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EXPONENTIAL ESTIMATES FOR THE CONJUGATE FUNCTION ON LOCALLY COMPACT ABELIAN GROUPS

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ABSTRACT. Let G be a locally compact Abelian group, with character group X. Suppose that X contains a measurable order P. For $f \in \mathcal{L}_2(G)$, the conjugate function of f is the function $\tilde{f}$ whose Fourier transform satisfies the identity $\hat{\tilde{f}}(\chi) = -i\,\mathrm{sgn}_P(\chi)\hat{f}(\chi)$, for almost all χ in X, where $\mathrm{sgn}_P(\chi) = -1, 0, 1$, according as $\chi \in (-P)\backslash\{0\}$, $\chi = 0$, or $\chi \in P\backslash\{0\}$. We prove that, when f is bounded with compact support, the conjugate function satisfies some weak type inequalities similar to those of the Hilbert transform of a bounded function with compact support in $\mathbb{R}$. As a consequence of these inequalities, we prove that $\tilde{f}$ possesses strong integrability properties, whenever f is bounded and G is compact. In particular, we show that, when G is compact and f is continuous on G, the function $\exp(|p\tilde{f}|)$ is integrable for all $p > 0$.

1. **Introduction.**

The Hilbert transform on $\mathbb{R}$. (1.1). In this essay we investigate some properties of the conjugate function of a bounded function on a locally compact Abelian group G. It is well-known that the conjugate function $\tilde{f}$ of a function f in $\mathcal{L}_\infty(\mathbb{T})$ may fail to be bounded. For example, the conjugate function of

$$f(x) = \frac{1}{2}(\pi - x) = \sum_{n=1}^{\infty} \frac{\sin\, nx}{n}$$

is the function

$$\tilde{f}(x) = -\sum_{n=1}^{\infty} \frac{\cos\, nx}{n} = \log\,(2\ \sin\ \tfrac{1}{2}x),$$

for $0 < x < 2\pi$. See [11], Vol. 1, p. 253, 2.2. Nevertheless, the conjugate function $\tilde{f}$ continues to have strong integrability properties. We have the following result: if $|f| \leqq 1$, then

$$\int_0^{2\pi} \exp\,(\lambda|\tilde{f}|)dx \leqq C_\lambda$$

for $0 < \lambda < \frac{1}{2}\pi$, where C_λ is a constant independent of f. See [11], Vol. 1, p. 254, Theorem (2.11). While this property obviously fails for the Hilbert transform on $\mathbb{R}$,

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an equivalent property of the conjugate function on $\mathbb{T}$ holds for the Hilbert transform on $\mathbb{R}$. We state it and other related properties below.

Let f be in $\mathcal{L}_p(\mathbb{R})$, where $p \in [1, \infty[$, and let $\alpha > 0$. The *truncated Hilbert transform*, the *Hilbert transform*, and the *maximal Hilbert transform*, are defined, respectively, by:

$$\mathrm{H}_\alpha f(x) = \frac{1}{\pi} \int_{\alpha \leqq |t|} f(x-t) \frac{1}{t}\, dt;$$

$$\mathrm{H} f(x) = \lim_{\alpha \downarrow 0} \mathrm{H}_\alpha f(x);$$

and

$$\mathrm{MH} f(x) = \sup_{0<\alpha} |\mathrm{H}_\alpha f(x)|.$$

We will establish analogues of the following properties of the Hilbert transform. Suppose that f is in $\mathcal{L}_\infty(\mathbb{R})$ with support contained in an interval of length A. Then there are constants c_1, c_2, c_3, and c_4, independent of f and A, such that:

$$(i) \qquad \lambda(\{x \in \mathbb{R} : |\mathrm{H} f(x)| > y\}) \leqq c_1 \frac{A \| f \|_\infty}{y} \exp\left(-c_2 \frac{y}{\| f \|_\infty}\right);$$

and

$$(ii) \qquad \lambda(\{x \in \mathbb{R} : |\mathrm{MH} f(x)| > y\}) \leqq c_3 \frac{A \| f \|_\infty}{y} \exp\left(-c_4 \frac{y}{\| f \|_\infty}\right),$$

where λ denotes Lebesgue measure on $\mathbb{R}$. See [7], Section 4.4, (4.4.1), (4.4.2), p. 119.

In Section 2, we will introduce the ergodic Hilbert transform and establish the exponential estimates for this transform. In Section 3, we will use a separation theorem for measurable orders to approximate the conjugate function by a sequence of ergodic Hilbert transforms. We then obtain our main results as a consequence of the properties of the ergodic Hilbert transform.

The modified Hilbert transform on $\mathbb{R}$. (1.2). For the sake of the transference methods used in Section 2, we need to modify the definition of the truncated Hilbert transform and use kernels with compact support. For f in $\mathcal{L}_p(\mathbb{R})$, $1 \leqq p < \infty$, let

$$\begin{aligned} \mathrm{T}_n f(x) &= \frac{1}{\pi} \int_{1/n \leqq |t| \leqq n} f(x-t) \frac{1}{t}\, dt \\ &= \frac{1}{\pi} \int_{1/n \leqq |t-x| \leqq n} f(t) \frac{1}{x-t}\, dt, \qquad n = 1, 2, \ldots. \end{aligned}$$

Suppose that f is in $\mathcal{L}_\infty(\mathbb{R})$ with support contained in an interval of length A, and let $y > 0$. Write

$$\begin{aligned} \mathrm{T}_n f(x) &= \frac{1}{\pi} \int_{1/n \leqq |x-t|} \frac{f(t)}{x-t}\, dt - \frac{1}{\pi} \int_{n \leqq |x-t|} \frac{f(t)}{x-t}\, dt \\ &= h_1(x) + h_2(x). \end{aligned}$$

We have

$$\{x : |T_n f(x)| \geqq y\} \subseteq \left\{x : |h_1(x)| \geqq \frac{y}{2}\right\} \cup \left\{x : |h_2(x)| \geqq \frac{y}{2}\right\}$$
$$\subseteq \left\{x : \mathrm{MH} f(x) \geqq \frac{y}{2}\right\};$$

from which follow the inequalities

$$(i) \qquad \lambda(\{x \in \mathbb{R} : |T_n f(x)| > y\}) \leqq c_3 \frac{A\| f \|_\infty}{y} \exp\left(-c_4 \frac{y}{2\| f \|_\infty}\right),$$

and

$$(ii) \qquad \lambda(\{x \in \mathbb{R} : \sup_{1 \leqq n} |T_n f(x)| > y\}) \leqq c_3 \frac{A\| f \|_\infty}{y} \exp\left(-c_4 \frac{y}{2\| f \|_\infty}\right)$$

where c_3 and c_4 are as in (1.1.*ii*).

2. **The transference result.**

The ergodic Hilbert transform. (2.1). In this section we will be dealing with a locally compact Abelian group G and a fixed, continuous, nonzero homomorphism ϕ from $\mathbb{R}$ into G. Denote by μ the Haar measure on G. The homomorphism ϕ generates a one-parameter group of measure-preserving transformations $\{U_t : t \in \mathbb{R}\}$ on the group G: for t in $\mathbb{R}$ and x in G, let $U_t x = x - \phi(t)$. Let $p \in [1, \infty[$, and let f be in $\mathcal{L}_p(G)$, we define the *ergodic Hilbert transform* on $\mathcal{L}_p(G)$ (*with respect to the homomorphism* ϕ) by:

$$H^\# f(x) = \lim_{\epsilon \downarrow 0} H^\#_{1/\epsilon} f(x);$$

where

$$H^\#_{1/\epsilon} f(x) = \frac{1}{\pi} \int_{\epsilon \leqq |t| \leqq 1/\epsilon} f(U_t x) \frac{1}{t} \, dt.$$

In [5], Theorem 2, it is shown that the above limit exists μ-almost everywhere on G, for more general one-parameter groups of measure-preserving transformations acting on a σ-finite measure space $\mathcal{N}$. Several properties of the ergodic Hilbert transform which is defined by the action of $\mathbb{R}$ via the homomorphism ϕ are discussed in [3], Section 6. The measure-theoretic problems arising in the application of Fubini's theorem to integrals involving the function $(s, x) \longmapsto U_s f(x)$ on the product space $\mathbb{R} \times \mathcal{N}$ are no longer relevant when $\mathcal{N} = G$, and $U_s f(x) = f(x - \phi(s))$. (The measurability of the function $(s, x) \longmapsto f(x - \phi(s))$ on the product space $\mathbb{R} \times G$ is discussed in Lemma (2.2) below.) Hence the restriction to σ-compact measure spaces is not needed in our discussion, and we may use the results [5] on arbitrary locally compact Abelian groups.

Denote by $H^\#_n$ the operator given by:

$$H^\#_n f(x) = \frac{1}{\pi} \int_{1/n \leqq |t| \leqq n} f(x - \phi(t)) \frac{1}{t} \, dt.$$

Whenever the integral expression for $H_n^{\#} f(x)$ is written, it will be assumed that x belongs to a subset of G whose complement has measure 0, such that the integral is finite for all $n = 1, 2, \dots$.

We will investigate properties of the operator $H^{\#}$ restricted to functions in $\mathcal{L}_\infty(G)$.

LEMMA (2.2). *Let ϕ be a continuous nonzero homomorphism from $\mathbb{R}$ into G. Let I be a bounded interval in $\mathbb{R}$. Suppose that f is in $\mathcal{L}_\infty(G)$. Then for μ-almost all x in G, the function $s \longmapsto 1_I(s)f(x-\phi)s))$ is in $\mathcal{L}_\infty(\mathbb{R})$; and*

$$(i) \qquad \|1_I(\cdot)f(x-\phi(\cdot))\|_{\infty,\mathbb{R}} \leqq \|f\|_{\infty,G}.$$

PROOF. The function $(s, x) \longmapsto 1_I(s)f(x-\phi)s))$ is $\lambda \times \mu$-measurable. This fact follows easily from [9], Lemma (20.6), p. 287. Assume that (i) fails. Then there is a subset A of G with $0 < \mu(A) < \infty$, and such that, for every $x \in A$, we have

$$\|1_I(\cdot)f(x-\phi(\cdot))\|_{\infty,\mathbb{R}} > \|f\|_{\infty,G}.$$

Thus for every $x \in A$, there is a subset $A(x)$ of $\mathbb{R}$ such that, $\lambda(A(x)) > 0$, and

$$|1_I(s)f(x-\phi(s))| > \|f\|_{\infty,G}$$

for all s in $A(x)$. The set

$$\{(s,x) \in \mathbb{R} \times G : \ |1_I(s)f(x-\phi(s))| > \|f\|_{\infty,G}\}$$

contains $B = \{(s,x) \in \mathbb{R} \times G : \ x \in A, \ s \in A(x)\}$. We have,

$$\lambda \times \mu(B) = \int_G \lambda(B_x) d\mu(x)$$

where B_x is an x-section of B : $B_x = \{s \in \mathbb{R} : \ (s, \ x) \in B\}$. Note that $B_x \supseteq A(x)$; so $\lambda(B_x) > 0$ for every x in A, and hence $\lambda \times \mu(B) > 0$, since $\mu(A) > 0$. Therefore, there is an s in $\mathbb{R}$ such that the set $B^s = \{x \in G : \ (s, \ x) \in B\}$ has positive μ-measure. Consequently, we have

$$\mu(\{x \in G : \ |1_I(s)f(x-\phi(s))| > \|f\|_{\infty,G}\}) > 0,$$

which implies that

$$\mu(\{x \in G : \ |f(x)| > \|f\|_{\infty,G}\}) > 0.$$

This is plainly a contradiction. □

Note that the inequalities $(1.2.i, ii)$ depend on the measure of an interval containing the support of f and not on the measure of the support of f itself. The following discussion is aimed at finding a substitute for the notion of an interval on the group G.

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Let C denote the component of the identity in G: C is a connected subset containing 0 and properly contained in no other connected subsets of G. In fact, C is a closed subgroup of G topologically isomorphic with $\mathbb{R}^a \times E$, where E is a compact connected subgroup, and the number a is the largest possible dimension of a subgroup of G topologically isomorphic with $\mathbb{R}^n$ for a nonnegative integer n. See [9], Theorems (7.1) and (9.14), pp. 60, and 95.

LEMMA (2.3). *Let A be a nonvoid compact subset of G. There is a compact subset B of G such that, $A \subseteq B$, and $\phi^{-1}(x - B)$ is an interval in $\mathbb{R}$ (possibly unbounded or void) for all x in G and any continuous homomorphism ϕ from $\mathbb{R}$ into G.*

PROOF. The group G is topologically isomorphic with $\mathbb{R}^a \times \Omega$, where a is a nonnegative integer, and Ω contains a compact open subgroup J. See [9], Theorem (24.30), p. 389. The component of the identity C is therefore topologically isomorphic with $\mathbb{R}^a \times E$ where E is a compact connected subgroup of G. Since $\phi(\mathbb{R})$ is connected, it is contained in C. The subgroup $\mathbb{R}^a \times J$ is both open and closed, hence it must contain the component of the identity $\mathbb{R}^a \times E$. Since A is compact, it can be covered by finitely many disjoint cosets $x_j + (\mathbb{R}^a \times J)$, $j = 1, \ldots n$. Write $A = \bigcup_{j=1}^{n}(A \cap (x_j + (\mathbb{R}^a \times J))) = \bigcup_{j=1}^{n} A_j$, where the A_j's are compact and mutually disjoint. If a is 0, take $B = \bigcup_{j=1}^{n}(x_j + J)$. Otherwise enlarge each A_j to a set of the form $x_j + (C_j \times J)$, where C_j is a compact and connected subset of $\mathbb{R}^a$, and take $B = \bigcup_{j=1}^{n}(x_j + (C_j \times J))$. We can always represent B by the latter formula by agreeing to take $C_j = \{0\}$, for all $j = 1, 2, \ldots, n$, in case $a = 0$. Then B is clearly compact and contains A. Let x be an arbitrary element of G, and let ϕ be any continuous homomorphism from $\mathbb{R}$ into G. Consider the set $x - B = \bigcup_{j=1}^{n}(x - x_j - (C_j \times J))$. Each $x - x_j - (C_j \times J)$ belongs to a coset of $\mathbb{R}^a \times J$. Since $\phi(\mathbb{R})$ is connected, it can intersect at most one coset of $\mathbb{R}^a \times J$. Hence either $\phi(\mathbb{R}) \cap (x - B)$ is void or it is equal to $\phi(\mathbb{R}) \cap (x - x_{j_0} - (C_{j_0} \times J))$ for exactly one $j_0 \in \{1, \ldots, n\}$. In the first case, $\phi^{-1}(x - B)$ is void. In the second case, the nonvoid set $\phi(\mathbb{R}) \cap (x - B) = \phi(\mathbb{R}) \cap (x - x_{j_0} - (C_{j_0} \times E))$, is clearly connected. We will show that the set $\phi^{-1}(\phi(\mathbb{R}) \cap (x - x_{j_0} - (C_{j_0} \times E)))$ is connected, from which it will follow that it is an interval in $\mathbb{R}$. Note that if ϕ were a homeomorphism of $\mathbb{R}$ onto $\phi(\mathbb{R})$, then we would be done. But according to [9], Theorem (9.1), p. 84, we need only consider one other case: $\phi(\mathbb{R})^-$ is a compact subgroup of G. In this case, $\phi(\mathbb{R})$ is necessarily contained in E. Hence, $\phi(\mathbb{R}) \cap (x - x_{j_0} - (C_{j_0} \times E))$ is equal to $\phi(\mathbb{R})$, since we are assuming that it is not void. Hence $\phi^{-1}(\phi(\mathbb{R}) \cap (x - x_{j_0} - (C_{j_0} \times E))) = \mathbb{R}$, and this completes the proof. □

REMARK (2.4). Suppose that f is in $\mathcal{L}_\infty(G)$ with compact support A. Let B be a compact subset of G containing A and such that $\phi^{-1}(x - B)$ is an interval in $\mathbb{R}$ for all x in G. Fix x in G, and let I denote an arbitrary interval in $\mathbb{R}$. Consider the function $s \longmapsto 1_I(s)1_B(x - \phi(s))$, and let J denote its support. We clearly have $J = I \cap \phi^{-1}(x - B)$. Since $\phi^{-1}(x - B)$ is an interval in $\mathbb{R}$, it follows that J is an interval in $\mathbb{R}$. We also have

$$\lambda(J) = \int_{\mathbb{R}} 1_I(s)1_B(x - \phi(s))ds.$$

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We are now in a position to establish our transference result. The argument presented in the first half of Theorem (2.5) below is the transference argument of Calderón [5]. For the sake of completeness, we will supply all the details of the proof.

THEOREM (2.5). *Let G be a locally compact Abelian group. Suppose that f is in $\mathcal{L}_\infty(G)$ with compact support A. Let B be as in* Lemma (2. 3), *and let ϕ be a continuous homomorphism of $\mathbb{R}$ into G. Then, for all $y > 0$, we have*

$$(i)\qquad \mu(\{x \in G : |\mathrm{H}_n^\# f(x)| > y\}) \leqq c_3\mu(B)\,\frac{\|f\|_\infty}{y}\exp\left(-c_4\,\frac{y}{2\|f\|_\infty}\right),$$

and

$$(ii)\qquad \mu(\{x \in G : \sup_{1\leqq n}|\mathrm{H}_n^\# f(x)| > y\}) \leqq c_3\mu(B)\,\frac{\|f\|_\infty}{y}\exp\left(-c_4\,\frac{y}{2\|f\|_\infty}\right)$$

where c_3 and c_4 are as in (1. 1. *ii*).

PROOF. Clearly, we need only establish (*ii*). For $y > 0$, define

$$S_y = \{x \in G : \sup_{1\leqq n}|\mathrm{H}_n^\# f(x)| > y\},$$

and

$$S_y^N = \{x \in G : \max_{1\leqq n\leqq N}|\mathrm{H}_n^\# f(x)| > y\};$$

then $S_y^N \uparrow S_y$, as $N \longrightarrow \infty$. Therefore it is enough to show that

$$(1)\qquad \mu(S_y^N) \leqq c_3\mu(B)\,\frac{\|f\|_\infty}{y}\exp\left(-c_4\frac{y}{2\|f\|_\infty}\right).$$

Let $I_n = [-n, -1/n] \cup [1/n, n]$, and let V be an arbitrary nonvoid open subset of $\mathbb{R}$ with compact closure such that $V - I_n$ is an interval in $\mathbb{R}$. For an arbitrary real number s in V, we have from the translation invariance of the measure μ

$$\begin{aligned}(2)\qquad \mu(S_y^N) &= \mu\left(\left\{x \in G : \max_{1\leqq n\leqq N}\left|\frac{1}{\pi}\int_{1/n\leqq|t|\leqq n} f(x-\phi(t))\,\frac{1}{t}\,dt\right| > y\right\}\right)\\ &= \mu\left(\left\{x-\phi(s) : \max_{1\leqq n\leqq N}\left|\frac{1}{\pi}\int_{1/n\leqq|t|\leqq n} f(x-\phi(t))\,\frac{1}{t}\,dt\right| > y\right\}\right)\\ &= \mu\left(\left\{x \in G : \max_{1\leqq n\leqq N}\left|\frac{1}{\pi}\int_{1/n\leqq|t|\leqq n} f(x+\phi(s-t))\,\frac{1}{t}\,dt\right| > y\right\}\right)\\ &= \mu\left(\left\{x \in G : \max_{1\leqq n\leqq N}\left|\frac{1}{\pi}\int_{1/n\leqq|t|\leqq n} f(x+\phi(s-t))\,1_{V-I_N}(s-t)\right.\right.\right.\\ &\qquad \left.\left.\left.\times\frac{1}{t}\,dt\right| > y\right\}\right).\end{aligned}$$

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The last inequality follows trivially from the one that preceeds it, since for $s \in V$, and $t \in I_n$, we have $1_{V-I_N}(s-t) = 1$. Let

$$\xi_y(s, x) = \begin{cases} 1 & \text{if } \max_{1 \leqq n \leqq N} \left| \frac{1}{\pi} \int_{I_n} f(x + \varphi(s-t)) 1_{V-I_N}(s-t) \frac{1}{t}\, dt \right| > y; \\ 0 & \text{otherwise;} \end{cases}$$

and

$$\xi'_y(s, x) = \begin{cases} 1 & \text{if } \sup_{1 \leqq n} \left| \frac{1}{\pi} \int_{I_n} f(x + \varphi(s-t)) 1_{V-I_N}(s-t) \frac{1}{t}\, dt \right| > y; \\ 0 & \text{otherwise.} \end{cases}$$

It is clear from the last equality in (2) that, for any s in V, we have

$$\begin{aligned} \mu(S_y^N) &= \int_G \xi_y(s, x) d\mu(x) \\ &= \frac{1}{\lambda(V)} \int_V \int_G \xi_y(s, x) d\mu(x) ds \\ &\leqq \frac{1}{\lambda(V)} \int_{\mathbb{R}} \int_G \xi_y(s, x) d\mu(x) ds \\ &\leqq \frac{1}{\lambda(V)} \int_G \int_{\mathbb{R}} \xi'_y(s, x) d\mu(x) ds. \end{aligned} \tag{3}$$

The penultimate inequality holds because we have enlarged the limits of integration of a nonnegative integrand. The last inequality follows from the relation $\xi'_y(s, x) \geqq \xi_y(s, x)$ which holds for $\lambda \times \mu$-almost all (s, x), and Fubini's theorem. Now note that

$$\int_{\mathbb{R}} \xi'_y(s, x) ds = \lambda(\{s \in \mathbb{R} : \sup_{1 \leqq n} |\mathrm{T}_n f(x + \phi(s)) 1_{V-I_N}(s)| > y\}), \tag{4}$$

where T_n is as in (1.2). The support of the function $s \longmapsto f(x + \phi(s)) 1_{V-I_N}(s)$ is contained in $(V - I_N) \cap \phi^{-1}(x - \operatorname{supp} f) \subseteq (V - I_N) \cap \phi^{-1}(B - x)$ which is an interval in $\mathbb{R}$ by the choice of V and B. Denote this interval by J. Using (1.2.*ii*), Lemma (2.2), and Remark (2.4) we get

$$\begin{aligned} \int_{\mathbb{R}} \xi'_y(s, x) ds &\leqq c_3 \frac{\lambda(J)}{y} \| f(x + \phi(\cdot)) 1_{V-I_N}(\cdot) \|_{\infty, \mathbb{R}} \\ &\quad \times \exp\left(- \frac{c_4 y}{2 \| f(x + \phi(\cdot)) 1_{V-I_N}(\cdot) \|_{\infty, \mathbb{R}}} \right), \\ &\leqq c_3 \frac{\lambda(J)}{y} \| f \|_{\infty, G} \exp\left(- \frac{c_4 y}{2 \| f \|_{\infty, G}} \right) \\ &\leqq \frac{c_3}{y} \| f \|_{\infty, G} \exp\left(- \frac{c_4 y}{2 \| f \|_{\infty, G}} \right) \int_{\mathbb{R}} 1_{V-I_N}(s) 1_B(x + \phi(s)) ds. \end{aligned} \tag{5}$$

Now use Fubini's theorem, (3), (5), and the translation invariance of μ to obtain

$$\begin{aligned}\mu(S_y^N) &\leqq \frac{1}{\lambda(V)}\int_G \frac{c_3}{y}\| f \|_{\infty,G} \exp\left(-\frac{c_4 y}{2\| f \|_{\infty,G}}\right) \\ &\quad\times \int_{\mathbb{R}} 1_{V-I_N}(s)1_B(x+\phi(s))ds\, d\mu(x) \\ &= \frac{1}{\lambda(V)}\frac{c_3}{y}\| f \|_{\infty,G} \exp\left(-\frac{c_4 y}{2\| f \|_{\infty,G}}\right) \\ &\quad\times \int_{\mathbb{R}}\int_G 1_{V-I_N}(s)1_B(x+\phi(s))ds\, d\mu(x) \\ &= \frac{\lambda(V-I_N)}{\lambda(V)}\mu(B)\frac{c_3}{y}\| f \|_{\infty,G} \exp\left(-\frac{c_4 y}{2\| f \|_{\infty,G}}\right).\end{aligned}$$

Choose V so that $\lambda(V-I_N)/\lambda(V)$ approaches 1. This implies (1) and completes the proof of (*ii*) □

The following theorem is a consequence of inequality (2.3.*i*), and the fact that $\mathrm{H}_n^{\#}f$ converges to $\mathrm{H}^{\#}f$ in the $\mathcal{L}_p(G)$-norm, and hence in measure, whenever f is in $\mathcal{L}_p(G)$, for $p \in]1, \infty[$. See [3], Theorem (6.5).

THEOREM (2.6). *Notation is as in* (2.1) *and* (2.5). *Let* $y > 0$; *then*

$$(i) \qquad \mu(\{x \in G : |\mathrm{H}^{\#}f(x)| > y\}) \leqq c_3\mu(B)\frac{\| f \|_\infty}{y} \exp\left(-c_4\frac{y}{2\| f \|_\infty}\right),$$

where c_3 *and* c_4 *are as in* (1.1.*ii*).

As noted in [6], Section 2, Calderón's argument extends easily to the set- up where $\mathbb{R}$ is replaced by an arbitrary amenable group, and in particular, a locally compact Abelian group. Several extensions in this direction are taken-up in [6]. However, Theorem (2.5) doesn't follow directly from any of these generalizations.

3. **The conjugate function on locally compact Abelian groups.** Throughout this section, G will denote a locally compact Abelian group, with character group X. We suppose that X contains a measurable subset P with the following properties: $P\cap(-P) = \{0\}; P\cup(-P) = X; P+P = P$. The set P is called an *order* on X. Define the function sgn_P on X by: $\mathrm{sgn}_P(\chi) = -1$, 0, or 1, according as $\chi \in (-P)\backslash\{0\}$, $\chi = 0$, or $\chi \in P\backslash\{0\}$. For $f \in \mathcal{L}_2(G)$, the *conjugate function* of f, (*with respect to the order* P) is the function $\tilde{f} \in \mathcal{L}_2(G)$ such that: $\hat{\tilde{f}}(\chi) = -i\ \mathrm{sgn}_P(\chi)\hat{f}(\chi)$ for almost all $\chi \in X$. For $1 < p < \infty$, a generalized version of M. Riesz's theorem asserts that there is a constant M_p such that $\| \tilde{f} \|_p \leqq M_p\| f \|_p$ for all f in $\mathcal{L}_2(G)\cap\mathcal{L}_p(G)$. The constant M_p is the same as the constant appearing in M. Riesz's theorem on the circle $\mathbb{T}$. See [3], Theorem (7.2). We will show that, for f in $\mathcal{L}_\infty(G)$, $\tilde{f}$ possesses properties identical to the ones given by Theorem (2.6) for the ergodic Hilbert transform.

The following characterization of orders will be crucial in establishing a link between the ergodic Hilbert transform and the conjugate function.

THEOREM (3.1). *Let K be a compact non-void subset of X. There is a real-valued homomorphism ψ on X such that, ψ is positive on $(K \cap P)\backslash(N \cup \{0\})$, and ψ is negative on $(K \cap (-P))\backslash(N \cup \{0\})$, where N is a subset of X with measure zero.*

See [3], Theorem (5.14).

LEMMA (3.2). *Let χ be a real-valued homomorphism on X, and let ϕ be its adjoint homomorphism. (See [9], Definition (24.37), p. 392.) For f in $\mathcal{L}_p(G)$, $1 < p \leqq \infty$, let $\mathrm{H}^{\#}f$ be the ergodic Hilbert transform of f (with respect to the homomorphism ϕ). Then*

$$(i) \qquad (\mathrm{H}^{\#}f)\hat{\ }(\chi) = -i \operatorname{sgn}(\psi(\chi))\hat{f}(\chi)$$

for almost all χ in X.

The proof is a straightforward computation. For details, see [3], Theorem (6.7).

A CONSTRUCTION (3.3). Suppose that g is in $\mathcal{L}_2(G)$. Its Fourier transform $\hat{g}$ vanishes off of a σ-compact subgroup X_0 of X. Write $X_0 = \bigcup_{n=1}^{\infty} K_n$ where each K_n is a compact subset of X_0 with nonvoid interior, and such that $K_n \subseteq K_m$ when $n \leqq m$. Apply Theorem (3.1) to each K_n and obtain a real-valued homomorphism ψ_n, and a null subset N of X (N is the union of all the null sets given by Theorem (3.1) for all the positive integers n) such that ψ_n is positive on $(P \cap K_n)\backslash(N \cup \{0\})$, and negative on $((-P)\cap K_n)\backslash(N \cup \{0\})$. Let ϕ_n denote the adjoint homomorphism of ψ_n. Denote by H^n the ergodic Hilbert transform corresponding to the homomorphism ϕ_n as in (2.1). The following theorem is a particular case of [3], Theorem (7.8). We present it here together with a simple proof.

THEOREM (3.4). *Notation is borrowed from (3.3). Suppose that f is in $\mathcal{L}_2(G)$ with Fourier transform vanishing off of X_0. Then the function $\mathrm{H}^n f$ converges in the $\mathcal{L}_2$-norm, and hence in measure, to the conjugate function $\tilde{f}$ of f.*

PROOF. Suppose first that $\hat{f}$ has a compact support K which is contained in X_0. Choose an integer n such that $K_m \supseteq K$, for all $m \geqq n$. From the construction (3.3), Lemma (3.2), and the definition of the conjugate function, it is easy to verify that the equalities

$$\hat{\tilde{f}}(\chi) = -i \operatorname{sgn}_P(\chi)\hat{f}(\chi) = -i \operatorname{sgn}(\psi(\chi))\hat{f}(\chi) = (\mathrm{H}^m f)\hat{\ }(\chi)$$

hold for all $m \geqq n$, and almost all χ in X. The uniqueness of the Fourier transform implies that $\mathrm{H}^m = \tilde{f}$, for all $m \geqq n$. Thus the conclusion of the theorem is true for all functions in $\mathcal{L}_2(G)$ with compactly supported Fourier transforms. Now let f be an arbitrary function in $\mathcal{L}_2(G)$ with Fourier transform supported in X_0. Given $\epsilon > 0$, choose g in $\mathcal{L}_2(G)$ with compactly supported $\hat{g}$, such that $\|f - g\|_2 < \epsilon/3$. We have

$$\begin{aligned}\|\mathrm{H}^n f - \tilde{f}\|_2 &= \|\mathrm{H}^n(f-g) + \mathrm{H}^n g - \tilde{g} + (\tilde{g} - \tilde{f})\|_2 \\ &\leqq \|\mathrm{H}^n(f-g)\|_2 + \|\mathrm{H}^n g - \tilde{g}\|_2 + \|\tilde{g} - \tilde{f}\|_2 \\ &\leqq \frac{\epsilon}{3} + \|\mathrm{H}^n g - \tilde{g}\|_2 + \frac{\epsilon}{3}.\end{aligned}$$

As $n \longrightarrow \infty$, $\|\mathrm{H}^n g - \tilde{g}\|_2 \longrightarrow 0$; which completes the proof of the theorem. □

As a consequence of Theorem (3.4), and Theorem (2.6), we obtain our main result.

THEOREM (3.5). *Let G be a locally compact Abelian group. Suppose that f is in $\mathcal{L}_\infty(G)$ with compact support A. Let B be as in* Lemma (2.3). *Then for all $y > 0$, we have*

$$(i) \qquad \mu(\{x \in G : |\tilde{f}(x)| > y\}) \leqq c_3\mu(B)\,\frac{\|f\|_\infty}{y}\,\exp\left(-c_4\,\frac{y}{2\|f\|_\infty}\right),$$

where c_3 and c_4 are as in $(1.1.ii)$.

For the remaining part of this section, we will suppose that G is compact.

In view of what is known about the constant M_p appearing in M. Riesz's theorem [$M_p \sim p$ as $p \uparrow \infty$], the strong integrability property of $\tilde{f}$, for f in $\mathcal{L}\infty(G)$, follows from an extrapolation theorem of S. Yano. See [11], Vol. II, Theorem (4.41), p. 119. However, we can obtain this result as a consequence of inequality (3.5.i).

THEOREM (3.6). *Notation is borrowed from* (3.5). *Let p be such that*

$$(i) \qquad p\|f\|_\infty < \frac{c_4}{2};$$

then

$$(ii) \qquad \int_G \exp\,(p|\tilde{f}(x)|)\,d\mu(x) < \infty.$$

PROOF. From (3.5.i), we have

$$\begin{aligned}\mu(\{x \in G : \exp\,(p|\tilde{f}(x)|) > y\}) &= \mu\left(\left\{x \in G : |\tilde{f}(x)| > \frac{\log y}{p}\right\}\right) \\ &\leqq c_3\,\frac{p\|f\|_\infty}{\log y}\,\exp\left(-c_4\,\frac{\log y}{2p\|f\|_\infty}\right) \\ &= c_3\,\frac{p\|f\|_\infty}{\log y}\,y^{-(c_4/2p\|f\|_\infty)}.\end{aligned}$$

Using [10], Corollary (21.72.i), p. 422, and some obvious estimates, we obtain

$$\begin{aligned}\int_G \exp\,(p|\tilde{f}(x)|)\,d\mu(x) &= \int_0^\infty \mu(\{x \in G : \exp\,(p|\tilde{f}(x)|) > y\}dy \\ &\leqq \int_0^e dy + \int_e^\infty c_3\,\frac{p\|f\|_\infty}{\log y}\,y^{-(c_4/2p\|f\|_\infty)}\,dy.\end{aligned}$$

Clearly, the last integral is finite if p satisfies (i). □

We end this essay by establishing on G yet another well-known property of the conjugate function on $\mathbb{T}$. The proof uses Theorem (3.5) and is exactly like the one on $\mathbb{T}$. For details, see [11], Vol. 1, theorem (2.11.ii), p. 253.

THEOREM (3.7). *Suppose that f is continuous on G; then* $\exp(\alpha|\tilde{f}|)$ *is integrable for all real numbers α.*

HISTORICAL NOTES. (3.8). Several properties of the conjugate function on $\mathbb{T}$ have been successfully established for the conjugate function on various abstract set-ups. Theorem (3.5) above is an example of such extensions. The property that attracted the most attention is the $\mathcal{L}_p$-boundedness of the conjugate function operator, for p in $]1, \infty[$. On $\mathbb{T}$ or $\mathbb{R}$, the result is M. Riesz's celebrated theorem. A fairly extensive history of this result appears in [3]. Recent abstract versions of M. Riesz's theorem appeared in [2], and [4]. Less complete are the results on the pointwise convergence related to the conjugate function on groups other than $\mathbb{T}$ or $\mathbb{R}$. We refer the interested reader to [1], [3] Section 7, and [8].

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