THE CONVERGENCE OF RAYLEIGH-RITZ APPROXIMATIONS

M. J. O'CARROLL and P. E. LUSH

(Received 26 September 1966; revised 5 October 1967)

1. Introduction

This paper is concerned with the convergence of Rayleigh-Ritz approximations to the solution of an elliptic boundary value problem. Although the work arose in connection with the aerofoil problem (and it is to this problem that the results obtained are immediately applied), the methods here employed are suitable for use on the wider class of problem mentioned above.

The subsonic aerofoil problem for two dimensional, steady, isentropic, irrotational flow can be expressed as a variational principle [1]. Thus, if we denote the velocity potential for a uniform stream by ϕ_∞, and the velocity potential for the corresponding incompressible flow by ϕ_0, the velocity potential ϕ maximizes

$$J[\phi] = \iint_R \{p - p_\infty + \rho_\infty \nabla\phi_0 \cdot \nabla(\phi - \phi_\infty)\} dx\, dy,$$

where R is the (infinite) region occupied by the fluid. The pressure p is a function of the density ρ only, and we are to express p in terms of ϕ by use of Bernoulli's equation. The admissible functions are of the form $\phi = \phi_0 + \chi$ where (i) for $r(=\sqrt{(x^2+y^2)})$ large, $|\chi| \leqq K'r^{-1}$, $|\nabla\chi| \leqq K'r^{-2}$, the constant K' being independent of the polar angle θ, and (ii) $\partial\phi/\partial n = 0$ on the obstacle C.

In [1] it was shown that the Rayleigh-Ritz procedure applied to the integral $J[\phi]$ yields a sequence of approximations $\{\phi_n\}$ to the velocity potential. If the functions ϕ_n are such that

$$|\nabla\phi_n| \leqq c < c^*, \tag{1.1}$$

where c^* is the sonic speed, then [2] the sequence $\{\phi_n\}$ converges uniformly in any bounded sub-domain of R, provided C is sufficiently regular. The regularity (of C) required is that any two points P, Q of R may be joined by N semi-circles lying wholly within R — the number N being bounded uniformly for all P, Q. For the case where C is convex, $N \leqq 5$.

In this paper we extend the class of obstacles for which the above argument holds to include those for which (i) C is a simple, closed, rectifiable Jordan curve, and (ii) C satisfies the

SMOOTHNESS CONDITION. If we denote by $S(X, r_0, \beta_0)$ the sector of a circle defined by centre X, radius r_0, and angle β_0 $(0 < \beta_0 < \pi/4)$ where β_0, r_0 (> 0) are given constants which are independent of X, then at each point X of C some sector $S(X, r_0, \beta_0)$ lies wholly within R.

As a by-product, we also show that the error involved in approximating to ϕ by use of ϕ_n is at most $K\{J[\phi]-J[\phi_n]\}^\nu$, where $K(=K(\nu))$ is a constant independent of n, and ν may be given any value in $0 < \nu < \frac{1}{2}$ — this represents a considerable improvement on [2] where the corresponding estimate involves $\nu = \frac{1}{4}$. To do this we need a form of a theorem due to Morrey ([3], summarized in [4]) that is applicable when P is on C and PQ is 'close' to C. For completeness, we give an ab inito presentation of this result.

2. A form of Morrey's theorem

Let S_r be a sector of a circle with radius r and angle β lying wholly within the region R, and denote by $D_W[z]$ the Dirichlet integral of z over any region W, viz.

$$D_W[z] = \iint_W (\nabla z)^2 dx\,dy.$$

Suppose that (i) for all such S_r

$$D_{S_r}[z] \leqq Lr^\alpha, \qquad \alpha \leqq 2, \tag{2.1}$$

where the constants L, α are independent of S_r, and (ii) z $(=z(x, y))$ is in class C_1. Let P, Q be any two points of R such that (iii) an isosceles triangle PQS with base PQ and base angles β, $0 < \beta \leqq \pi/4$, lies wholly within R. Denote the mid-point of PQ by T, and let X be any point on TS, then

$$|z(P)-z(Q)| \leqq |z(P)-z(X)|+|z(X)-z(Q)|. \tag{2.2}$$

Let (r, θ) be polar coordinates with origin at P, then, if $r_x = PX$,

$$|z(P)-z(X)| \leqq \int_0^{r_x} |z_r|dr,$$

with $|z(X)-z(Q)|$ being given by a similar expression.

Now the left hand side of (2.2) is independent of θ, so that on integration with respect to θ, (2.2) becomes

$$\beta|z(P)-z(Q)| \leqq \iint_{PTS} |z_r|dr\,d\theta + \cdots,$$

where a similar term involving an integral over triangle QTS is not written at length. Writing the integral over PTS as

https://doi.org/10.1017/S1446788700006467 Published online by Cambridge University Press

$$\iint_{PTS} r^{-1+\gamma}\{|z_r|r^{-\gamma}\} r dr d\theta \qquad (= I_1, \text{ say}),$$

where $\gamma > 0$, Schwarz' inequality then gives

$$I_1 \leqq \{PS^{2\gamma} \cdot \beta/(2\gamma)\}^{\frac{1}{2}} \left\{\iint_{S_\rho} (\nabla z)^2 r^{-2\gamma} r dr d\theta\right\}^{\frac{1}{2}} \tag{2.3}$$

In (2.3), the region of integration in each integral on the right hand side has been extended to the sector S_ρ $(\rho = PS)$ where S_r denotes $S(P, r, \beta)$ for $r \leqq \rho - S_\rho$ is in the triangle PSQ since $\beta \leqq \pi/4$.

Writing $I(r) = D_{S_r}[z]$, the integral on the right hand side of (2.3) becomes

$$\begin{aligned} \int_0^{I(\rho)} r^{-2\gamma} dI &\leqq \int_0^{I(\rho)} (L/I)^{2\gamma/\alpha} dI, \text{ by (2.1)}, \\ &= L^{2\gamma/\alpha} I(\rho)^{1-2\gamma/\alpha}/(1-2\gamma/\alpha), \end{aligned} \tag{2.4}$$

provided $2\gamma < \alpha$. Now $I(\rho) \leqq D_{PSQ}[z]$, $(= I$, say) and since $\beta \leqq \pi/4$, $PS \leqq PQ$ we have finally the

THEOREM. *If the function $z(x, y)$ satisfies conditions (i) and (ii), then for any two points P, Q of the region R satisfying (iii), and for $0 < \gamma < \alpha/2$,*

$$|z(P) - z(Q)| \leqq L_1 I^\nu PQ^\gamma/\beta^{\frac{1}{2}}, \qquad \nu = \tfrac{1}{2} - \gamma/\alpha \tag{2.5}$$

where

$$L_1 = L^{\gamma/\alpha}[2\alpha/\{\gamma(\alpha - 2\gamma)\}]^{\frac{1}{2}} \text{ and } I = D_{PSQ}[z].$$

We recover Morrey's form of the theorem by setting $\beta = \pi/4$, $\gamma = \alpha/4$, $I \leqq L \cdot PQ^\alpha$ in (2.5).

3. Application of the theorem

Let C (supposed a closed, rectifiable, Jordan curve) be given by $x = f(t)$, $y = g(t)$, $t_1 \leqq t \leqq t_2$, where $f(t)$ and $g(t)$ are continuous functions of bounded variation. In [5], [6] it is shown that for such a function $f(t)$, the set of values of X, such that $f(t) = X$ has infinitely many solutions for t in the range $t_1 \leqq t \leqq t_2$, has measure zero. Enclose C in a square $a \leqq x \leqq b$, $a \leqq y \leqq b$. It follows that for any arbitrarily chosen $\delta (> 0)$ we can find a set of lines $x = x_i$, $i = 0, \cdots, N$, with

$$x_{i+1} - x_i = \delta, \; x_0 \leqq a \leqq x_1, \; x_{N-1} \leqq b \leqq x_N, \qquad N = N(\delta),$$

such that each line has a finite number of intersections with C. Likewise we may construct a similar set of lines $y = y_i$, $i = 0, \cdots, N'$ — the precise values of N, N' are immaterial to the rest of the argument. Thus C can be covered by a chain of squares such that each square contains at least one point of C in its interior or on its boundary.

Let G be the set of sides (taken to include their end points) of these squares which lie in the exterior of C and which do not contain any points of C, then G consists of (i) a simple closed polygon G_1 which contains C and the remainder of G in its interior, and possibly (ii) simple closed polygons $G_2, G_3, \cdots$ in the interior of G_1, and (iii) segments not forming a polygon.

Let δ_1 be the side of the largest square of any orientation that can fit into a sector of radius r_0 and angle β_0, and set $\delta = \delta_1/2$. At any point X of C, the sector $S(X, r_0, \beta_0)$ that lies within R, irrespective of its orientation and position relative to the grid covering C, contains in its interior one of the grid squares of side $\delta_1/2$. By the construction of G, this square is either (i) in the closure of the exterior of G_1 or (ii) in the closure of the interior of one of $G_2, G_3, \cdots$. In either case the sector intersects the (non-simple) polygon $G_1, G_2, \cdots$, and thus any point X on C can be connected to this polygon by (the base of) an isosceles triangle of angle $\beta_0/2$ lying in R. This argument also shows that possibility (iii) of the last paragraph is essentially trivial.

Let Q be a point on G_2 and, since R is connected, join Q to G_1 by a curve ρ in R which does not intersect G_2. Now the distance of ρ from C must be greater than some number d (say), for otherwise ρ and C, being compact, would have a point in common. Since there are at most NN' squares of side δ, there is only a finite number of $G_2, G_3, \cdots$, and so there is a number $d_{\min}$ such that $d \geqq d_{\min}$ for all polygons $G_2, G_3, \cdots$. By the argument used above, ρ can be covered by a finite chain of squares of side $d_{\min}/2$ lying wholly in R, obtained by taking a grid with the same orientation as that of side δ. Keeping only a simply connected set of these squares, we can join G_2 to G_1 in such a way that the polygon formed by G_1, G_2 and the boundary of this new chain of squares is simple. The exterior of this polygon is the union of the exterior of G_1, the interior of G_2 and the 'channel' resulting from the covering ρ. The argument of the last paragraph regarding the connection of any point on C to this polygon by means of an isosceles triangle is still applicable. As there is only a finite number of $G_2, G_3, \cdots$, it follows that we can enclose C in a simple polygon with μ sides (call it G_μ) such that the sector $S(X, r_0, \beta_0)$ at each point X of C intersects G_μ.

Let E be the intersection of the exterior of C and the interior of G_μ, and let Y be any point in E. Call d_1 the distance of Y from C, then a circle of radius d_1 lies in R, and we can then connect Y to a point X on C by (the base of) an isosceles triangle of base angle β, $\beta_0/2 \leqq \beta < \pi/4$, lying in R. As in the last paragraph we can connect X to G_μ by an isosceles triangle, and any point in the exterior of G_μ can be joined to G_μ by an isosceles triangle of base angle β, $\beta_0/2 \leqq \beta < \pi/4$, lying in R. If we erect on each side of G_μ an isosceles triangle of base angle β, $\beta_0/2 \leqq \beta < \pi/4$, lying in the exterior of G_μ, then any two points in R, P and Q say, can be joined by a

https://doi.org/10.1017/S1446788700006467 Published online by Cambridge University Press

chain of not more than $(\mu+3)$ non-overlapping isosceles triangles of angles $\beta, \beta_0/2 \leqq \beta < \pi/4$, each of which lies wholly within R. Moreover, since μ depends only upon the geometrical properties of C, it is independent of P and Q.

Now let P and Q be any two points in a bounded sub-region of R, and call the segments of the polygonal arc formed by the bases of the chain of triangles that connects them $P_{i-1}P_i$, $i = 1, \cdots, M$, $P_0 = P$, $P_M = Q$, $M \leqq \mu+3$. Application of the Theorem of § 2 together with Hölder's inequality then gives

$$|z(P)-z(Q)| \leqq \sqrt{2}L_1 M^{1-\nu} d^\gamma \left(\sum_1^M I_i\right)^\nu / \beta_0^{\frac{1}{2}}, \qquad \nu = \tfrac{1}{2}-\gamma/\alpha,$$

after putting $\beta \geqq \beta_0/2$ and $P_{i-1}P_i \leqq d$ where d is the diameter of the sub-region being considered. But $\sum_1^M I_i$ is less than $D_R[z]$ so that

$$|z(P)-z(Q)| \leqq K_1\{D_R[z]\}^\nu, \tag{3.1}$$

where the constant K_1 is independent of P and Q.

4. Conclusions

Let ϕ_n be a Rayleigh-Ritz approximation to the function ϕ defined in § 1, then, as in [2],

$$D_R[\phi_{m+n}-\phi_n] < \varepsilon, \tag{4.1}$$

for $m > M(\varepsilon)$. If we put $z = \phi_{m+n}-\phi_n$ in (3.1), we deduce that if $\{\phi_n\}$ converges at any point Q, then $\{\phi_n\}$ converges uniformly in any bounded sub-domain of R. The assumption that $\{\phi_n\}$ should converge at any point Q is essentially vacuous since ϕ_n is determined only to an additive constant which we may select so that $\phi_n(Q) = 0$. Finally, as in [2], (1.1) implies that

$$D_R[\phi_{m+n}-\phi_n] \leqq K_2\{J[\phi_{m+n}]-J[\phi_n]\}, \tag{4.2}$$

where K_2 is a constant independent of m, n, and (3.1) then becomes

$$|\phi(P)-\phi_n(P)| \leqq K\{J[\phi]-J[\phi_n]\}^\nu,$$

on letting $m \to \infty$. Here K is a constant independent of P, n, and $0 < \nu < \frac{1}{2}$.

The assumption that $|\nabla\phi_n| \leqq c^*$ implies $D_{S_r} \leqq Lr^2$, and Morrey's theorem then gives

$$|\phi_n(P)-\phi_n(Q)| \leqq L_2 PQ,$$

for $PQ \leqq R^*$ where R^* is the distance of P from C. The constant L_2 is independent of n, and the Hölder continuity of ϕ then follows by taking the limit $n \to \infty$.

Finally, the argument presented in §§ 2—4 does not depend on the particular form of the integrand of $J[\phi]$, in particular, (4.1) and (4.2) depend essentially upon the elliptic character of the stationary value of $J[\phi]$. Thus the results obtained above apply to any proper elliptic boundary value problem.

References

[1] P. E. Lush and T. M. Cherry, 'The variational method in hydrodynamics', *Quart. J. Mech. and Appl. Math.* 9. (1956), 6—21.
[2] P. E. Lush, 'The convergence of Rayleigh-Ritz approximations in hydrodynamics', *J. Aust. Math. Soc.* 3 (1963), 99—103.
[3] C. B. Morrey, *Problems in the Calculus of Variations and Related Topics* (Univ. of Calif. Publ. in Math. 1, 1943).
[4] M. Shiffman, 'On the existence of subsonic flows of a compressible fluid', *J. Rat. Mech. and Anal.* 1 (1952), 605—652.
[5] I. I. Priwalow, *Einführung in die Functionentheorie* (Teil II B. G. Teubner Verlag, 1959).
[6] T. Estermann, 'Über die totale Variation einer stetigen Funktion und den Cauchyschen Integralsatz', *Math. Zeitschrift* 37 (1933), 556—60.

University of New England
Armidale

https://doi.org/10.1017/S1446788700006467 Published online by Cambridge University Press