ij}(x)=\mathfrak{M}\{(D_hy_i)\alpha_{h\ell}(x,y,z)D_\ell[y_j-\chi^j(x,y,z)]\}$$

and that from the definition of the correctors χ^j, $1\le j\le n$, we have that

$$\mathfrak{M}\{[D_h\chi^i(x,y,z)]\alpha_{h\ell}(x,y,z)D_\ell[y_j-\chi^j(x,y,z)]\}=0.$$

Hence

$$q_{ij}(x)=\mathfrak{M}\{D_h[y_i-\chi^i(x,y,z)]\alpha_{h\ell}(x,y,z)D_\ell[y_j-\chi^j(x,y,z)]\}\tag{5.7}$$

and the proposition follows.

LEMMA 5.9. *The operator $L_0'=-\sum_{1\le i,j\le n_1}\partial_i q_{ij}(x)\partial_j$ is an elliptic operator with constant coefficients in $\mathbb{R}^{n_1}$.*

PROOF. Let $\xi=(\xi_1,\ldots,\xi_{n_1})\in\mathbb{R}^{n_1}$, $\xi\ne 0$ and (*cf.* Proposition 5.7)

$$f(y,z)=\xi_1[y_1-\chi^1(y,z)]+\cdots+\xi_{n_1}[y_{n_1}-\chi^{n_1}(y,z)].$$

Then, from (5.7) we have that

$$\sum_{1\le i,j\le n_1}q_{ij}\xi_i\xi_j=\mathfrak{M}\{[D_hf(y,z)]\alpha_{h\ell}(y,z)D_\ell f(y,z)\}$$

and from Proposition 5.6 that

$$\mathfrak{M}\{[D_\ell f(y,z)]\alpha_{\ell\mu}(y,z)D_\mu f(y,z)\}=\mathfrak{M}\{(E_1f)^2+\cdots+(E_pf)^2\}.$$

So to prove the lemma it is enough to prove that

$$\mathfrak{M}\{(E_1f)^2+\cdots+(E_pf)^2\}\ne 0$$

To do this, since the function $(E_1f)^2+\cdots+(E_pf)^2$ is a finite sum of C^∞ functions $\varphi(y,z)$ periodic with respect to the variable y with $z\in M$, M being compact, it is enough to prove that there is an open $U\subseteq \mathbb{O}\times M$ and $1\le i\le p$ such that $E_if(y,z)\ne 0$, $(y,z)\in U$. This follows from the observation that if $E_if(y,z)=0$, $\forall(y,z)\in\mathbb{O}\times M$ then, since the vector fields $E_1,\ldots,E_p$ satisfy Hörmander's condition, we would have that $f(y,z)=c$, $\forall(y,z)\in\mathbb{O}\times M$ and hence that

$$\xi_1y_1+\cdots+\xi_{n_1}y_{n_1}=\xi_1\chi^1(y,z)+\cdots+\xi_{n_1}\chi^{n_1}(y,z)+c$$

which is absurd since the second member of the above equality is a sum of functions periodic with respect to y.

It follows from the above proposition that there are linearly independent vector fields $Y_1, \ldots, Y_{n_1}$ in $\mathbb{R}^{n_1}$ with constant coefficients such that $L_0' = -(Y_1^2 + \cdots + Y_{n_1}^2)$. Let us denote by $W_1, \ldots, W_{n_1}$ respectively the images of $Y_1, \ldots, Y_{n_1}$ under the linear isomorphism of $\mathbb{R}^{n_1}$ with $\mathfrak{a}_1$ that maps $\partial_i \longrightarrow X_i$, $1 \leq i \leq n_1$. Then it follows from Proposition 5.8 3) (recall that Q_H has been identified with $\mathbb{R}^n$ and that ${}_HW$ denotes the left invariant vector field on ${}_HQ$ satisfying ${}_HW(e) = W$, *cf.* Section 5.3) that

$$L_0 = -({}_HW_1^2 + \cdots + {}_HW_{n_1}^2)$$

i.e. L_0 is a left invariant sub-Laplacian on Q_H, which is also invariant with respect to the natural dilation structure of Q_H (*cf.* [12]).

The homogenised operator L_H is defined to be the image of L_0 by the natural map $Q_H \longrightarrow N_H = Q_H / C$ (*cf.* Section 5.3).

5.7 *The homogenization formula.* Now we can state the following:

PROPOSITION 5.10. *Let u_0 be as in Proposition 5.2, $D_H \subseteq N_H$ as in Corollary 5.3 and L_H as above. Then u_0 can be viewed as defined on D_H and then it satisfies $(\partial/\partial t + L_H)u_0 = 0$ in $(-1,1) \times D_H$.*

Observe that it is enough to prove the above proposition in the case when G is simply connected. This observation simplifies the situation since when G is simply connected and $\mathbb{O} = \mathbb{R}^n$. We shall not give though the details of the proof because it is exactly the same with the proof of the homogenization formula in the classical case of uniformly elliptic second order differential operators with periodic coefficients (*cf.* [4]).

The only modification is that, since in our case we deal with hypoelliptic and not uniformly elliptic operators we have to replace D with some set of the type $U \times M$, where U is a very regular, in the sense of J. M. Bony [6], neighborhood of 0 in $\mathbb{O}$, *i.e.* it is such that

(i) $U = B_1 \cap B_2$, where B_1 and B_2 are two Euclidean balls of $\mathbb{R}^n$ and

(ii) if $x \in \partial U$, hence $x \in B_i$ for some $i \in \{1,2\}$, $\nu = (\nu_n, \ldots, \nu_1)$ is the vertical unit vector to the ball B_i at the point x and the operators L_ε, $0 < \varepsilon \leq 1$ are written in divergence form as $L_\varepsilon = -\partial_i a_{ij}^\varepsilon \partial_j$ then

$$\sum_{1 \leq i,j \leq n} a_{ij}^\varepsilon(x,z)\nu_i \nu_j > 0.$$

Observe that since D can be scaled down to a subset of $U \times M$, we can indeed replace it by $U \times M$.

To see that not only 0 but every $y = (y_n, \ldots, y_1) \in \mathbb{O}$ has such a very regular neighborhood U let us observe that a_{ij}^ε =const. $1 \leq i,j \leq k$. Hence, if $\xi \neq 0, \xi = (\xi_n, \ldots, \xi_1)$, $\xi_{k+1} = \cdots = \xi_n = 0$, then

$$\sum_{1 \leq i,j \leq n} a_{ij}^\varepsilon \xi_i \xi_j > 0, \; 0 < \varepsilon \leq 1.$$

So the intersection $U = B_1 \cap B_2$ of the balls B_1 and B_2 of radius $M + \delta$, centered at the points $y + M\xi$ and $y - M\xi$ respectively, for M large and δ small enough is a very regular neighborhood of y.

Apart from this modification the energy proof of the homogenization formula (*cf.* [4]) carries through without any change at all.

6. **The functions F_j, $1 \leq j \leq n_2$, F_{ij}, $1 \leq i,j \leq n_1$.** We shall use the notations of Section 5. In particular $D_n, \ldots, D_1, D_0, \ldots, D_{-r}$ denote, respectively the vector fields $\frac{\partial}{\partial y_n}, \ldots, \frac{\partial}{\partial y_1}, Z_0, \ldots, Z_{-r}$ and $\partial_n, \ldots, \partial_1$ the vector fields $\frac{\partial}{\partial x_n}, \ldots, \frac{\partial}{\partial x_1}$.

Whenever the indices i,j appear, in this section, we assume that, at the same time, we also have $\mathbb{O}_i = \mathbb{R}$.

The functions $\psi^{ij}(y,z)$, $1 \leq i,j \leq n_1$ and $\psi^j(y,z)$, $n_1 < j \leq n_2$, called *second order correctors*, are defined to be C^∞ functions that are finite sums of functions $\varphi(y,z)$ which are periodic with respect to y and that satisfy

$$\begin{aligned} A(x)\psi^{ij} &= -\alpha_{ij} - \alpha_{ij} + \alpha_{i\mu} D_\mu \chi^j + D_\ell(\alpha_{\ell i}\chi^j) \\ &\quad + \alpha_{j\mu} D_\mu \chi^i + D_\ell(\alpha_{\ell j}\chi^i) + q_{ij} + q_{ij}, \quad \mathfrak{M}(\psi^{ij}) = 0. \end{aligned}$$

$$A(x)\psi^j = -\partial_\ell \alpha_{\ell j} - D_\ell \beta_{\ell j} + D_\ell(\alpha_{\ell\mu}\partial_\mu \chi^j) + \partial_\ell(\alpha_{\ell\mu} D_\mu \chi^j) + \partial_\ell q_{\ell j}, \quad \mathfrak{M}(\psi^j) = 0$$

Notice that the second members of the above equations are indeed finite sums of C^∞ functions $\varphi(y,z)$, periodic with respect to y and with zero mean and therefore the functions ψ^{ij}, $1 \leq i,j \leq n_1$ and $\psi^j, n_1 < j \leq n_2$ are well defined.

We put

$$\begin{aligned} F_j(x,y,z) &= x_j - \chi^j(y,z), \\ F_j^\varepsilon(x,z) &= \varepsilon F_j(\tau_{\varepsilon^{-1}}x, \tau_{\varepsilon^{-1}}x, z), \quad 1 \leq j \leq n_1, \ 0 < \varepsilon \leq 1. \end{aligned}$$

$$\begin{aligned} F_j(x,y,z) &= x_j - \chi^j(x,y,z) - \psi^j(y,z), \\ F_j^\varepsilon(x,z) &= \varepsilon^2 F_j(\tau_{\varepsilon^{-1}}x, \tau_{\varepsilon^{-1}}x, z), \quad n_1 < j \leq n_2, \ 0 < \varepsilon \leq 1. \end{aligned}$$

$$\begin{aligned} F_{ij}(x,y,z) &= x_i x_j - x_i \chi^j(y,z) - x_j \chi^i(y,z) - \psi^{ij}(y,z), \\ F_{ij}^\varepsilon(x,z) &= \varepsilon^2 F_{ij}(\tau_{\varepsilon^{-1}}x, \tau_{\varepsilon^{-1}}x, z), \quad 1 \leq i,j \leq n_1, \ 0 < \varepsilon \leq 1. \end{aligned}$$

Then we have

$$\begin{aligned} L_\varepsilon F_j^\varepsilon(x,z) &= L_H x_j, \quad 1 \leq i < n_2, \ 0 < \varepsilon \leq 1, \\ L_\varepsilon F_{ij}^\varepsilon(x,z) &= L_H x_i x_j, \quad 1 \leq i,j \leq n_1, \ 0 < \varepsilon \leq 1 \end{aligned} \tag{6.1}$$

and

$$F_j^\varepsilon(x,z) \longrightarrow x_j, \ 1 \leq j \leq n_2, \ F_{ij}^\varepsilon(x,z) \longrightarrow x_i x_j, \quad 1 \leq i,j \leq n_1 \tag{6.2}$$

as $\varepsilon \longrightarrow 0$, uniformly on the compact subsets of $\mathbb{O} \times M$.

7. **The rescaling argument and the Harnack inequalities.** In this section, we shall adapt a rescaling argument of M. Avellaneda and F. H. Lin [2], [3] and then we shall use this argument to prove certain Harnack inequalities for the positive solutions of the equation $(\partial/\partial t + L)u = 0$. In particular, we shall prove Theorem 1.

We use the notation of Section 5.

The functions F_j^ε, $1 \le j \le n_2$ and F_{ij}^ε, $1 \le i, j \le n_1$ are as in Section 6. The balls D_t and D are as in Section 4.

From Lemma 7.1 through Lemma 7.4, when we use the indices i, j at the same time we assume that $\mathbb{O}_i = \mathbb{O}_j = \mathbb{R}$.

LEMMA 7.1. *For all $\mu \in (0,1)$ there are $\theta \in (0,1)$, $\varepsilon_0 \in (0,1)$ and $c > 0$ such that for all $0 < \epsilon \le \varepsilon_0$ and all functions u_ε satisfying*

$$\Big(\frac{\partial}{\partial t} + L_\varepsilon\Big)u_\varepsilon = 0 \ \ in\ (-1,1) \times D, \ \ \|u_\varepsilon\|_\infty \le 1$$

we have that

(7.1)

$$\sup_{(t,x,z)\in(-\theta^2,\theta^2)\times D_\theta} \Big| u_\epsilon(t,x,z) - A_0^\varepsilon - \sum_{1\le j\le n_1} A_j^\varepsilon F_j(x,z) - A^\varepsilon t - \sum_{1\le i,j\le n_1} A_{ij}^\varepsilon F_{ij}^\varepsilon(x,z) - \sum_{n_1<j\le n_2} A_j^\varepsilon F_j^\varepsilon(x,z)\Big| < \theta^{2+\mu}$$

where A_0^ε, A^ε, A_j^ε, $1 \le j \le n_2$, A_{ij}^ε, $1 \le i, j \le n_1$ are constants satisfying

$$|A_0^\varepsilon| < c, \ |A^\varepsilon| < c, \ |A_j^\varepsilon| < c, \ 1 \le j \le n_2, \ |A_{ij}^\varepsilon| < c, \ 1 \le i, j \le n_1$$

and

$$\Big(\frac{\partial}{\partial t} + L_\varepsilon\Big)\Big[A^\varepsilon t + \sum_{1\le i,j\le n_1} A_{ij}^\varepsilon F_{ij}^\varepsilon(x,z) + \sum_{n_1<j\le n_2} A_j^\varepsilon F_j^\varepsilon(x,z)\Big] = 0$$

PROOF. First we observe that there is $\mu' > \mu$ and $c > 0$ such that for all $\theta \in (0,1)$ and u satisfying

$$\Big(\frac{\partial}{\partial t} + L_H\Big)u = 0 \text{ in } (-1,1) \times D_H, \quad \|u\|_\infty \le 1$$

we have that

(7.2)

$$\sup_{(t,x,z)\in(-\theta^2,\theta^2)\times D_\theta} \Big| u(t,x,z) - A_0^0 - \sum_{1\le j\le n_1} x_j - A^0 t - \sum_{1\le i,j\le n_1} A_{ij}^0 x_i x_j - \sum_{n_1<j\le n_2} A_j^0 x_j \Big| < c\theta^{2+\mu'}$$

where A_0^0, A^0, A_j^0, $1 \le j \le n_2$, A_{ij}^0, $1 \le i, j \le n_1$ are constants satisfying

$$|A_0^0| < c, \ |A^0| < c, \ |A_j^0| < c, \ 1 \le j \le n_2, \ |A_{ij}^0| < c, \ 1 \le i, j \le n_1$$

and

$$\Big(\frac{\partial}{\partial t} + L_H\Big)\Big[A^0 t + \sum_{1\le i,j\le n_1} A^0_{ij}x_ix_j + \sum_{n_1<j\le n_2} A^0_j x_j\Big] = 0.$$

This follows from the fact that the homogenised operator L_H is hypoelliptic (*cf.* [6]).

Let us fix these values of θ and c. If (7.1) weren't true then there would a sequence of functions $u_{\varepsilon_m}, \varepsilon_m \longrightarrow 0$, $(m \longrightarrow \infty)$ not satisfying (7.1). We can assume, by extracting a subsequence if necessary that, $u_{\varepsilon_m} \longrightarrow u_0$, $(m \longrightarrow \infty)$ uniformly on the compact subsets of $(-1,1)\times D$, and then u_0 would satisfy (7.2).

Let us take $A^{\varepsilon_m} = A^0$, $A_0^{\varepsilon_m} = A_0^0$, $A_j^{\varepsilon_m} = A_j^0$, $1\le j\le n_2$, $A_{ij}^{\varepsilon_m} = A_{ij}^0$, $1\le i,j\le n_1$. Then using the assumption that the functions u_{ε_m} do not satisfy (7.1) and passing to the limit we have that

$$\theta^{2+\mu} < \sup_{(t,x,z)\in(-\theta^2,\theta^2)\times D_\theta}\Big|u(t,x,z) - A_0^0 - \sum_{1\le j\le n_1} A_j^0 x_j - A^0 t - \sum_{1\le i,j\le n_1} A^0_{ij}x_ix_j - \sum_{n_1<j\le n_2} A_j^0 x_j\Big| < c\theta^{2+\mu'}$$

for all $\theta \in (0,1)$ which is absurd. Hence the lemma.

LEMMA 7.2. *Let θ, μ and ε_0 be as in Lemma 7.1. Then there is a constant $c>0$ such that for all $m\in\mathbb{N}$ and $\varepsilon\in(-1,1)$ such that $\varepsilon \le \theta^{m-1}\varepsilon_0$ and all u_ε satisfying*

$$\Big(\frac{\partial}{\partial t} + L_\varepsilon\Big)u_\varepsilon = 0 \text{ in } (-1,1)\times D, \quad \|u_\varepsilon\|_\infty \le 1$$

we have that

$$\sup_{(t,x,z)\in(-\theta^{2m},\theta^{2m})\times D_{\theta^m}}\Big|u_\epsilon(t,x,z) - A_0^{\varepsilon,m} - \sum_{1\le j\le n_1} A_j^{\varepsilon,m}F_j^\epsilon(x,z) - A^{\varepsilon,m}t - \sum_{1\le i,j\le n_1} A_{ij}^{\varepsilon,m}F_{ij}^\epsilon(x,z) - \sum_{n_1<j\le n_2} A_j^{\varepsilon,m}F_j^\epsilon(x,z)\Big| < \theta^{m(2+\mu)} \tag{7.3}$$

where $A_0^{\varepsilon,m}$, $A^{\varepsilon,m}$, $A_j^{\varepsilon,m}$, $1\le j\le n_2$, $A_{ij}^{\varepsilon,m}$, $1\le i,j\le n_1$ are constants satisfying

$$|A_0^{\varepsilon,m}| < c,\ |A^{\varepsilon,m}| < c,\ |A_j^{\varepsilon,m}| < c,\ 1\le j\le n_2,\ |A_{ij}^{\varepsilon,m}| < c,\ 1\le i,\ j\le n_1$$

$$\Big(\frac{\partial}{\partial t} + L_\varepsilon\Big)\Big[A^{\varepsilon,m}t + \sum_{1\le i,j\le n_1} A_{ij}^{\varepsilon,m}F_{ij}^{\varepsilon,m}(x,z) + \sum_{n_1<j\le n_2} A_j^{\varepsilon,m}F_j^{\varepsilon,m}(x,z)\Big] = 0.$$

PROOF. The lemma will be proved by induction. For $m=1$ we are in the case of Lemma 7.1. So assume that (7.3) is true for some $m\in\mathbb{N}$. We put

$$w_\varepsilon(x,z) = \theta^{m(2+\mu)}\Big[u_\epsilon(\theta^{2m}t, \tau_{\theta^m}x, z) - A_0^{\varepsilon,m} - \sum_{1\le j\le n_1} A_j^{\varepsilon,m}F_j^\epsilon(\tau_{\theta^m}x,z) - A^{\varepsilon,m}\theta^{2m}t \\ \sum_{1\le i,j\le n_1} A_{ij}^{\varepsilon,m}F_{ij}^\epsilon(\tau_{\theta^m}x,z) - \sum_{n_1<j\le n_2} A_j^{\varepsilon,m}F_j^\epsilon(\tau_{\theta^m}x,z)\Big]. \tag{7.4}$$

Then we have that

$$\Big(\frac{\partial}{\partial t}+L_{\varepsilon\theta^{-m}}\Big)w_\varepsilon=0 \text{ in } (-1,1)\times D, \quad \|w_\varepsilon\|_\infty\le 1.$$

Therefore it follows from Lemma 7.1 that, for $\varepsilon\theta^{-m}\le\varepsilon_0$ we have that

(7.5)

$$\sup_{(t,x,z)\in(-\theta^2,\theta^2)\times D_\theta}\Big|w_\epsilon(t,x,z)-B_0^\varepsilon-\sum_{1\le j\le n_1}B_j^\varepsilon F_j^{\varepsilon\theta^{-m}}(x,z)-B^\varepsilon t$$
$$-\sum_{1\le i,j\le n_1}B_{ij}^\varepsilon F_{ij}^{\varepsilon\theta^{-m}}(x,z)-\sum_{n_1<j\le n_2}B_j^\varepsilon F_j^{\varepsilon\theta^{-m}}(x,z)\Big|<\theta^{2+\mu}$$

with

$$|B_0^\varepsilon|<c,\ |B^\varepsilon|<c,\ |B_j^\varepsilon|<c, 1\le j\le n_2, |B_{ij}^\varepsilon|<c\ , 1\le i,\ j\le n_1$$
$$\Big(\frac{\partial}{\partial t}+L_{\varepsilon\theta^{-m}}\Big)\Big[B^\varepsilon t+\sum_{1\le i,j\le n_1}B_{ij}^\varepsilon F_{ij}^{\varepsilon\theta^{-m}}(x,z)+\sum_{n_1<j\le n_2}B_j^\varepsilon F_j^{\varepsilon\theta^{-m}}(x,z)\Big]=0.$$

Let us put

$$A_0^{\varepsilon,m+1}=A_0^{\varepsilon,m}+\theta^{m(2+\mu)}B_0^\varepsilon A^{\varepsilon,m+1}=A^{\varepsilon,m}+\theta^{m\mu}B^\varepsilon,$$
$$A_j^{\varepsilon,m+1}=A_j^{\varepsilon,m}+\theta^{m(1+\mu)}B_j^\varepsilon,\quad 1\le j\le n_1,$$
$$A_j^{\varepsilon,m+1}=A_j^{\varepsilon,m}+\theta^{m\mu}B_j^\varepsilon,\quad n_1<j\le n_1,$$
$$A_{ij}^{\varepsilon,m+1}=A_{ij}^{\varepsilon,m}+\theta^{m\mu}B_{ij}^\varepsilon,\quad 1\le i,j\le n_1.$$

Then putting (7.4) and (7.5) together we have that

$$\sup_{(t,x,z)\in(-\theta^2,\theta^2)\times D_{\theta^m}}\theta^{-m(2+\mu)}\Big|u_\epsilon(\theta^{2m}t,\tau_{\theta^m}x,z)-A_0^{\varepsilon,m+1}-\sum_{1\le j\le n_1}A_j^{\varepsilon,m+1}F_j^\varepsilon(\tau_{\theta^m}x,z)$$
$$-A^{\varepsilon,m+1}t-\sum_{1\le i,j\le n_1}A_{ij}^{\varepsilon,m+1}F_{ij}^\varepsilon(\tau_{\theta^m}x,z)-\sum_{n_1<j\le n_2}A_j^{\varepsilon,m+1}F_j^\varepsilon(\tau_{\theta^m}x,z)\Big|<\theta^{2+\mu}$$

and from this that

$$\sup_{(t,x,z)\in(-\theta^{2(m+1)},\theta^{2(m+1)})\times D_{\theta^{m+1}}}\Big|u_\epsilon(t,x,z)-A_0^{\varepsilon,m+1}-\sum_{1\le j\le n_1}A_j^{\varepsilon,m+1}F_j^\varepsilon(x,z)-A^{\varepsilon,m+1}t$$
$$-\sum_{1\le i,j\le n_1}A_{ij}^{\varepsilon,m+1}F_{ij}^\varepsilon(x,z)-\sum_{n_1<j\le n_2}A_j^{\varepsilon,m+1}F_j^\varepsilon(x,z)\Big|<\theta^{(m+1)(2+\mu)}$$

which proves the inductive step and the lemma follows.

COROLLARY 7.3. *Let ε_0 be as in Lemma 7.2. Then there is $c>0$ such that for all $\varepsilon\in(0,\varepsilon_0]$ and all u_ε satisfying*

$$\Big(\frac{\partial}{\partial t}+L_\varepsilon\Big)u_\varepsilon=0 \ in\ (-1,1)\times D,\quad \|u_\varepsilon\|_\infty\le 1$$

we have that

$$\sup_{(t,x,z)\in(-(\frac{\varepsilon}{\varepsilon_0})^2,(\frac{\varepsilon}{\varepsilon_0})^2)\times D_{\frac{\varepsilon}{\varepsilon_0}}} \Big| u_\epsilon(t,x,z) - A_0^\varepsilon - \sum_{1\leq j\leq n_1} A_j^\varepsilon F_j^\varepsilon(x,z)\Big| < c\Big(\frac{\varepsilon}{\varepsilon_0}\Big)^2 \tag{7.6}$$

where A_0^ε, A^ε, A_j^ε, $1\leq j\leq n_1$ *are constants satisfying*

$$|A_0^\varepsilon| < c,\ |A^\varepsilon| < c,\ |A_j^\varepsilon| < c,\ 1\leq j\leq n_1.$$

LEMMA 7.4. *There is a constant* $c > 0$ *such that for all* u *satisfying*

$$\Big(\frac{\partial}{\partial t} + L\Big)u = 0 \ in\ (-R^2,R^2)\times D_R, \quad R\geq 1$$

we have that

$$\sup_{(t,x,z)\in(-1,1)\times D} \Big| u(t,x,z) - A_0 - \frac{1}{R}\sum_{1\leq j\leq n_1} A_j[(x_j - \chi^j(x,z)]\Big| < \frac{c}{R^2}\|u\|_\infty \tag{7.7}$$

where A_0, A_j, $1\leq j\leq n_1$ *are constants satisfying*

$$|A_0| < c,\ |A_j| < c,\ \ 1\leq j\leq n_1.$$

PROOF. The lemma follows from Corollary 7.3 and the observation that if u satisfies

$$\Big(\frac{\partial}{\partial t} + L\Big)u = 0 \text{ in } (-R^2,R^2)\times D_R, \quad R\geq 1$$

then the function u_ε defined by $u_\varepsilon(t,x,z) = u(R^2t,\tau_R x,z)$, $\varepsilon = 1/R$ satisfies

$$\Big(\frac{\partial}{\partial t} + L_\varepsilon\Big)u_\varepsilon = 0 \text{ in } (-1,1)\times D.$$

Let us recall that we have fixed a basis $\{X_n,\dots,X_1,Z_0,\dots,Z_{-r}\}$ of $\mathfrak{g}$, with $\{X_n,\dots,X_1\}$ a basis of $\mathfrak{q}$ as in Proposition 3.3 and $\{Z_0,\dots,Z_{-r}\}$ a basis of $\mathfrak{m}$. We have also identified the elements of $\mathfrak{g}$ with the left invariant vector fields on G and if $X\in\mathfrak{g}$ then we denote by ${}_NX$ the left invariant vector field on G_N (G and G_N have been identified as differential manifolds) satisfying ${}_NX(e) = X(e)$ (*cf.* Section 3). Note that ${}_NX_i = X_i$, $1\leq i\leq k$.

We put

$$H_i = Z_i + \sum_{k<j\leq n_1} Z_i(\chi^j)_N X_j, \quad -r\leq i\leq 0,\ \ H_i = X_i + \sum_{k<j\leq n_1} X_i(\chi^j)_N X_j, \quad 1\leq i\leq k.$$

We have the following proposition, which follows from (7.7) and the fact that L is hypoelliptic (*cf.* [6]).

PROPOSITION 7.5. *There is a constant $c > 0$ such that*

$$|X_i u(0,e)| \leq \frac{c}{R}\|u\|_\infty, \quad 1 \leq i \leq n_1,$$
$$|Z_i u(0,e)| \leq \frac{c}{R}\|u\|_\infty, \quad -r \leq i \leq 0$$
$$\left|\frac{\partial}{\partial t}u(0,e)\right| \leq \frac{c}{R^2}\|u\|_\infty, \; |X_i u(0,e)| \leq \frac{c}{R^2}\|u\|_\infty, \quad n_1 < i \leq n$$
$$|X_j H_i u(0,e)| \leq \frac{c}{R^2}\|u\|_\infty, \quad -r \leq i \leq k, 1 \leq j \leq n,$$
$$|Z_j H_i u(0,e)| \leq \frac{c}{R^2}\|u\|_\infty, \quad -r \leq i \leq k, -r \leq j \leq 0$$

for all u satisfying

$$\left(\frac{\partial}{\partial t} + L\right)u = 0 \text{ in } (-R^2, R^2) \times S_E(e,R), \quad R \geq 1.$$

We shall need the following result of L. Saloff-Coste [19]:

THEOREM 7.6 (*cf.* [19]). *Given any $0 < a < b < 1$, $0 < \gamma < 1$ there is a constant $c > 0$ such that for all $(s,g) \in \mathbb{R} \times G$, $R > 0$ and every positive $0 \leq u \in C^\infty([s-R^2,s] \times \bar{S}_E(g,R)$ satisfying $\partial/\partial t + L)u = 0$ in $(s-R^2,s) \times S_E(g,R)$ we have*

$$u(t,y) \leq cu(s,g), \; (t,y) \in [s-bR^2, s-aR^2] \times S_E(g,\gamma R).$$

An immediate consequence of Proposition 7.5 and Theorem 7.6 is the following result, a particular case of which is Theorem 1.

THEOREM 7.7. *For every integer $\ell \geq 0$ and $0 < a < b < 1$ there is a constant $c > 0$ such that for all $t \geq 1$*

$$\left|\frac{\partial^\ell}{\partial t^\ell} Z_i u(at,x)\right| \leq ct^{-\ell-\frac{1}{2}}u(bt,x), \quad -r \leq i \leq 0$$
$$\left|\frac{\partial^\ell}{\partial t^\ell} X_i u(at,x)\right| \leq ct^{-\ell-\frac{1}{2}}u(bt,x), \quad 1 \leq i \leq n$$
$$\left|\frac{\partial^\ell}{\partial t^\ell} X_i u(at,x)\right| \leq ct^{-\ell-1}u(bt,x), \quad n_1 < i \leq n$$
$$\left|\frac{\partial^\ell}{\partial t^\ell} Z_{iN} X_j u(at,x)\right| \leq ct^{-\ell-1}u(bt,x), \quad -r \leq i \leq 0, \; k < j \leq n$$
$$\left|\frac{\partial^\ell}{\partial t^\ell} X_{iN} X_j u(at,x)\right| \leq ct^{-\ell-1}u(bt,x), \quad 1 \leq i \leq n, \; k < j \leq n$$
$$\left|\frac{\partial^\ell}{\partial t^\ell} X_i H_j u(at,x)\right| \leq ct^{-\ell-1}u(bt,x), \quad 1 \leq i \leq n, \; -r \leq j \leq k$$
$$\left|\frac{\partial^\ell}{\partial t^\ell} Z_i H_j u(at,x)\right| \leq ct^{-\ell-1}u(bt,x), \quad 1 \leq i \leq n, \; -r \leq j \leq k$$

for all $u \geq 0$ such that $(\partial/\partial t + L)u = 0$ in $(0,t) \times S_E(x,\sqrt{t})$.

8. **The proof of Theorem 2.** *It is easy to see that the Riesz transforms $R_i = E_i L^{-\frac{1}{2}}$, $1 \le i \le p$ and their adjoints $R_i^* = L^{-\frac{1}{2}} E_i$, $1 \le i \le p$ are bounded in L^2. This follows from the observation that*

$$\sum_{1 \le i \le p} \|R_i \varphi\|_2^2 = -\sum_{1 \le i \le p} (E_i^2 L^{-\frac{1}{2}} \varphi, L^{-\frac{1}{2}} \varphi) = (L^{\frac{1}{2}} \varphi, L^{-\frac{1}{2}} \varphi) = \|\varphi\|_2^2.$$

So we only need to prove that they are bounded from L^1 to weak-L^1. Then by interpolation we can prove that they are bounded on $L^q, 1 < q < 2$ and by duality on $L^q, 2 < q < \infty$ (cf. [19]).

We put $\gamma(t) = dg$-measure $\big(S_E(e,t)\big)$.

We denote by $X^y K(x,y)$ the derivative of the kernel $K(x,y), x, y \in G$ with respect to the vector field X and the variable y.

We say that the kernel $K(x,y)$ satisfies standard estimates if there is a constant $c > 0$ such that

$$(8.1)\quad |K(x,y)| \le \frac{c}{\gamma\big(d(x,y)\big)},\ |E_i^y K(x,y)| \le \frac{c}{d_E(x,y)\gamma\big(d_E(x,y)\big)}, \quad x, y \in G,\ 1 \le i \le p.$$

We recall the following Gaussian estimate for the heat kernel $p_t(x,y)$ (*i.e.* the fundamental solution of the equation $(\partial/\partial t + L)u = 0$) due to N. Th. Varopoulos [24] (which we state here in a less sharp form):
there are constants $c_1, c_2 > 0$ such that

$$(8.2)\quad c_1 \gamma(\sqrt{t})^{-1} \exp\Big(-\frac{d_E^2(x,y)}{c_1 t}\Big) \le p_t(x,y) \le c_2 \gamma(\sqrt{t})^{-1} \exp\Big(-\frac{d_E^2(x,y)}{c_2 t}\Big), \quad x, y \in G,\ t > 0.$$

Moreover since the operator L is self adjoint the heat kernel $p_t(x,y)$ is symmetric, *i.e.* $p_t(x,y) = p_t(y,x)$, $x, y \in G, t > 0$.

We also recall the following small time Harnack inequalities also due to N. Th. Varopoulos [24]:

For all integers $\ell, \mu \ge 0$ and $0 < a < b < 1$ there is a constant $c > 0$ such that

$$(8.3)\qquad \Big|\frac{\partial^\ell}{\partial t^\ell} E_{i_1} \cdots E_{i_\mu} u(at,x)\Big| \le c t^{-\ell-\frac{\mu}{2}} u(bt,x), \quad x \in G,\ 0 < t \le 1$$

for all $u \ge 0$ satifying $(\partial/\partial t + L)u = 0$ in $(0,t) \times S_E(x,\sqrt{t})$.

Let $T_t, t > 0$ be the semigroup of operators associated to L, *i.e.* $T_t\varphi(x) = \int p_t(x,y)\varphi(y)\,dt$.

Then

$$R_i(\varphi) = \int_0^\infty t^{-\frac{1}{2}} E_i T_t(\varphi)\,dt,\ R_i^*(\varphi) = \int_0^\infty t^{-\frac{1}{2}} T_t(E_i\varphi)\,dt.$$

Hence the kernels $K_i(x,y)$ and $K_i^*(x,y)$ of the operators R_i and R_i^* respectively, $1 \le i \le p$, are given by

$$(8.4)\qquad K_i(x,y) = \int_0^\infty t^{-\frac{1}{2}} E_i^x p_t(x,y)\,dt,\ K_i^*(x,y) = \int_0^\infty t^{-\frac{1}{2}} E_i^y p_t(x,y)\,dt.$$

Moreover

$$(8.5)\qquad E_j^y K_i(x,y) = \int_0^\infty t^{-\frac{1}{2}} E_j^y E_i^x p_t(x,y)\,dt\ E_j^y K_i^*(x,y) = \int_0^\infty t^{-\frac{1}{2}} E_j^y E_i^y p_t(x,y)\,dt.$$

It follows from (8.3) and (8.4) and Theorem 1 that there is $c > 0$ such that

$$(8.6)\qquad |K_i(x,y)| \le \frac{c}{\gamma\big(d_E(x,y)\big)},\ |K_i^*(x,y)| \le \frac{c}{\gamma\big(d_E(x,y)\big)},\quad x,y \in G.$$

The operators R_i, $1 \le i \le p$. We observe that the function $u(t,y) = E_i^x p_t(x,y)$ satisfies $(\partial/\partial t + L)u = 0$. Hence applying Theorem 1 twice we can see that the kernel $K_i(x,y)$ satisfies the standard estimates (8.1). Now, applying the theory of singular integral operators on spaces of homogeneous type developed in [9] we get that the operators R_i, $1 \le i \le p$ are bounded from L^1 to weak-L^1.

The operators R_i^*, $1 \le i \le p$. The problem in this case is that the estimates (8.1) are not satisfied by the kernels $K_i^*(x,y)$ of the operators R_i^*, $1 \le i \le p$. This is due to the fact that as we have seen in Section 1, the inequalities (8.2) are not true for $\ell \ge 2, t > 1$ and therefore we do not have the appropriate estimates for the $E_j^y E_i^y p_t(x,y)$, $1 \le i,j \le p$.

To get around this difficulty we shall consider the operators

$$T_{i0}(\varphi) = \int_0^1 t^{-\frac{1}{2}} T_t(E_i\varphi)\,dt,\ T_{i1}(\varphi) = \int_1^\infty t^{-\frac{1}{2}} T_t(E_i\varphi)\,dt$$

whose kernels, also denoted by $T_{i0}(x,y)$ and $T_{i1}(x,y)$, are given by

$$T_{i0}(x,y) = \int_0^1 t^{-\frac{1}{2}} E_i^x p_t(x,y)\,dt,\ T_{i1}(x,y) = \int_1^\infty t^{-\frac{1}{2}} E_i^y p_t(x,y)\,dt.$$

Clearly $R_i^* = T_{i0} + T_{i1}$. We shall prove that both T_{i0} and T_{i1} are bounded from L^1 to weak-L^1, $1 \le i \le p$.

The operators T_{i0}, $1 \le i \le p$. We observe that the kernel of the operator T_{i0} is integrable at infinity and singular near the diagonal. Actually it is the part of the kernel of the operator R_i^* that is singular near the diagonal. Hence it is bounded on L^2. Also using the inequalities (8.3) we see that the kernel $T_{i0}(x,y)$ satisfies the standard estimates (8.1). So, arguing in the same way as in the case of the operators R_i, we can prove that the operators T_{i0} are bounded from L^1 to weak-L^1.

The operators T_{i1}, $1 \le i \le p$. From now on, as in Section 7, when we use the index j we assume that at the same time $\mathbb{O}_j = \mathbb{R}$.

We shall need the following lemma whose proof will be given at the end of this section.

LEMMA 8.1. *For all $k < j \le n$ the operator $W_j(\varphi) = \int_1^\infty t^{-\frac{1}{2}} T_t({}_N X_j \varphi)\,dt$, whose kernel, also denoted by $W_j(x,y)$, is given by $W_j(x,y) = \int_1^\infty t^{-\frac{1}{2}}\, {}_N X_j^y p_t(x,y) dt$ is bounded on L^2 and since $W_j(x,y)$ (which is a kernel integrable near the diagonal and singular at infinity) satisfies the standard estimates (8.1), W_j is bounded from L^1 to weak-L^1.*

The L^1 to weak-L^1 boundedness of T_{i1} follows easily from the above lemma. Indeed, since the functions $E_i\chi^j, k < j \le n_1$ are bounded, the operators $W_j E_i \chi^j(f) = W_j\big((E_i\chi^j)f\big)$ are bounded on L^2 and from L^1 to weak-L^1.

Let us put

$$T = T_{i1} + \sum_{k<j\le n_1} W_j E_i \chi^j, \ H = E_i + \sum_{k<j\le n_1} (E_i\chi^j)_N X_j.$$

Then T is bounded on L^2 and its kernel, also denoted by $T(x,y)$ is given by

$$T(x,y) = \int_1^\infty t^{-\frac{1}{2}} H^y p_t(x,y)\, dt.$$

Now it follows from Theorem 7.7 that $T(x,y)$ satisfies the standard estimates (8.1). Hence T is bounded from L^1 to weak-L^1. This, together with the fact that the operators $W_jE_i\chi^j$, $k<j\le n_1$ are bounded from L^1 to weak-L^1, implies that the operator T_{i1} is bounded from L^1 to weak-L^1, which completes the proof of Theorem 2.

9. **The proof of Lemma 8.1.** Of course Lemma 8.1 can be proved using some version of the T1 Theorem for spaces of homogeneous type (*cf.* [7], [10], [17]). We shall not follow this approach though. Instead we shall try to explain how the proof given in G. David and J. L. Journé [11] (in particular Section III of that paper) can be adapted in our case. The reader is referred to that paper for the omitted details.

To simplify things we shall work with the control distance $d(x,y)$ asssociated to the the basis $\{X_n,\ldots,X_1,Z_0,\ldots,Z_{-r}\}$ of $\mathfrak{g}$ instead of the control distance $d_E(x,y)$ associated to the fields $\{E_1,\ldots,E_p\}$ (*cf.* Section 4). For $t\ge 1$ the estimates (8.2) are still valid with $d_E(x,y)$ replaced by $d(x,y)$.

We put $S(x,t)=\{y\in G, d(x,y)<t\}, t>0$.

The kernel $W_j(x,y)$ is integrable near the diagonal and singular at infinity. Furthermore it is a standard kernel, since it follows from (8.2) and Theorem 7.7 that there is a constant $c>0$ such that

$$|W_j(x,y)| \le \frac{c}{\gamma\big(d(x,y)\big)}, \tag{8.7}$$

$$|Y^y W_j(x,y)| \le \frac{c}{d(x,y)\gamma\big(d(x,y)\big)},\ |Y^x W_j(x,y)| \le \frac{c}{d(x,y)\gamma\big(d(x,y)\big)}$$

for all $x,y\in G$, $d(x,y)\ge 1$.

Let W_j^* the adjoint of W_j. Then we have

$$W_j 1 = 0,\ W_j^* 1 = 0. \tag{8.8}$$

Indeed, that $W_j1=0$ follows from (8.7) (*cf.* [11]). To see that $W_j^*1=0$ let us write W_j as the limit, as $A\to\infty$, of the operators W_{Aj} whose kernels are given by $\int_1^A t^{-\frac{1}{2}}{}_N X_j^y p_t(x,y)\,dt$. Then, that $W_j^*1=0$, follows from (8.7) and the observation that $W_{Aj}^*1=0$, $A>1$.

Let d be as in Section 4.

Let $f\in C_o^\infty\big((-1,1)\big), f\ge 0, \int f(x)\,dx=1$ and for $g=xz, z\in M, x=(x_n,\ldots,x_1)\in Q$ (*cf.* Section 3) put $h(g)=\prod_{1\le i\le n, \mathfrak{O}_i=\mathbb{R}} f(x_i)$ and

$$\varphi_i(g) = \frac{1}{2^{-di}} h\big(\tau_{2^{-i}}(g)\big), \quad i\ge 0,\ i\in\mathbb{Z}.$$

Let U be an open neighborhood of e which is the diffeomorphic image of a convex neighborhood U' of 0 under the exponential map

$$(x_n, \ldots, x_1, z_0, \ldots, z_{-r}) \longrightarrow \exp(x_n X_n + \cdots + x_1 X_1 + z_0 Z_0 + \cdots + z_{-r} Z_{-r}).$$

Let $D = n + r + 1$.

Let $h' \in C_o^\infty(U), h' \geq 0, \int h'(g)dg = 1$.

For $g \in U, g = \exp(x_n X_n + \cdots + x_1 X_1 + z_0 Z_0 + \cdots + z_{-r} Z_{-r}), (x_n, \ldots, x_1, z_0, \ldots, z_{-r}) \in 2^i U'$ we put

$$\varphi_i(g) = \frac{1}{2^{-Di}} h'\big(\exp[2^{-i}(x_n X_n + \cdots + x_1 X_1 + z_0 Z_0 + \cdots + z_{-r} Z_{-r})]\big), \quad i < 0,\ i \in \mathbb{Z}$$

and $\varphi_i(g) = 0$ for all the other $g \in G$.

We put

$$\psi_i = \psi_i - \psi_{i+1}, \quad i \in \mathbb{Z}.$$

We also put

$$\begin{gathered} p_i(s) = 1,\ 0 \leq s \leq 2^i,\ i < 0,\ i \in \mathbb{Z} \\ p_i(s) = 2^i,\ 2^i < s \leq 1,\ i < 0,\ i \in \mathbb{Z} \\ p_i(s) = \frac{2^i}{s^{d+1}},\ s > 1,\ i < 0,\ i \in \mathbb{Z} \\ p_i(s) = 2^{-di},\ 0 < s \leq 2^i,\ i \geq 0,\ i \in \mathbb{Z} \\ p_i(s) = \frac{2^i}{s^{d+1}},\ s > 2^i,\ i \geq 0,\ i \in \mathbb{Z}. \end{gathered}$$

Using the notations of [11], we denote by S_i, T_i, T_i', T_i'' the operators whose kernels also denoted by $S_i(x,y)$, $T_i(x,y)$, $T_i'(x,y)$, $T_i''(x,y)$, are given by

$$\begin{gathered} S_i(x,y) = \iint \varphi_i(v) W_j(xv, yw) \varphi_i(w)\, dv\, dw \\ T_i(x,y) = \iint \varphi_i(v) W_j(xv, yw) \psi_i(w)\, dv\, dw,\ T_i'(x,y) = \iint \psi_i(v) W_j(xv, yw) \varphi_i(w)\, dv\, dw \\ T_i''(x,y) = \iint \psi_i(v) W_j(xv, yw) \psi_(w)\, dv\, dw. \end{gathered}$$

We have that

$$\sum_{-M \leq i \leq N} T_i + T_i' + T_i'' = S_{-M} - S_{N+1}, \quad N, M \in \mathbb{N}.$$

Observe that if K is a compact $K \subseteq G$, then there is $c > 0$, $c = c(K, h, h')$ such that

$$(8.9) \qquad |S_i(x,y)| \leq \iint \varphi_i(v) \frac{c}{\big(1 + d(v,w)\big)^{D-1}} |_N X_j \varphi_i(w)|\, du\, dw \leq c 2^{-di}, \quad x, y \in K.$$

Hence the operators S_i converge weakly to 0 as $i \longrightarrow \infty$. On the other hand, as $i \longrightarrow -\infty$, the S_i converge to W_j. So, the operator $\sum_{-M \leq i \leq N} T_i + T_i' + T_i''$ converges weakly to W_j, as $i \longrightarrow \infty$. It follows that to prove that W_j is bounded on L^2, it is enough to prove that the operators $T_i, T_i', T_i'', i \in \mathbb{Z}$ are bounded on L^2 and that the sums $\sum_{-M\le i\le N} T_i$, $\sum_{-M\le i\le N} T_i'$ and $\sum_{-M\le i\le N} T_i''$ converge strongly to bounded operators. To do this we have to apply Cotlar's lemma to the sequences of operators $\{T_i\}$, $\{T_i'\}$ and $\{T_i''\}$ (*cf.* [11]). For this, we need the following estimates for the kernel $T_i(x,y)$ (it can be proved in the same way that the kernels $T_i'(x,y)$ and $T_i''(x,y)$ satisfy similar estimates too)

$$|T_i(x,y)| \le cp_i\big(d(x,y)\big) \tag{8.10}$$

$$\begin{aligned}|T_i(x,y) - T_i(x',y)| + |T_i(y,x) - T_i(y,x')|\\ \le c\min\Big(1, \frac{d(x,x')}{2^i}\Big)\big[p_i\big(d(x,y)\big) + p_i(d(x',y))\big]\end{aligned} \tag{8.11}$$

$$\int T_i(x,y)\,dy = 0, \quad x \in G \tag{8.12}$$

$$\int T_i(x,y)\,dx = 0, \quad y \in G \tag{8.13}$$

(8.13) follows from (8.8) and (8.12) from the fact that $\int \psi(w)\,dw = 0$.

To prove (8.10) we shall distinguish different cases. When $i < 0$, $d(x,y) \le 2^i 10$ then (8.10) follows from the fact that the kernel $T_i(x,y)$ is integrable near the diagonal.

When $i < 0$, $2^i 10 < d(x,y) \le 1$ then (8.10) follows from the observation that since $\int \psi_i(w)\,dw = 0$,

$$\begin{aligned}|T_i(x,y)| &= \Big|\iint \varphi_i(v)[W_j(xv,yw) - W_j(xv,y)]\psi_i(w)\,dv\,dw\Big|\\ &\le c2^i \iint \varphi_i(v)|\psi_i(w)|\,dv\,dw = c2^i.\end{aligned}$$

When $i \ge 0$, $d(x,y) \le 2^i 10$ then (8.10) follows from (8.9).

When $i \ge 0$, $d(x,y) > 2^i 10$ or $i < 0$, $d(x,y) > 1$ then (8.10) follows from the observation that since $\int_i \psi(w)dw = 0$

$$\begin{aligned}|T_i(x,y)| &= \Big|\iint \varphi_i(v)[W_j(xv,yw) - W_j(xv,y)]\psi_i(w)\,dv\,dw\Big|\\ &\le c\frac{2^i}{d(x,y)^{D+1}} \iint \varphi_i(v)\psi_i(w)\,dv\,dw = c\frac{2^i}{d(x,y)^{D+1}}.\end{aligned}$$

To prove (8.11) we observe that we can assume that $d(x,x') < 2^i$, because otherwise it follows from (8.10). The next thing to observe is that if $Y \in \{X_n,\ldots,X_1,Z_0,\ldots,Z_{-r}\}$ and Y_R is the right invariant vector field on G such that $Y_R(e) = Y(e)$ then

$$|Y^y T_i(x,y)| = \Big|\iint \varphi_i(v) W_j(xv,yw) Y_R \psi_i(w)\,dv\,dw\Big|$$

and arguing in the same way as for (8.10) we get

$$|Y^y T_i(x,y)| \le c2^i p_i\big(d(x,y)\big). \tag{8.14}$$

Similarly

$$(8.15)\qquad |Y^x T_i(x,y)| = \int\!\!\int (Y_R\varphi_i)(v) W_j(xv, yw)\psi_i(w)\,dv\,dw| \leq c2^i p_i\big(d(x,y)\big).$$

Now to prove (8.11) it is enough to to join the points x and x' with a piecewise smooth curve $\gamma(t)$ of length $|\gamma| \leq 2^i 2$ (*cf.* Section 4) and then use (8.14) and (8.15).

Once we have (8.10), (8.11), (8.12) and (8.13), then we can prove (*cf.* [11]) that there is $c > 0$ such that

$$||T_i T_\ell^*||_{L^2,L^2} + ||T_i^* T_\ell||_{L^2,L^2} \leq c2^{-|i-\ell|}, \quad i,\ell \in \mathbb{Z}$$

an estimate which allows the application of Cotlar's lemma to the sequence of operators $\{T_i\}$. For the sequences of operators $\{T_i'\}$ and $\{T_i''\}$ we can argue in a similar way.

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