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TENSOR PRODUCTS OF LOG-HYPONORMAL AND OF CLASS $A(s, t)$ OPERATORS

KÔTARÔ TANAHASHI[1]

Department of Mathematics, Tohoku Pharmaceutical University, Sendai 981-8558, Japan
e-mail: tanahasi@tohoku-pharm.ac.jp

and MUNEO CHŌ[2]

Department of Mathematics, Kanagawa University, Yokohama 221-8686, Japan
e-mail: chiyom01@kanagawa-u.ac.jp

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Abstract. Let A (resp. B) be a bounded linear operator on a complex Hilbert space $\mathcal{H}$ (resp. $\mathcal{K}$). We show that the tensor product $A \otimes B$ is log-hyponormal if and only if A and B are log-hyponormal, and that a similar result holds for class $A(s, t)$ operators.

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1. Introduction. Let $\mathcal{H}, \mathcal{K}$ be complex Hilbert spaces and $\mathcal{H} \otimes \mathcal{K}$ the tensor product of $\mathcal{H}, \mathcal{K}$; i.e., the completion of the algebraic tensor product of $\mathcal{H}, \mathcal{K}$ with the inner product $\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle x_1, x_2 \rangle \langle y_1, y_2 \rangle$ for $x_1, x_2 \in \mathcal{H}$, $y_1, y_2 \in \mathcal{K}$. Let $B(\mathcal{H})$ (resp. $B(\mathcal{K})$) be the algebra of all bounded linear operators on $\mathcal{H}$ (resp. $\mathcal{K}$). Let $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$. $A \otimes B \in B(\mathcal{H} \otimes \mathcal{K})$ denotes the tensor product of A and B; i.e., $(A \otimes B)(x \otimes y) = Ax \otimes By$ for $x \in \mathcal{H}$, $y \in \mathcal{K}$.

Let S and $T \in B(\mathcal{H})$. T is said to be *non-negative* if $T \geq 0$; i.e., $\langle Tx, x \rangle \geq 0$ for all $x \in \mathcal{H}$. $S \leq T$ means $T - S$ is non-negative, and $S < T$ means $T - S$ is non-negative and invertible. T is said to be *p-hyponormal* $(0 < p)$ if $(TT^*)^p \leq (T^*T)^p$. If $p = 1$, T is said to be *hyponormal*, and if $p = 1/2$, T is said to be *semi-hyponormal*. T is said to be *log-hyponormal* if T is invertible and $\log(TT^*) \leq \log(T^*T)$. If T is p-hyponormal and $0 < q < p$, then T is q-hyponormal. Invertible p-hyponormal operators are log-hyponormal.

Let $T = U|T|$ be the polar decomposition of $T \in B(\mathcal{H})$ and $\tilde{T}_{s,t} = |T|^s U |T|^t$ be the Aluthge transform for $s, t > 0$. T is called a *class* $A(s, t)$ operator if $|\tilde{T}_{s,t}|^{\frac{2t}{s+t}} \geq |T|^{2t}$. T is called a *class* $wA(s, t)$ operator if T is a class $A(s, t)$ operator and $|T|^{2s} \geq |(\tilde{T}_{s,t})^*|^{\frac{2s}{s+t}}$. A class $A(1, 1)$ operator is called a *class A operator* and a class $wA(1/2, 1/2)$ operator is called a *w-hyponormal operator* ([**2**, **8**]). T is said to be *paranormal* if $\|Tx\|^2 \leq \|T^2x\| \|x\|$ for $x \in \mathcal{H}$. It is known that class A operators are paranormal.

D. Xia [**16**] investigated properties of hyponormal and semi-hyponormal operators. A. Aluthge [**1**] introduced p-hyponormal operators and investigated properties of a p-hyponormal operator by its Aluthge transform. The idea of log-hyponormal operator is due to T. Ando [**3**] and the first paper in which log-hyponormality appeared is [**6**]. See

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[**2**, **14**, **15**] for properties of log-hyponormal operators. M. Ito [**8**] proved p-hyponormal operators and log-hyponormal operators are class $wA(s, t)$ operators for all $s, t > 0$. (See [**7**, **14**, **17**, **18**] for related results.) M. Ito and T. Yamazaki [**9**] proved that class $A(s, t)$ operators are class $wA(s, t)$ operators, and investigated the relations between these classes of operators.

There are many properties which are preserved under tensor product. For example, H. Jinchuan [**11**] proved that $A \otimes B$ is normal if and only if A, B are normal, where A, B are non-zero operators. Similar results were obtained for subnormal operators by B. Magajna [**12**], hyponormal operators by J. Stochel [**13**], p-hyponormal operators by B. P. Duggal [**4**], class A operators by I. H. Jeon and B. P. Duggal [**10**] and p-quasihyponormal operators by D. R. Farenick and I. H. Kim [**5**]. But T. Ando [**3**] proved that there exist paranormal operators A and B such that $A \otimes B$ is not paranormal. In this paper, we show that the tensor product $A \otimes B$ is log-hyponormal if and only if A and B are log-hyponormal, and that a similar result holds for class $A(s, t)$ operators.

2. Results. The following key lemma is due to J. Stochel [**13**].

LEMMA 1. [**12**] *Let $A_1, A_2 \in B(\mathcal{H})$, $B_1, B_2 \in B(\mathcal{K})$ be non-negative operators. If A_1 and B_1 are non-zero, then the following assertions are equivalent.*

(1) $A_1 \otimes B_1 \leq A_2 \otimes B_2$.

(2) *There exists $c > 0$ such that $A_1 \leq cA_2$ and $B_1 \leq c^{-1}B_2$.*

The proofs of the following elementary properties are easy.

LEMMA 2. *Let $A = U_A|A|$ and $B = U_B|B|$ be the polar decompositions of $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$, respectively. Then the following assertions hold.*

(1) $|A \otimes B| = |A| \otimes |B|$.

(2) $A \otimes B = (U_A \otimes U_B)(|A| \otimes |B|)$ *is the polar decomposition of $A \otimes B$.*

(3) $(\widetilde{A \otimes B})_{s,t} = \tilde{A}_{s,t} \otimes \tilde{B}_{s,t}$ *for $s, t > 0$.*

THEOREM 3. *Let $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ be non-zero operators. Then $A \otimes B$ is a class $A(s, t)$ operator if and only if A and B are class $A(s, t)$ operators for $s, t > 0$.*

Proof. Let A and B be class $A(s, t)$ operators. Then

$$|\tilde{A}_{s,t}|^{\frac{2t}{s+t}} \geq |A|^{2t},$$

$$|\tilde{B}_{s,t}|^{\frac{2t}{s+t}} \geq |B|^{2t}.$$

Hence

$$\begin{aligned} \left|(\widetilde{A \otimes B})_{s,t}\right|^{\frac{2t}{s+t}} &= |\tilde{A}_{s,t} \otimes \tilde{B}_{s,t}|^{\frac{2t}{s+t}} \\ &= |\tilde{A}_{s,t}|^{\frac{2t}{s+t}} \otimes |\tilde{B}_{s,t}|^{\frac{2t}{s+t}} \geq |A|^{2t} \otimes |B|^{2t} \\ &= (|A| \otimes |B|)^{2t} = |A \otimes B|^{2t}, \end{aligned}$$

by Lemmas 1 and 2. Hence $A \otimes B$ is a class $A(s, t)$ operator.

Conversely let $A \otimes B$ be a class $A(s, t)$ operator. Then there exists $c > 0$ such that

$$|A|^{2t} \leq c|\tilde{A}_{s,t}|^{\frac{2t}{s+t}},$$

$$|B|^{2t} \leq c^{-1}|\tilde{B}_{s,t}|^{\frac{2t}{s+t}},$$

by Lemma 1. Let $x \in \mathcal{H}$ be a unit vector. Then

$$\begin{aligned}\||A|^t x\|^2 &= \langle |A|^{2t} x, x|\rangle \le c\big\langle |\tilde{A}_{s,t}|^{\frac{2t}{s+t}} x, x\big\rangle \\ &\le c\big\| |\tilde{A}_{s,t}|^{\frac{t}{s+t}} \big\|^2 = c\|\tilde{A}_{s,t}\|^{\frac{2t}{s+t}} = c\||A|^s U |A|^t\|^{\frac{2t}{s+t}} \le c\||A|^t\|^2,\end{aligned}$$

where $A = U|A|$ is the polar decomposition of A. Hence $\||A|^t\|^2 \le c\||A|^t\|^2$ and $1 \le c$.

Similarly we have $1 \le c^{-1}$ because $|B|^{2t} \le c^{-1}|\tilde{B}_{s,t}|^{\frac{2t}{s+t}}$. Thus $c = 1$. This implies that A and B are class $A(s, t)$ operators. □

LEMMA 4. *Let $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ be invertible non-negative operators. Then*

$$\log(A \otimes B) = (\log A) \otimes I + I \otimes (\log B),$$

where I denotes the identity operator.

Proof. Let $A = \int_0^\infty \lambda dE(\lambda)$ and $B = \int_0^\infty \mu dF(\mu)$ be the spectral decompositions of A and B, respectively. Then

$$A \otimes B = \int_0^\infty \int_0^\infty \lambda\mu dG(\lambda, \mu),$$

where

$$G(\sigma \times \tau) = E(\sigma) \otimes F(\tau)$$

for all Borel sets $\sigma, \tau \subset [0, \infty)$. Hence

$$\begin{aligned}\log(A \otimes B) &= \int_0^\infty \int_0^\infty \log(\lambda\mu)\, dG(\lambda, \mu) \\ &= \int_0^\infty \int_0^\infty (\log\lambda + \log\mu)\, dG(\lambda, \mu) \\ &= (\log A) \otimes I + I \otimes (\log B).\end{aligned}$$

□

THEOREM 5. *Let $A \in B(\mathcal{H})$, $B \in B(\mathcal{K})$. Then $A \otimes B$ is* log*-hyponormal if and only if A and B are* log*-hyponormal.*

Proof. Let A and B be log-hyponormal. Then A and B are invertible and

$$\log|A| \ge \log|A^*|,$$
$$\log|B| \ge \log|B^*|.$$

Hence $A \otimes B$ is invertible and

$$\begin{aligned}&\log|A \otimes B| - \log|(A \otimes B)^*| \\ &\quad = \log(|A| \otimes |B|) - \log(|A^*| \otimes |B^*|) \\ &\quad = (\log|A|) \otimes I + I \otimes (\log|B|) - (\log|A^*|) \otimes I - I \otimes (\log|B^*|) \\ &\quad = (\log|A| - \log|A^*|) \otimes I + I \otimes (\log|B| - \log|B^*|) \ge 0,\end{aligned}$$

by Lemmas 1 and 4. Thus $A \otimes B$ is log-hyponormal.

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Conversely let $A \otimes B$ be log-hyponormal. Since $A \otimes B$ is invertible and

$$\sigma(A \otimes B) = \{\lambda\mu | \lambda \in \sigma(A), \mu \in \sigma(B)\},$$

we have that A and B are invertible and

$$\begin{aligned} &\log|A \otimes B| - \log|(A \otimes B)^*| \\ &\quad = (\log|A| - \log|A^*|) \otimes I + I \otimes (\log|B| - \log|B^*|) \geq 0. \end{aligned}$$

Hence

$$\begin{aligned} &\langle((\log|A| - \log|A^*|) \otimes I)\, x \otimes y, x \otimes y\rangle \\ &\quad \geq -\langle(I \otimes (\log|B| - \log|B^*|))\, x \otimes y, x \otimes y\rangle \end{aligned}$$

for $x \in \mathcal{H}$, $y \in \mathcal{K}$, and

$$\begin{aligned} &\langle(\log|A| - \log|A^*|)\, x, x\rangle \\ &\quad \geq -\langle(\log|B| - \log|B^*|)\, y, y\rangle \end{aligned}$$

for unit vectors $x \in \mathcal{H}$, $y \in \mathcal{K}$. This implies that there exists a real number $c \in \mathbb{R}$ such that

$$\begin{aligned} &\inf_{\|x\|=1} \langle(\log|A| - \log|A^*|)\, x, x\rangle = c \\ &\quad \geq \sup_{\|y\|=1} \langle -(\log|B| - \log|B^*|)\, y, y\rangle \\ &\quad = -\inf_{\|y\|=1} \langle(\log|B| - \log|B^*|)\, y, y\rangle. \end{aligned}$$

Hence

$$\begin{aligned} \log|A| - \log|A^*| &\geq cI, \\ \log|B| - \log|B^*| &\geq -cI. \end{aligned}$$

Since

$$\log(|kA|) - \log(|(kA)^*|) = \log|A| - \log|A^*|$$

for all $k > 0$, we may assume that $I < |A|, e^c|A^*|, |B|, e^{-c}|B^*|$ by taking kA, kB instead of A, B for some large $k > 0$. Then

$$\begin{aligned} \left\|(\log|A|)^{\frac{1}{2}} x\right\|^2 &= \langle(\log|A|)x, x\rangle \\ &\geq \langle(\log(e^c|A^*|))\, x, x\rangle \\ &= \left\|(\log(e^c|A^*|))^{\frac{1}{2}}\, x\right\|^2 \end{aligned}$$

for $x \in \mathcal{H}$. Hence

$$\begin{aligned} (\log\|A\|)^{\frac{1}{2}} &= (\log\||A|\|)^{\frac{1}{2}} \\ &= \left\|(\log|A|)^{\frac{1}{2}}\right\| \geq \left\|(\log(e^c|A^*|))^{\frac{1}{2}}\right\| \\ &= (\log(e^c\|A^*\|))^{\frac{1}{2}} = (\log(e^c\|A\|))^{\frac{1}{2}} \end{aligned}$$

and $1 \geq e^c$.

Similarly we have that $1 \geq e^{-c}$ by $\log|B| - \log|B^*| \geq -cI$. Thus $c = 0$ and this implies that A and B are log-hyponormal. □

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REFERENCES

1. A. Aluthge, On p-hyponormal operators for $0 < p < 1$, *Integral Equations Operator Theory* **13** (1990), 307–315.

2. A. Aluthge and D. Wang, An operator inequality which implies paranormality, *Math. Inequalities and Applications* **2** (1999), 113–119.

3. T. Ando, Operators with a norm condition, *Acta Sci. Math. Szeged* **33** (1972), 169–178.

4. B. P. Duggal, Tensor products of operators – strong stability and p-hyponormality, *Glasgow Math. J.* **42** (2000), 371–381.

5. D. R. Farenick and I. H. Kim, Tensor products of quasihyponormal operators, *Proceedings of Kotac* **4** (2002), 113–119.

6. M. Fujii, C. Himeji and A. Matsumoto, Theorems of Ando and Saito for p-hyponormal operators, *Math. Japonica* **39** (1994), 595–598.

7. T. Huruya, A note on p-hyponormal operators, *Proc. Amer. Math. Soc.* **125** (1997), 3617–3624.

8. M. Ito, Some classes of operators with generalized Aluthge transformations, *SUT J. Math.* **35** (1999), 149–165.

9. M. Ito and T. Yamazaki, Relations between two inequalities $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \leq B^r$ and $A^p \leq (A^{\frac{p}{2}}B^rA^{\frac{r}{2}})^{\frac{p}{p+r}}$ and their applications, preprint.

10. I. H. Jeon and B. P. Duggal, On operators with an absolute value condition, preprint.

11. H. Jinchuan, On the tensor products of operators, *Acta Math. Sinica* **9** (1993), 195–202.

12. B. Magajna, On subnormality of generalized derivations and tensor products, *Bull. Austral. Math. Soc.* **31** (1985), 235–243.

13. Jan Stochel, Seminormality of operators from their tensor products, *Proc. Amer. Math. Soc.* **124** (1996), 435–440.

14. K. Tanahashi, On log-hyponormal operators, *Integral. Equations Operator Theory* **34** (1999), 364–372.

15. K. Tanahashi, Putnam's inequality for log-hyponormal operators, *Integral. Equations Operator Theory* to appear.

16. D. Xia, *Spectral theory of hyponormal operators* (Birkhauser-Verlag, 1983).

17. M. Yanagida, Powers of class $wA(s,t)$ operators associated with generalized Aluthge transformations, preprint.

18. T. Yoshino, The p-hyponormality of the Aluthge transform, *Interdisciplinary Information Sciences* **3** (1997), 91–93.