Euclidean Proof of Pascal's Theorem.*

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Note on the expression for the area of a triangle in Cartesian Coordinates, and a general proof of the Addition Theorem in Trigonometry connected therewith.

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In the Euclidean theory of areas, where convex polygons alone are considered, there is no question as to the sign of an area. The element of area is the rectangle, and an area is signless, or always positive.

When an extension of the notion of an area is sought which will apply to any closed polygon noncrossed or crossed, a definition is given which, naturally, must be such as to affect in no way the meaning we have for a Euclidean area. The element of area is usually a Euclidean triangle, but the convention is made that the area of a triangle ABC is to be considered positive, if on tracing out the perimeter with the vertices in that order the area lies always to the left, but otherwise negative.

The area bounded by a closed broken line ABCD...KA is then *defined* as the algebraic sum of the triangles OAB, OBC,...OKA, O being any point in the plane.

It then follows that Area ABC...KA + Area AK...CBA is zero, and that if the polygon be convex the area is positive when the vertices are taken counterclockwise, and otherwise negative. It may also be proved that the position of O is immaterial.

In the Analytical Geometry we have the corresponding problem: Given the coordinates of the vertices of a polygon taken in order, can we express the area as a function of these coordinates? If O be taken at the origin, then it is clear that we can express the area provided we can determine the area of any triangle OPQ, where P and Q are any two points (x_1, y_1), (x_2, y_2).

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We then have the theorem that the area of the triangle OPQ is $\frac{1}{2}(x_1y_2 - x_2y_1)$ in magnitude and sign.

Several of the proofs of this most important fundamental theorem in areas given in English text-books are singularly inadequate, and furnish little more than verifications in very simple cases. In continental text-books the formula is often made to depend on the trigonometrical expression of the area of a triangle in terms of its two sides and the included angle, along with the Addition Theorem; but the use of the Addition Theorem so early in the Analytical Geometry which is itself the basis of Trigonometry appears objectionable. (For a third proof due to Lucas, see Niewenglowski's *Géométrie Analytique.*)

The following demonstration is elementary, and depends on the analytical expression for the area of a rectangle of a very simple kind. Moreover, the language employed applies in all possible cases.

Theorem I. If P be any point (x, y), M and N its projections on the x-axis and y-axis respectively, then the area of the rectangle OMPN is equal to the product xy in magnitude and sign.

This result is easily verified, and is due to the positions of the axes relative to each other.

Cor. Hence $\quad \text{ONPM} = -\text{OMPN} = -xy.$

Theorem II. If P_1, P_2 are the points (x_1, y_1), (x_2, y_2) respectively, then the triangle $OP_1P_2 = \frac{1}{2}(x_1y_2 - x_2y_1)$ in magnitude and sign.

Construct the rectangle P_1RP_2S whose diagonal is P_1P_2, and whose axes are parallel to the axes of reference. Join RO, (Fig. 1). Then, by definition, with, say, R as origin of triangles of area,

$$\begin{aligned}
\triangle OP_1P_2 &= \triangle ROP_1 + \triangle RP_1P_2 + \triangle RP_2O \\
&= \tfrac{1}{2}[RN_2N_1P_1 + RP_1SP_2 + RP_2M_2M_1] \\
&= \tfrac{1}{2}[(ON_1P_1M_1 + OM_1RN_2) \\
&\qquad + (OM_1P_1N_1 + ON_1SM_2 + OM_2P_2N_2 + ON_2RM_1) \\
&\qquad + (OM_1RN_2 + ON_2P_2M_2)] \\
&= \tfrac{1}{2}(ON_1SM_2 + OM_1RN_2), \quad \because\ ON_1P_1M_1 + OM_1P_1N_1 = 0, \text{ etc.} \\
&= \tfrac{1}{2}(-\overline{ON_1}\,.\,\overline{OM_2} + \overline{OM_1}\,.\,\overline{ON_2}) \\
&= \tfrac{1}{2}(-x_2y_1 + x_1y_2).
\end{aligned}$$

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Cor. 1. From this formula follows immediately an expression for the area of any closed polygon $P_1P_2P_3\ldots$ in terms of the coordinates of the vertices, viz., $\Sigma OP_1P_2 = \frac{1}{2}\Sigma(x_1y_2 - x_2y_1)$.

Cor. 2. The application to trigonometry is now pretty obvious.

If P_1 be any point in the part OX of the x-axis and P_2 any other point in the plane (Fig. 2), then the area of the triangle OP_1P_2 is equal to $\frac{1}{2} \cdot \overline{OP_1} \cdot \overline{M_2P_2}$ in magnitude and sign.

Hence if $OP_1 = r_1$, $OP_2 = r_2$, and $P_1\widehat{O}P_2 = \theta$, we deduce $M_2P_2 = r_2\sin\theta$ in magnitude and sign, and $\therefore$ $\triangle OP_1P_2 = \frac{1}{2}r_1r_2\sin\theta$. This result can not be affected by any rotation of the figure round O through an angle θ_1. If the Cartesian coordinates of P_1 and P_2 then become (x_1y_1), (x_2y_2), while their polar coordinates, with OX for initial direction, are (r_1, θ_1), (r_2, θ_2), then $\triangle OP_1P_2 = \frac{1}{2}r_1r_2\sin(\theta_2 - \theta_1)$ in magnitude and sign.

Hence $$x_1y_2 - x_2y_1 = r_1r_2\sin(\theta_2 - \theta_1).$$

Replace the Cartesian coordinates by their equivalents in polars and we obtain

$$r_1\cos\theta_1 \cdot r_2\sin\theta_2 - r_2\cos\theta_2 \cdot r_1\sin\theta_1 = r_1r_2\sin(\theta_2 - \theta_1),$$

or, removing the common factors r_1 and r_2, we deduce the trigonometrical identity

$$\sin(\theta_2 - \theta_1) = \sin\theta_2\cos\theta_1 - \cos\theta_2\sin\theta_1.$$

Since there is no restriction in the θ's this formula is general, and might be utilised as a base for the addition theorem in trigonometry. Such a process would be as natural as the inverse process which is usually adopted.

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