SIMPLE CONTINUITY INEQUALITIES FOR RUIN PROBABILITY IN THE CLASSICAL RISK MODEL

BY

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ABSTRACT

A simple technique for continuity estimation for ruin probability in the compound Poisson risk model is proposed. The approach is based on the contractive properties of operators involved in the integral equations for the ruin probabilities. The corresponding continuity inequalities are expressed in terms of the Kantorovich and weighted Kantorovich distances between distribution functions of claims. Both general and light-tailed distributions are considered.

KEYWORDS

Risk model, infinite-time ruin probability, continuity inequalities, contractive operators, Kantorovich's metric.

1. INTRODUCTION

Throughout the paper, a compound Poisson risk process is considered:

$$R_t = u + ct - \sum_{j=1}^{N_t} X_j, \quad t \geq 0, \tag{1.1}$$

where N_t is a Poisson process with *rate* λ, independent of $\{X_j\}$, c is the *premium rate* and, $u \geq 0$ is the *initial reserve*.

A common distribution function of the i.i.d. positive random variables, representing claim sizes, is denoted by F. Let

$$\psi(u) := P\left(\inf_{t\geq 0} R_t < 0 | R_0 = u\right), \quad u \geq 0, \tag{1.2}$$

be the *ruin probability* in the infinite time.

Typically, in applications, the parameters λ and F that govern the risk model are unknown. Therefore, they must be approximated by parameters $\widetilde{\lambda}$, $\widetilde{F}$. Such

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approximation can be the result of theoretical considerations or of statistical procedures. (See e.g. Benouaret and Aïssanni (2010), Enikeeva *et al.* (2001), Kalashnikov (2000) for a discussion of this issue.)

Under this scenario, one could consider a risk process as in (1.1), but one that is governed by the parameters $\widetilde{\lambda}$ and $\widetilde{F}$. That is, the approximations $\widetilde{\lambda}$ and $\widetilde{F}$ define an approximated risk process. We denote the ruin probability for this process by $\widetilde{\psi}(u)$, $u \geq 0$. Our main objective is the comparison between ψ and $\widetilde{\psi}$.

Specifically, assuming that

$$\mu := \int_0^\infty x dF(x) < \infty, \quad \widetilde{\mu} := \int_0^\infty x d\widetilde{F}(x) < \infty$$

hold, and the net profit conditions:

$$\rho := \frac{\lambda\mu}{c} < 1, \quad \widetilde{\rho} := \frac{\widetilde{\lambda}\widetilde{\mu}}{c} < 1, \tag{1.3}$$

we developed a simple technique to prove the following inequalities:

$$\sup_{u \geq 0} |\psi(u) - \widetilde{\psi}(u)| \leq \frac{\widetilde{\mu}}{c - \lambda\mu}|\lambda - \widetilde{\lambda}| + \frac{\lambda}{c - \lambda\mu}\int_0^\infty |F(x) - \widetilde{F}(x)| dx, \tag{1.4}$$

$$\sup_{u \geq 0} e^{\alpha u}|\psi(u) - \widetilde{\psi}(u)| \leq \frac{K}{\beta}\left[\int_0^\infty e^{x\beta} d\widetilde{F}(x) - 1\right]|\lambda - \widetilde{\lambda}| + K\lambda \int_0^\infty e^{x\beta}|F(x) - \widetilde{F}(x)| dx, \tag{1.5}$$

where the values of constants $\alpha > 0$, $\beta > 0$, and $K = K(c, \widetilde{\lambda}, \alpha) < \infty$ will be specified in Theorem 2.4 of Section 2.

Inequality (1.4) is obtained in a general setting, whereas (1.5) is proved for the light-tailed distributions F and $\widetilde{F}$. In the last case, the exponential vanishing of $\psi(u)$ and $\widetilde{\psi}(u)$ was used.

In particular, bounds (1.4) and (1.5) could be useful in the typical situations where the ruin probability ψ cannot be calculated in the model (1.1). In such a case, the distribution of claim sizes F is approximated by some distribution function $\widetilde{F}$, for which the ruin probability $\widetilde{\psi}$ can be found.

Beginning with the classical Cramér–Lundberg approximation, several methods for ruin probability estimation have been proposed (see, for instance, the book by Asmussen and Albrecher (2010) and the references therein). However, to get such approximations it is necessary to have either the values of two, three or more moments of F, or certain characteristics of its moment generating function (or perhaps other parameters of F). A similar situation occurs in the well-developed theory of upper or two-sided bounds of ruin probability (see, for instance, the bounds considered in the books by Asmussen and Albrecher

(2010), Rolski *et al.* (1999) and the papers by Cai and Dickson (2002), Kartashov (2001), Kalashnikov (1996) and Kalashnikov (1997)).

In our setting, in order to apply inequality (1.4), it is sufficient to know the first moments of claim size distributions (or some bounds for these moments). In Section 3, we briefly discuss how (1.4) can be used to evaluate the convergence rate of non-parametric estimators of ruin probabilities.

There are few papers in the literature that offer continuity inequalities for ruin probability. We found the following papers: Benouaret and Aïssani (2010), Enikeeva *et al.* (2001), Rusaityte (2001) and Kalashnikov (2000). The method developed in these works is also suitable for more general risk models (e.g. for the Sparre Andersen model).

The method used in the first two papers mentioned above is based on a rather complicated technique expressing the ruin probabilities in terms of stationary distribution of related Markov chains. To compare stationary distributions of two "close" chains, the appropriated ergodicity conditions and results from Kartashov (1986), Meyn and Tweedie (1993), and Yu (2005) were used. Such an approach allows the estimation of the ruin probability continuity for rather general risk processes. However, the inequalities obtained in the aforementioned papers have the following peculiarities:

i. They are local, i.e. the inequalities hold only if the parameters of risk processes compared are close enough.
ii. The inequalities are not linear with respect to the distances between the parameters of the models.

In a recent paper by Benouaret and Aïssani (2010), the strong total variation metrics were used and in Enikeeva *et al.* (2001) the deviation of $\widetilde{\psi}$ from ψ is measured by means of $\int_0^\infty e^{\alpha u}|\psi(u) - \widetilde{\psi}(u)|du$.

Rusaityte (2001) studied the continuity of ruin probability for Markov modulated risk models with investments. Applying regenerative techniques and under suitable Lyapunov-type conditions, Rusaityte obtained the continuity inequalities for ruin probability. For the case of light tails, the deviations of claim size distributions is measured by the weighted uniform metric.

Another direction for the study of continuity of ruin probabilities is based on sensitivity analysis (see e.g. Asmussen and Albrecher (2010), Chan and Yang (2005)). In this theory, assuming that F belongs to some parametric family, derivatives of the ruin probability with respect to the parameters are calculated. This allows local estimations of the ruin probability alterations to be obtained with respect to disturbances of the parameters of F and λ.

2. Main results

2.1. The general case

Let $\mathfrak{F}$ denote the set of distribution functions F of all positive random variables.

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The function $\mathbb{K} : \mathfrak{F} \times \mathfrak{F} \longrightarrow [0, \infty]$,

$$\mathbb{K}(F, G) := \int_0^\infty |F(x) - G(x)| dx, \tag{2.1}$$

is called the *Kantorovich metric* in $\mathfrak{F}$.

It is well known that if $\int_0^\infty x dF(x) < \infty$ and $\int_0^\infty x dG(x) < \infty$ then $\mathbb{K}(F, G) < \infty$, and for a sequence $\{F; F_n\} \in \mathfrak{F}$ with $\int_0^\infty x dF_n(x) < \infty$, $\int_0^\infty x dF(x) < \infty$, we have $\mathbb{K}(F_n, F) \longrightarrow 0$ if and only if

i. $F_n \Rightarrow F$ (converges weakly); and
ii. $\int_0^\infty x dF_n(x) \longrightarrow \int_0^\infty x dF(x)$ (as $n \longrightarrow \infty$).
(See e.g. Rachev and Rüschendorf (1998))

Theorem 2.1.

$$\sup_{u \geq 0} |\psi(u) - \widetilde{\psi}(u)| \leq K \left[\lambda \mathbb{K}(F, \widetilde{F}) + |\lambda - \widetilde{\lambda}| \widetilde{\mu} \right], \tag{2.2}$$

where $\mathbb{K}$ *is the Kantorovich metric defined in* (2.1)*, and*

$$K = \min \left\{ \frac{1}{c - \mu\lambda}, \frac{1}{c - \widetilde{\mu}\widetilde{\lambda}} \right\}. \tag{2.3}$$

Inequality (2.2) holds for every distribution functions of claim sizes F, $\widetilde{F}$ (provided that the averages are finite), particularly for the so-called heavy-tailed distributions. Unfortunately, we could not find explicit formulas for calculating ruin probabilities in some particular cases of heavy tails. Thus, to give a numerical illustration of (2.2) we consider claim sizes with the Gamma distribution.

Example 2.2. *For* $\varepsilon > 0$ *let* F *and* $\widetilde{F} = \widetilde{F}_\varepsilon$ *have the densities* $Gamma(\alpha = 2, \beta = 1)$ *and* $Gamma(\alpha = 2, \beta = 1 + \varepsilon)$*, respectively. Using the exact expressions for the ruin probabilities* $\psi(u)$ *and* $\widetilde{\psi}(u)$*, given for instance in Yuanjiang et al. (2003), we have calculated numerically* $\sup_{u \geq 0} |\psi(u) - \widetilde{\psi}_\varepsilon(u)|$ *and the term* $\frac{\lambda}{c - \mu\lambda} \mathbb{K}(F, \widetilde{F}_\varepsilon)$ *on the right-hand side of* (2.2)*. For* $\lambda = \widetilde{\lambda} = 1$ *the results are presented in the following tables.*

TABLE 1

$c = 3, \quad \rho = 2/3.$

ε	$\sup_{u \geq 0} \lvert\psi(u) - \widetilde{\psi}_\varepsilon(u)\rvert$	$\frac{\lambda}{c - \mu\lambda} \mathbb{K}(F, \widetilde{F}_\varepsilon)$
0.5	0.2931	0.6667
0.2	0.1594	0.3333
0.1	0.0908	0.1818
0.01	0.0104	0.0198

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TABLE 2

$c = 10, \quad \rho = 1/5.$

ε	$\sup_{u \geq 0} \lvert \psi(u) - \widetilde{\psi}_\varepsilon(u) \rvert$	$\frac{\lambda}{c - \mu\lambda} \mathbb{K}(F, \widetilde{F}_\varepsilon)$
0.5	0.0685	0.0833
0.2	0.0345	0.0417
0.1	0.0189	0.0227
0.01	0.0021	0.0025

As we can see from these tables, inequality (2.2) works better for *small* ρ (i.e. in the light traffic case). In this example, for $c = 100$ ($\rho = 1/50$ and $\varepsilon = 0.01$) the left-hand side of (2.2) is 0.000198, while its right-hand side is 0.000202.

From the point of view of statistical applications, it is preferable to have a distance as "weak"as possible on the right-hand side of (2.2). The convergence in $\mathbb{K}$ is equivalent to the weak convergence plus the convergence of the first moments. The next question is then: Can the Kantorovich metric $\mathbb{K}$ in (2.2) be replaced by, say, the uniform distance

$$\mathcal{U}(F, \widetilde{F}) := \sup_{x \geq 0} |F(x) - \widetilde{F}(x)|?$$

(For continuous limiting distributions, the convergence in $\mathcal{U}$ coincides with the weak convergence). The answer is negative, since the convergence of the first moments is important for the stability of ruin probabilities. Indeed, we know that, if $\lambda = \widetilde{\lambda}$ then

$$|\psi(0) - \widetilde{\psi}(0)| = \frac{\lambda}{c} |\mu - \widetilde{\mu}| .$$

The following example is offered to show what happens for $u > 0$.

Example 2.3. *We denote by X and $\widetilde{X}$ the random variables with distribution functions F and $\widetilde{F}$, respectively. Let $\lambda = \widetilde{\lambda} = 1$, $c = 3$, $X = 1$ and for $n = 1, 2, \ldots$*

$$\widetilde{X} = \widetilde{X}_n = \begin{cases} 1, & \textit{with probability } p_1 = 1 - 1/n, \\ n, & \textit{with probability } p_2 = 1/n. \end{cases}$$

Then $\mu = EX = 1$, $\widetilde{\mu}_n = E\widetilde{X}_n = 2 - 1/n$, and $|\psi(0) - \widetilde{\psi}_n(0)| = 1/3 - 1/3n \longrightarrow 1/3$.

On the other hand, it easy to see that $\mathcal{U}(F, \widetilde{F}_n) \longrightarrow 0$ as $n \longrightarrow \infty$ and particularly $\widetilde{F}_n \Longrightarrow F$.

To ensure that $u = 0$ is not an exceptional value, we take $u = 3$ and, using the formulas for $\psi(u)$ when X is a random variable taking values in a finite set (see

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e.g. Chapter 4 in Kass et al. (2001)), we obtain

$$\psi(3) \approx 0.0018, \quad \textit{and}$$

$$\widetilde{\psi}_n(3) = 1 - \left[e - \frac{2}{3}e^{2/3}\left(1 - \frac{1}{n}\right) + \frac{1}{18}e^{1/3}\left(1 - \frac{1}{n^2}\right)\right] \longrightarrow 0.5009,$$

as $n \longrightarrow \infty$.

2.2. The case of light-tailed distributions of claim sizes

Assumption 1.

i. *There exists a number* $r_* \in (0, \infty]$ *such that* $\mathcal{J}(r) := \int_0^\infty e^{rx} dF(x) < \infty$ *for* $r < r_*$, *and* $\mathcal{J}(r) \longrightarrow \infty$ *as* $r \uparrow r_*$.
ii. *For some* $\widetilde{r}_* \in (0, \infty]$ *the same is true for the distribution function* $\widetilde{F}$.

It is well known that under the above assumption *adjustment coefficients* $\gamma \in (0, r_*)$, $\widetilde{\gamma} \in (0, \widetilde{r}_*)$ exist, and Lundberg inequalities:

$$\psi(u) \leq e^{-\gamma u}, \quad \widetilde{\psi}(u) \leq e^{-\widetilde{\gamma} u}, \quad u \geq 0 \tag{2.4}$$

hold (see e.g. Asmussen and Albrecher (2010), Rolski *et al.* (1999)).

These inequalities show that it is not effective in practically interesting cases of large initial capitals u to measure the deviation of $\widetilde{\psi}$ from ψ under the uniform metric $\sup_{u\geq 0} |\psi(u) - \widetilde{\psi}(u)|$ (as in Theorem 2.1). It is better to fix $\alpha \in (0, \gamma_*)$, where $\gamma_* := \min(\gamma, \widetilde{\gamma})$, and consider the distance

$$d_\alpha(\psi, \widetilde{\psi}) := \sup_{u\geq 0} e^{\alpha u} |\psi(u) - \widetilde{\psi}(u)|. \tag{2.5}$$

In the Theorem 2.4 below, we obtain the upper bound for $d_\alpha(\psi, \widetilde{\psi})$ in (2.5), where α can be chosen to be arbitrarily close to γ_* (but different to it). The price for such improvement of the deviation measure is that we have to replace the Kantorovich metric on the right side of inequality (2.2), by the following *weighted Kantorovich metric* $\mathbb{K}_\beta$ on the space $\mathfrak{F}$ of all distribution functions of positive random variables. For a given $\beta > 0$,

$$\mathbb{K}_\beta(F, G) := \int_0^\infty e^{\beta x} |F(x) - G(x)| \, dx. \tag{2.6}$$

The sufficient condition of finiteness of $\mathbb{K}_\beta(F, G)$ is $\int_0^\infty e^{\beta x} dF(x) < \infty$ and $\int_0^\infty e^{\beta x} dG(x) < \infty$.

On the other hand, under such restrictions, $\mathbb{K}(F, F_n) \longrightarrow 0$ if and only if F_n converges to F weakly and $\int_0^\infty e^{\beta x} dF_n(x) \longrightarrow \int_0^\infty e^{\beta x} dF(x)$ as $n \longrightarrow \infty$.

Given that we could not find supporting references for the above, we provide a proof in the Appendix.

Now let $\alpha \in (0, \gamma_*)$ be an arbitrary fixed number and β be an arbitrary but fixed upper bound of γ, such that

$$\min(r_*, \widetilde{r}_*) > \beta \geq \gamma. \tag{2.7}$$

Theorem 2.4. *Under Assumption 1,*

$$\sup_{u\geq 0} e^{\alpha u}|\psi(u)-\widetilde{\psi}(u)| \leq \frac{1}{\beta c(1-M_\alpha)} \times \left[\int_0^\infty e^{x\beta}d\widetilde{F}(x)-1\right]|\lambda-\widetilde{\lambda}| + \frac{\lambda}{c(1-M_\alpha)}\mathbb{K}_\beta(F,\widetilde{F}), \tag{2.8}$$

where $\mathbb{K}_\beta$ *is the weighted Kantorovich metric defined in* (2.6)*, and*

$$M_\alpha := \frac{\widetilde{\lambda}}{c}\int_0^\infty e^{\alpha x}[1-\widetilde{F}(x)]dx < 1. \tag{2.9}$$

The following inequality that was obtained for the case $\widetilde{\lambda}=\lambda$ provides the bound with greater accuracy than (2.8).

Proposition 2.5. *Let* $\widetilde{\lambda}=\lambda$*. Then, under hypotheses of Theorem 2.4,*

$$\sup_{u\geq 0} e^{\alpha u}|\psi(u)-\widetilde{\psi}(u)| \leq \frac{\lambda}{c(1-M_\alpha)} \times \left[\sup_{u\geq 0} e^{\alpha u}\int_u^\infty |F(x)-\widetilde{F}(x)|dx + \sup_{u\geq 0} e^{-(\gamma-\alpha)u}\int_0^u e^{\gamma x}|F(x)-\widetilde{F}(x)|dx\right]. \tag{2.10}$$

Remark 2.6.

a. *We cannot take* $\alpha=\gamma_*=\min(\gamma,\widetilde{\gamma})$*, since in* (2.8) *and* (2.9) *it can be* $M_\alpha=1$.
b. *To prove inequality* (2.8)*, we have essentially used Lundberg's inequalities* (2.4)*. It is hardly credible that* (2.8) *could, somehow be derived from inequality* (2.2)*. In a certain sense, inequalities* (2.2) *and* (2.8) *are "immeasurable". For instance, we can see from a simple example with exponential distributions that when taking in* (2.8) $\beta=0$*,* $\lambda=\widetilde{\lambda}$ *the resulting inequality is false for* α *close enough to* γ.

2.3. Remarks on approximation by empirical distributions

As an example of application of inequality (2.2), we can consider the important situation from a practical point of view. That is, when one needs to estimate the ruin probability $\psi(u)$ in the risk model (1.1) *without assuming any preliminary information* on intensity of claims λ and/or the distribution function of claim sizes F. Instead, it is assumed that i.i.d. observations $X_1, X_2, \ldots, X_n$ of the random variable X, distributed according to F, are available. Also, that statistical data $\tau_1, \tau_2, \ldots, \tau_n$ on intervals between claims arrivals are acceptable.

Among other possible approaches to non-parametric statistical estimation of ruin probability (see e.g. Frees (1986), Bening and Korolev (2003), Mnatsakanov *et al.* (2008), Marceau and Rioux (2001)), one can consider the following one. The estimators

$$\hat{\lambda}_n := \frac{n}{\tau_1 + \cdots + \tau_n}, \quad (n = 1, 2, \dots)$$

and the empirical distribution functions

$$\widehat{F}_n(x) := \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}(x), \quad x \geq 0,$$

are calculated, and they are used to approximate the unknown λ and F.

In this case, the calculation of the ruin probability $\widetilde{\psi} \equiv \widetilde{\psi}_n$ requires only adequate computer facilities, since the random variable $\widetilde{X}_n$ with the distribution $\hat{F}_n$ takes n values $X_1, X_2, \dots, X_n$ with probabilities equal to $1/n$. There are well-known explicit formulas to calculate the ruin probability in the classic model if a claim size is a random variable with a finite number of values (see e.g. the book by Kass *et al.* (2001)).

It is known that if $\int_0^\infty x dF(x) < \infty$, then $\int_0^\infty |F(x) - \widehat{F}_n(x)| dx \longrightarrow 0$ almost surely as $n \longrightarrow \infty$ (see for instance Rachev (1991)). Also $\hat{\lambda}_n \longrightarrow \lambda$ almost surely. Therefore by (2.2), $\widetilde{\psi}_n(u)$ is a consistent estimate of $\psi(u)$, and with probability one $\sup_{u \geq 0} |\psi(u) - \widetilde{\psi}_n(u)| \longrightarrow 0$ as $n \longrightarrow \infty$.

Additionally, if we assume that for some $\delta > 0$, $\mu_{2+\delta} := \int_0^\infty x^{2+\delta} dF(x) < \infty$, then from the results of Bobkov and Ledoux (2014) it can be derived that

$$E \int_0^\infty |F(x) - \widehat{F}_n(x)| dx \leq \left[1 + \frac{2}{\delta} \mu_{2+\delta}^{1/2} \right] \frac{1}{\sqrt{n}}, \quad n = 1, 2, \dots. \tag{2.11}$$

Using the asymptotic normality, we can see that

$$E|\lambda - \widetilde{\lambda}_n| \leq c(\lambda) \frac{1}{\sqrt{n}}, \quad n = 1, 2, \dots \tag{2.12}$$

Assuming the existence of some known upper bounds for λ, μ and $\widetilde{\mu}$, we can substitute these bounds in (2.2). Combining such modification of inequality (2.2) with (2.11) and (2.12), we can obtain an upper bound for average deviations $\widetilde{\psi}_n(u)$ from $\psi(u)$.

3. Proofs

We will use notation: $\overline{F}(x) := 1 - F(x)$; $\overline{\widetilde{F}}(x) := 1 - \widetilde{F}(x)$, $x \geq 0$.

The following integral equations for ruin probabilities are commonly known (see e.g. Rolski *et al.* (1999)),

$$\psi(u) = \frac{\lambda}{c}\left(\int_u^\infty \overline{F}(t)dt + \int_0^u \psi(u-t)\overline{F}(t)dt\right), \tag{3.1}$$

$$\widetilde{\psi}(u) = \frac{\widetilde{\lambda}}{c}\left(\int_u^\infty \overline{\widetilde{F}}(t)dt + \int_0^u \psi(u-t)\overline{\widetilde{F}}(t)dt\right), \quad u \geq 0. \tag{3.2}$$

Let $\mathfrak{X}$ be the space of all functions $x : [0, \infty) \longrightarrow [0, 1]$ endowed with the uniform metric $\nu(x, y) := \sup_{u\geq 0} |x(u) - y(u)|$. Then $(\mathfrak{X}, \nu)$ is a complete metric space.

It is easy to check that for the operators

$$Tx(u) = \frac{\lambda}{c}\left(\int_u^\infty \overline{F}(t)dt + \int_0^u x(u-t)\overline{F}(t)dt\right), \quad u \geq 0, \tag{3.3}$$

$$\widetilde{T}x(u) = \frac{\widetilde{\lambda}}{c}\left(\int_u^\infty \overline{\widetilde{F}}(t)dt + \int_0^u x(u-t)\overline{\widetilde{F}}(t)dt\right), \quad u \geq 0, \tag{3.4}$$

we have $T\mathfrak{X} \subset \mathfrak{X}$ and $\widetilde{T}\mathfrak{X} \subset \mathfrak{X}$.

Moreover, these operators are contractive on $\mathfrak{X}$ with modules $\rho = \lambda\mu/c$ and $\widetilde{\rho} = \widetilde{\lambda}\widetilde{\mu}/c$, respectively. Indeed, by (3.3), for every $x, y \in \mathfrak{X}$,

$$\begin{aligned}
\nu(Tx, Ty) &= \frac{\lambda}{c}\sup_{u\geq 0}\left|\int_0^u x(u-t)\overline{F}(t)dt - \int_0^u y(u-t)\overline{F}(t)dt\right| \\
&\leq \frac{\lambda}{c}\sup_{u\geq 0}\int_0^u \overline{F}(t)\sup_{s\in[0,u]}|x(s)-y(s)|dt \\
&\leq \frac{\lambda}{c}\nu(x, y)\int_0^\infty \overline{F}(t)dt = \frac{\lambda\mu}{c}\nu(x, y).
\end{aligned}$$

The inequality

$$\nu(\widetilde{T}x, \widetilde{T}y) \leq \frac{\widetilde{\lambda}\widetilde{\mu}}{c}\nu(x, y) \tag{3.5}$$

is similarly verified.

According to (3.1), (3.2), ψ and $\widetilde{\psi}$ are the unique fixed points of T and $\widetilde{T}$: $\psi = T\psi$ and $\widetilde{\psi} = \widetilde{T}\widetilde{\psi}$.

Now,

$$\begin{aligned}
\nu(\psi, \widetilde{\psi}) = \nu(T\psi, \widetilde{T}\widetilde{\psi}) &\leq \nu(T\psi, T\widetilde{\psi}) + \nu(T\widetilde{\psi}, \widetilde{T}\widetilde{\psi}) \\
&\leq \rho\nu(\psi, \widetilde{\psi}) + \nu(T\widetilde{\psi}, \widetilde{T}\widetilde{\psi}),
\end{aligned}$$

or

$$\nu(\psi, \widetilde{\psi}) \leq \frac{1}{1-\rho}\nu(T\widetilde{\psi}, \widetilde{T}\widetilde{\psi}) = \frac{c}{c-\lambda\mu}\nu(T\widetilde{\psi}, \widetilde{T}\widetilde{\psi}). \tag{3.6}$$

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In view of (3.3) and (3.4), for each $x \in \mathfrak{X}$ we have

$$\begin{aligned}
\nu(Tx, \widetilde{T}x) \leq & \sup_{u\geq 0}\left|\frac{\lambda}{c}\left(\int_u^\infty \overline{F}(t)dt + \int_0^u x(u-t)\overline{F}(t)dt\right)\right. \\
& \left. - \frac{\lambda}{c}\left(\int_u^\infty \overline{\widetilde{F}}(t)dt + \int_0^u x(u-t)\overline{\widetilde{F}}(t)dt\right)\right| \\
& + \sup_{u\geq 0}\left|\frac{\lambda-\widetilde{\lambda}}{c}\left(\int_u^\infty \overline{\widetilde{F}}(t)dt + \int_0^u x(u-t)\overline{\widetilde{F}}(t)dt\right)\right| \\
\leq & \frac{\lambda}{c}\int_0^\infty |\overline{F}(t) - \overline{\widetilde{F}}(t)|dt + \frac{|\lambda-\widetilde{\lambda}|}{c}\int_0^\infty (1-\widetilde{F}(t))dt \\
= & \frac{\lambda}{c}\mathbb{K}(F, \widetilde{F}) + \frac{|\lambda-\widetilde{\lambda}|}{c}\widetilde{\mu}.
\end{aligned}$$

Combining the last inequality with (3.6), we obtain

$$\nu(\psi, \widetilde{\psi}) \leq \frac{1}{c-\lambda\mu}\left[\lambda\mathbb{K}(F, \widetilde{F}) + |\lambda-\widetilde{\lambda}|\widetilde{\mu}\right].$$

To prove the similar inequality with $\frac{1}{c-\widetilde{\lambda}\widetilde{\mu}}$ instead of $\frac{1}{c-\lambda\mu}$ inequality (3.5) is used. Thus, we have proved inequality (2.2).

Regarding the proof of Theorem 2.4, first we note that $\alpha < \min(\gamma, \widetilde{\gamma}) < \min(r_*, \widetilde{r}_*)$, where $r_*, \widetilde{r}_*$ are numbers from Assumption 1. Hence, the constant

$$M_\alpha := \frac{\widetilde{\lambda}}{c}\int_0^\infty e^{\alpha t}\overline{\widetilde{F}}(t)dt, \tag{3.7}$$

is finite, as well as the function $I(r) := \frac{\widetilde{\lambda}}{c}\int_0^\infty e^{rt}\overline{\widetilde{F}}(t)dt$, $r \in [0, \widetilde{r}_*)$. We have that $I(0) = \dfrac{\widetilde{\lambda}\widetilde{\mu}}{c} < 1$ and $I(r)$ is strictly increasing with $I(\widetilde{\gamma}) = 1$, where $\widetilde{\gamma}$ is the adjustment coefficient in the approximating risk model (see e.g. Rolski *et al.* (1999)). Therefore, inequality (2.9) holds.

In the space $\mathfrak{X}$, we define the following metric ν_α (taking values in $[0, \infty]$):

$$\nu_\alpha(x, y) := \sup_{u\geq 0} e^{\alpha u}|x(u) - y(u)|, \quad x, y \in \mathfrak{X}.$$

Now we show that the operator $\widetilde{T}$ in (3.4) satisfies the inequality:

$$\nu_\alpha(\widetilde{T}x, \widetilde{T}y) \leq M_\alpha \nu_\alpha(x, y); \quad x, y \in \mathfrak{X}, \tag{3.8}$$

where $M_\alpha < 1$ is the constant from (3.7).

By (3.4), we obtain that

$$\begin{aligned}\nu_\alpha(\widetilde{T}x, \widetilde{T}y) &= \sup_{u\geq 0} e^{\alpha u}\frac{\widetilde{\lambda}}{c}\left|\int_0^u x(u-t)\overline{\widetilde{F}}(t)dt - \int_0^u y(u-t)\overline{\widetilde{F}}(t)dt\right| \\ &\leq \frac{\widetilde{\lambda}}{c}\sup_{u\geq 0}\int_0^u \overline{\widetilde{F}}(t)|x(u-t)-y(u-t)|e^{\alpha(u-t)}e^{\alpha t}dt \\ &\leq \nu_\alpha(x, y)\frac{\widetilde{\lambda}}{c}\int_0^\infty e^{\alpha t}\overline{\widetilde{F}}(t)dt.\end{aligned}$$

Now by the triangle inequality,

$$\begin{aligned}\nu_\alpha(\psi, \widetilde{\psi}) &= \nu_\alpha(T\psi, \widetilde{T}\widetilde{\psi}) \\ &\leq \nu_\alpha(\widetilde{T}\widetilde{\psi}, \widetilde{T}\psi) + \nu_\alpha(\widetilde{T}\psi, T\psi) \\ &\leq M_\alpha \nu_\alpha(\psi, \widetilde{\psi}) + \nu_\alpha(\widetilde{T}\psi, T\psi).\end{aligned} \tag{3.9}$$

Since $\alpha < \min(\gamma, \widetilde{\gamma})$, from (2.4), we see that $\nu_\alpha(\psi, \widetilde{\psi}) < \infty$ and, therefore, we can subtract $M_\alpha \nu_\alpha(\psi, \widetilde{\psi})$ from both parts of (3.9). This leads to the inequality:

$$\nu_\alpha(\psi, \widetilde{\psi}) \leq \frac{1}{1-M_\alpha}\nu_\alpha(\widetilde{T}\psi, T\psi). \tag{3.10}$$

Using definitions (3.3) and (3.4), we bound the last term in (3.10) as follows.

$$\begin{aligned}\nu_\alpha(T\psi, \widetilde{T}\psi) \leq &\sup_{u\geq 0} e^{\alpha u}\left|\frac{\lambda}{c}\left(\int_u^\infty \overline{F}(t)dt + \int_0^u \psi(u-t)\overline{F}(t)dt\right)\right. \\ &\left. - \frac{\lambda}{c}\left(\int_u^\infty \overline{\widetilde{F}}(t)dt + \int_0^u \psi(u-t)\overline{\widetilde{F}}(t)dt\right)\right| \\ &+ \sup_{u\geq 0} e^{\alpha u}\left|\left[\frac{\lambda}{c} - \frac{\widetilde{\lambda}}{c}\right]\left(\int_u^\infty \overline{\widetilde{F}}(t)dt + \int_0^u \psi(u-t)\overline{\widetilde{F}}(t)dt\right)\right| \end{aligned} \tag{3.11}$$

$$=: I_1 + I_2.$$

For the first term I_1 on the right-hand side of (3.11), we have

$$I_1 \leq \frac{\lambda}{c}\left[\sup_{u\geq 0}\left(e^{\alpha u}\int_u^\infty \left|\overline{F}(t) - \overline{\widetilde{F}}(t)\right| dt + e^{\alpha u}\int_0^u \psi(u-t)\left|\overline{F}(t) - \overline{\widetilde{F}}(t)\right| dt\right)\right]. \tag{3.12}$$

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In view of (2.4) $\psi(u-t) \le e^{-\gamma u+\gamma t}$. Therefore,

$$I_1 \le \frac{\lambda}{c}\left[\sup_{u\ge 0}\left(\int_u^\infty e^{\alpha t}\left|\overline{F}(t)-\overline{\widetilde{F}}(t)\right|dt + e^{\alpha u}e^{-\gamma u}\int_0^u e^{\gamma t}\left|\overline{F}(t)-\overline{\widetilde{F}}(t)\right|dt\right)\right]. \tag{3.13}$$

Since $\alpha < \gamma \le \beta$, (3.13) yields

$$I_1 \le \frac{\lambda}{c}\int_0^\infty e^{\beta t}\left|\overline{F}(t)-\overline{\widetilde{F}}(t)\right|dt. \tag{3.14}$$

Similarly, for the term I_2 in (3.11), we have

$$\begin{aligned} I_2 &\le \frac{|\lambda-\widetilde{\lambda}|}{c}\left[\sup_{u\ge 0}\left(e^{\alpha u}\int_u^\infty \overline{\widetilde{F}}(t)dt + e^{\alpha u}e^{-\gamma u}\int_0^u e^{\gamma t}\overline{\widetilde{F}}(t)dt\right)\right] \\ &\le \frac{|\lambda-\widetilde{\lambda}|}{c}\int_0^\infty e^{\beta t}\left[1-\overline{\widetilde{F}}(t)\right]dt. \end{aligned} \tag{3.15}$$

The last integral is equal to $\frac{1}{\beta}[\int_0^\infty e^{\beta t}d\widetilde{F}(t) - 1]$.

Combining inequalities (3.10), (3.11), (3.14) and (3.15), we obtain inequality (2.8) in Theorem 2.4.

Finally, inequality (2.10) in Proposition 2.5 is a consequence of (3.10), (3.12) and (3.13).

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APPENDIX A

Proposition A.1. *Let $\beta > 0$ and $\mathbb{K}_\beta$ be the metric in $\mathfrak{F}$ defined in* (2.6). *Then*

a. *If $\int_0^\infty e^{\beta x} dF(x) < \infty$ and $\int_0^\infty e^{\beta x} dG(x) < \infty$, then $\mathbb{K}_\beta(F, G) < \infty$.*
b. *Let $\int_0^\infty e^{\beta x} dF(x) < \infty$ and $\int_0^\infty e^{\beta x} dF_n(x) < \infty$, $n \geq 1$.*
Then $\mathbb{K}_\beta(F_n, F) \longrightarrow 0$ if and only if

$$\text{(i)}\, F_n \Rightarrow F;$$

$$\text{(ii)} \int_0^\infty e^{\beta x} dF_n(x) \longrightarrow \int_0^\infty e^{\beta x} dF(x), \quad \textit{as} \;\; n \longrightarrow \infty.$$

Proof.

a. This part follows immediately from the definition of $\mathbb{K}_\beta$.
b. Let $F_n \Rightarrow F$ and $\int_0^\infty e^{\beta x} dF_n(x) \longrightarrow \int_0^\infty e^{\beta x} dF(x)$.

For each $c > 0$,

$$\begin{aligned} \mathbb{K}_\beta(F_n, F) &= \int_0^\infty e^{\beta t} |F_n(t) - F(t)| dt \\ &\leq \int_0^c e^{\beta t} |F_n(t) - F(t)| dt + \int_c^\infty e^{\beta t} [1 - F_n(t)] dt + \int_c^\infty e^{\beta t} [1 - F(t)] dt \qquad \text{(A.1)} \\ &=: I_{1,n} + I_{2,n} + I. \end{aligned}$$

Fixing an arbitrary $\varepsilon > 0$, we choose $c > 0$ such that $I < \varepsilon$. The weak convergence $F_n \Rightarrow F$ implies that $F_n(x) \longrightarrow F(x)$ almost everywhere on $[0, \infty)$. Hence by the Dominated Convergence Theorem, there is N such that in (A.1) $I_{1,n} < \varepsilon$ for $n \geq N$.

We get that

$$\begin{aligned} \int_0^\infty e^{\beta t} dF_n(t) - \int_0^\infty e^{\beta t} dF(t) &= \beta \int_0^c e^{\beta t} [F(t) - F_n(t)] dt \\ &+ \left[\beta \int_c^\infty e^{\beta t} [1 - F_n(t)] dt - \beta \int_c^\infty e^{\beta t} [1 - F(t)] dt \right]. \qquad \text{(A.2)} \end{aligned}$$

The left-hand side of (A.2) approaches zero as $n \longrightarrow \infty$. For $n \geq N$, the absolute value of the first summand on the right-hand side of (A.2) is less than $\beta\varepsilon$.

Thus, for all $n \geq N_1 \geq N$ the absolute value of the expression in square brackets in (A.2) is less than $\varepsilon + \varepsilon\beta$. Joining all this together, and taking into account (A.1), we find that

$$\mathbb{K}(F_n, F) \longrightarrow 0 \;\; \text{as} \;\; n \longrightarrow \infty. \qquad \text{(A.3)}$$

We now assume that (A.3) holds. Since $\mathbb{K}_\beta(F_n, F) \geq \mathbb{K}(F_n, F)$ ($\mathbb{K}$ is the Kantorovich metric), $F_n \Rightarrow F$ (the convergence in the Kantorovich metric implies weak convergence, see Rachev and Rüschendorf (1998)).

Finally,

$$\left| \int_0^\infty e^{\beta t} dF_n(t) - \int_0^\infty e^{\beta t} dF(t) \right| \leq \beta \int_0^\infty e^{\beta t} |F_n(t) - F(t)| dt \longrightarrow 0,$$

as $n \longrightarrow \infty$. ∎