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ON THE BEHAVIOUR OF FUNCTIONS WITH FINITE WEIGHTED DIRICHLET INTEGRAL NEAR THE BOUNDARY

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1. Introduction

L. Carleson ([6]) proved the following theorem:

Let u be a finite continuous function in the unit open ball B with center zero in the complex plane. If u satisfies the following condition:

$$\int_B |\operatorname{grad} u|^2 (1-|z|)^\alpha dxdy < +\infty \qquad (z = x+iy,\ 0 \leq \alpha < 1)\,,$$

then the radial limit $\lim_{r\to 1} u(re^{i\theta})$ $(\theta \in \partial B)$ exists on ∂B except for a set of C_α-capacity zero, where the C_α-capacity is the capacity on the real line with respect to the kernel of order $\alpha, r^{-\alpha}$. This is generalization of Beurling's theorem ([2]). The above theorem is proved by using the Fourier series and hence the original proof cannot be immediately applied to the same problem on the higher dimensional Euclidean space.

Let $\boldsymbol{R}^p$ denote the p-dimensional Euclidean space. The elements of $\boldsymbol{R}^p$ are denoted by $x=(x_1,\cdots,x_p)$, $s=(s_1,\cdots,s_p)\cdots$ etc. The distance between x and 0 is denoted by $|x|$. Let $\boldsymbol{R}_+^{p+1}$ denote the upper half space of $\boldsymbol{R}^{p+1}$. In particular, the elements of $\boldsymbol{R}_+^{p+1}$ are denoted by $(x;y)$, $(s;t)$ $\cdots$ etc, where $x, s \in \boldsymbol{R}^p$ and $0 < y, t < +\infty$. We may consider $\boldsymbol{R}^p$ as the boundary of $\boldsymbol{R}_+^{p+1}$ by the ordinary embedding. The $C_\alpha^{(p)}$-capacity is the capacity on $\boldsymbol{R}^p$ with respect to the kernel of order $\alpha, r^{\alpha-p}$. We shall prove the following

THEOREM 1. *Let $p \geq 2$, $0 \leq \alpha < 1$ and u be a locally integrable function in $\boldsymbol{R}_+^{p+1}$. If u satisfies the following condition*:

$$\iint_{\boldsymbol{R}_+^{p+1}} |\operatorname{grad} u|^2 y^\alpha dxdy < +\infty\,, \tag{1}$$

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where partial derivatives of u are in the sense of distributions, then there exists a locally integrable function v in $\boldsymbol{R}_+^{p+1}$ such that $u = v$ almost everywhere (a.e.) in $\boldsymbol{R}_+^{p+1}$ and $\lim_{r\to 0} v(x\,;r)$ exists on R^p except for a set of $C_{1-\alpha}^{(p)}$-capacity zero.

Applying this theorem to a locally integrable function u in the open unite ball B_p with center 0 in R^p ($p \geq 3$) satisfying analogous condition to (1), we shall obtain a generalized form of the Carleson's theorem in R^p.

Next, we shall examine the behaviour of harmonic functions satisfying (1) near the boundary. We introduce a more extended conception than the non-tangential limit. For $\gamma \geq 1$, $m > 0$ and $s \in R^p$, define

$$R(m, s, \gamma) = \{(x\,; y)\,; |s - x|^\gamma < my\}\,.$$

We say that a function $f(x\,; y)$ on $\boldsymbol{R}_+^{p+1}$ has a $T(\gamma)$-limit L at s provided with

$$\lim_{\substack{(x;y)\to(s;0)\\(x;y)\in R(m,s,\gamma)}} f(x\,; y) = L$$

for any $m > 0$. We shall show the following

THEOREM 2. *Let $p \geq 2$, $-1 < \alpha < 1$ and $0 < \beta \leq 1 - \alpha$. If a harmonic function h satisfies* (1), *then h has $T\left(\dfrac{p - (\beta/2)}{p - (1-\alpha)/2}\right)$-limits on R^p except for a set of $C_\beta^{(p)}$-capacity zero.*

Finally, we shall deal with the rectangular limit to the boundary of harmonic functions on $\boldsymbol{R}_+^{p+1}$. Let

$$\Gamma_m(s\,; t) = R(m, s, 1) \cap \{(x\,; y)\,; y < t\}\,.$$

For $-1 < \alpha \leq 1$ and a harmonic function h in $\boldsymbol{R}_+^{p+1}$, define

$$S_\alpha(s\,; t) = \iint_{\Gamma_1(s;t)} |\mathrm{grad}\, h(x\,; y)|^2 \frac{1}{y^{p-\alpha}} dxdy\,.$$

The third main theorem is the following

THEOREM 3. *Let $p \geq 2$, $-1 < \alpha < 1$, E be a measurable set on $\boldsymbol{R}^p$ and h harmonic in $\boldsymbol{R}_+^{p+1}$. If $S_\alpha(s\,; 1) < +\infty$ for any $s \in E$, then $\lim_{r\to 0} h(s\,; r)$ exists on E except for a set of $C_{1-\alpha}^{(p)}$-capacity zero.*

This theorem is suggested by the following theorem which was obtained by A. P. Carderón and E. M. Stein (See [4], [11], [12]).

Let $p \geq 2$, E be a measurable set on $\boldsymbol{R}^p$ and h harmonic in $\boldsymbol{R}_+^{p+1}$. Then the following three conditions are equivalent.

(i) h has a non-tangential limit for almost all point $s \in E$.

(ii) h is non-tangentially bounded for *a.a.* point $s \in E$.

(iii) $S_1(s; 1) < +\infty$ for *a.a.* point $s \in E$.

2. The proof of Theorem 1.

For the proof of Theorem 1, we prepare the following three lemmas.

LEMMA 1. *Let* $0 \leq \alpha < 1$. *Then there exists a constant* c_1 *such that*

$$\int_0^\infty \frac{1}{y^\alpha} dy \int_{R^p} \frac{1}{\sqrt{|s_1 - x|^2 + (t_1 - y)^2}^p \sqrt{|s_2 - x|^2 + (t_2 - y)^2}^p} dx$$
$$\leq c_1 \frac{1}{|s_1 - s_2|^{p-1+\alpha}}$$

for any $s_1, s_2 \in \boldsymbol{R}^p$ *and* $0 < t_1, t_2 < +\infty$.

The method of the proof is the same as in the case of $p = 1$, so we omit the proof. See P. 56 in [6].

LEMMA 2. (i) *Let* $0 \leq \alpha < 1$. *Then there exists a positive constant* c_2 *such that*

$$\int_\tau^\rho (\xi^2 |a(t)|^2 + |a'(t)|^2) t^\alpha dt \geq c_2 \xi^{1-\alpha} |a(\rho) - a(\tau)|^2$$

for any $\xi, r > 0$, $0 < \tau < \rho < r$ *and any finite continuous function* $a(t)$ *of BL-type in the open interval* $(0, r)$.

(ii) *Let* $-1 < \alpha < 1$. *Then there exists a positive constant* c_3 *such that*

$$\int_0^\rho (\xi^2 |a(t)|^2 + |a'(t)|^2) t^\alpha dt \geq c_3 \xi^{1-\alpha} |a(\rho) - a(0)|^2$$

for any $\xi, r > 0$, $0 < \rho < r$ *and any finite continuous function* $a(t)$ *in* $[0, 1)$ *of BL-type in* $(0, r)$.

The proof is analogous to in Lemma 5 of [6], so we omit the proof.

For a measure μ on $\boldsymbol{R}^p$ $(p \geq 2)$, the potential of μ of order α $(0 < \alpha < p)$ is defined by

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$$U_\alpha^\mu(x) = A(p,\alpha)\int \frac{1}{|x-s|^{p-\alpha}}d\mu(s) = k_\alpha * \mu(x) ,$$

where $A(p,\alpha) = \pi^{\alpha-p/2}\dfrac{\Gamma((p-\alpha)/2)}{\Gamma(\alpha/2)}$ and $k_\alpha = A(p,\alpha)r^{\alpha-p}$. Let m_ε be the unit measure on $\boldsymbol{R}^p$ uniformly distributed in the open ball $B(0,\varepsilon)$ with center 0 and radius ε. The following proposition is well-known ([9]).

PROPOSITION. *Let $p \geq 2$ and $0 < \alpha < p$. Then there exists a constant c_4 depending on p and α such that for any potential U_α^μ,*

$$U_\alpha^\mu * m_\varepsilon(x) \leq c_4 U_\alpha^\mu(x) .$$

Let μ be a measure on R^p. Set

$$h_\mu(x;y) = \frac{2}{\sigma_{p+1}}\int \frac{y}{(|x-s|^2+y^2)^{(p+1)/2}}d\mu(s)$$

in $\boldsymbol{R}_+^{p+1}$ provided the right integral is defined, where σ_{p+1} is the surface area of the unit sphere in $\boldsymbol{R}^{p+1}$. By the properties of maximal functions ([13]) and the preceding proposition, we obtain

$$h_\mu(x;y) \leq \sup_{\varepsilon>0} U_\alpha^\mu * m_\varepsilon(x) \leq c_4 U_\alpha^\mu(x) \tag{2}$$

for any potential U_α^μ.

LEMMA 3. *Let u be a temperate distribution in $\boldsymbol{R}^p$ $(p \geq 2)$. Suppose that the Fourier transformation $\tilde{u}$ of u is a function such that*

$$\int |\xi|^\alpha |\tilde{u}(\xi)|^2 d\xi < +\infty \qquad (0 < \alpha < p) .$$

Then u is a signed measure on $\boldsymbol{R}^p$, and $h_u(x;y)$ is defined and harmonic in $\boldsymbol{R}_+^{p+1}$. $\lim_{r\to 0} h_u(x;r)$ exists except for a set of $C_\alpha^{(p)}$-capacity zero.

Proof. By [1], we can set $u = U_{\alpha/2}^\mu$ in the sense of distributions, where μ is a square integrable function in $\boldsymbol{R}^p$. The potential $U_{\alpha/2}^{|\mu|}$ is not identically infinite and hence it is locally integrable. Consequently, $u = U_{\alpha/2}^\mu$ in the sense of measures. By (2), $h_u = h_{U_{\alpha/2}^\mu}$ is defined and harmonic in $\boldsymbol{R}_+^{p+1}$. Since $C_\alpha^{(p)}(\{x \in \boldsymbol{R}^p ; U_{\alpha/2}^{|\mu|} = +\infty\}) = 0$, it is sufficient to show that

$$\lim_{r\to 0} h_{U_{\alpha/2}^{\mu+}}(x;r) = U_{\alpha/2}^{\mu+}(x)$$

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everywhere. This holds evidently at any point $x \in \boldsymbol{R}^p$ where $U_{\alpha/2}^{\mu^+}(x) = +\infty$. Suppose $U_{\alpha/2}^{\mu^+}(x) < +\infty$. For any $\eta > 0$, there exists a $\delta > 0$ such that $U_{\alpha/2}^{\mu_1}(x) < \eta$, where μ_1 is the restriction of μ^+ to the open ball $B(x, \delta)$. Put $\mu_0 = \mu^+ - \mu_1$. By (2),

$$\begin{aligned}\overline{\lim_{r\to 0}}\, |h_{U_{\alpha/2}^{\mu^+}}(x; r) - U_{\alpha/2}^{\mu^+}(x)| &\leq \overline{\lim_{r\to 0}}\, |h_{U_{\alpha/2}^{\mu_0}}(x; r) - U_{\alpha/2}^{\mu_0}(x)| \\ &+ \overline{\lim_{r\to 0}}\, h_{U_{\alpha/2}^{\mu_1}}(x; r) + U_{\alpha/2}^{\mu_1}(x) \leq \overline{\lim_{r\to 0}}\, |h_{U_{\alpha/2}^{\mu_0}}(x; r) - U_{\alpha/2}^{\mu_0}(x)| \\ &+ \eta(c_4 + 1) .\end{aligned}$$

Since $U_{\alpha/2}^{\mu_0}$ is finite continuous in $B(x, \delta)$,

$$\lim_{r\to 0} h_{U_{\alpha/2}^{\mu_0}}(x; r) = U_{\alpha/2}^{\mu_0}(x) .$$

Let η tend to zero. We obtain

$$\lim_{r\to 0} h_{U_{\alpha/2}^{\mu^+}}(x; r) = U_{\alpha/2}^{\mu^+}(x) .$$

This completes the proof.

We remark on some transformation. We introduce an infinitely differentiable function on $\boldsymbol{R}^1$ such that $f(t) = 0$ on $t \leq 1$, $f(t) = 1$ on $t \geq 2$, $0 < f(t) < 1$ on $1 < t < 2$ and $0 \leq f(t) \leq 1$ on $\boldsymbol{R}^1$. Consider the domains

$$D = \{(x; y) ; |x| < 4 \quad \text{and} \quad f(|x|) < y < f(|x|) + 2\}$$

and

$$M = D \cap \{(x; y) ; |x| > 3 \quad \text{or} \quad y > f(|x|) + 1\}$$

Define a mapping Φ from D to $\boldsymbol{R}^{p+1}$ by

$$\Phi(x; y) = (x; y - f(|x|)) .$$

Set

$$U = \{(x; y) ; |x| < 4 \quad \text{and} \quad 0 < y < 2\}$$

and

$$T = U \cap \{(x; y) ; |x| > 3 \quad \text{or} \quad y > 1\} .$$

It is obvious that $\Phi(D) = U$ and $\Phi(M) = T$. Set $u_1(x; y) = u(x; y + f(|x|))$ in U, where u is the function in Theorem. Then u_1 is a locally integrable function in U such that

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$$\iint_U |\text{grad}\, u_1(x\,;\,y)|^2\, y^\alpha dxdy < +\infty\ .$$

Since

$$\int_{1/2}^{\infty} dy \int |\text{grad}\, u(x\,;\,y)|^2\, dx < +\infty\ ,$$

we can assume that u is bounded on M. (See Theorem 4 (p. 125) in [7]) Hence u_1 is bounded on T. Let u_2 be a function in $\boldsymbol{R}_+^{p+1}$ such that $u_2 = u_1$ in $U - \bar{T}$, $u_2 = 0$ on $\boldsymbol{R}_+^{p+1} - U$ and

$$\iint_{\boldsymbol{R}_+^{p+1}} |\text{grad}\, u_2(x\,;\,y)|^2\, y^\alpha dxdy < +\infty\ .$$

Then $u(x\,;\,y) = u_2(x\,;\,y)$ in $|x| < 1$ and $0 < y < 1$. Hence $\lim_{r\to 0} u(x\,;\,r) = \lim_{r\to 0} u_2(x\,;\,r)$ in $|x| < 1$ provided one of the two limits exist. If, for u_2, there exists our desired function v', we obtain obviously, for u, the function v in Theorem. Consequently, we can assume that u is supported by $|x| \leq 1$ and $0 < y \leq 1$.

There exists a sequence $(u_n)_{n=1}^{\infty}$ of continuously differentiable functions in $\boldsymbol{R}_+^{p+1}$ such that $u_n = 0$ on $|x| \geq 3/2$ or $y \geq 3/2$ for all n, $u_n \to u$ as $n \to \infty$ a.e. in $\boldsymbol{R}_+^{p+1}$ and

$$\iint_{D,y>\eta} |\text{grad}\,(u_n - u)|^2\, dxdy = \varepsilon_n(\eta) \to 0 \qquad \text{as } n \to \infty$$

for each $\eta > 0$. There exists a sequence $(\eta_k)_{k=1}^{\infty}$ such that $\eta_k < 2$, $\eta_k \downarrow 0$ and

$$\int |u_n(x\,;\,\eta_k) - u(x\,;\,\eta_k)|\, dx \to 0 \qquad \text{as } n \to \infty$$

for each k. (Choose a subsequence of $(u_n)_{n=1}^{\infty}$, if necessary.) Put $D_k = \{(x\,;\,y)\,;\,|x| < 2, \eta_k < y < 2\}$. Applying the Green's formula to u_n and

$$g_{(s;t)}(x\,;\,y) = \frac{1}{\sqrt{|s-x|^2 + (t-y)^2}^{\,p-1}}$$

in D_k, we obtain

$$\begin{aligned} u_n(s\,;\,t) = {} & \frac{1}{\sigma_{p+1}(p-1)} \int_{\partial D_k} v_n \frac{\partial}{\partial n} g_{(s;t)} d\sigma(x\,;\,y) \\ & - \frac{1}{\sigma_{p+1}(p-1)} \iint_{D_k} (\text{grad}\, u_n, \text{grad}\, g_{(s;t)}) dxdy \end{aligned}$$

for $(s;t)\in D_k$, where $\partial/\partial n$ is the outer normal derivative on ∂D_k and $d\sigma$ the surface element on ∂D_k. Since $u_n = 0$ on $|x| \geq 3/2$ or $y \geq 3/2$,

$$\begin{aligned} u_n(s;t) = &\frac{2}{\sigma_{p+1}}\int u_n(x;\eta_k)\frac{t-\eta_k}{\sqrt{|s-x|^2+(t-\eta_k)^2}^{\,p+1}}dx \\ &-\frac{1}{\sigma_{p+1}(p-1)}\iint_{D_k}(\mathrm{grad}\, u_n, \mathrm{grad}\, g_{(s;t)})dxdy \end{aligned} \tag{3}$$

For a vanishing sequence $(a_n(\eta_k))_{n=1}^{\infty}$ of positive numbers such that

$$\sum_{n=1}^{\infty}\frac{\varepsilon_n(\eta_k)}{a_n(\eta_k)^2} < +\infty\ , \tag{4}$$

Put

$$A_{n,k} = \left\{(s;t)\in \boldsymbol{R}_+^{p+1};\ \iint_{D_k}|(\mathrm{grad}\,(u_n-u), \mathrm{grad}\, g_{(s;t)})|\,dxdy > a_n(\eta_k)\right\}.$$

Then

$$C_2^{(p+1)}(A_{n,k}) \leq c\frac{1}{a_n(\eta_k)^2}\iint_{D_k}|\mathrm{grad}\,(u_n-u)|^2\,dxdy = c\frac{\varepsilon_n(\eta_k)}{a_n(\eta_k)^2}\,.$$

By (4),

$$\iint_{D_k}|(\mathrm{grad}\,(u_n-u), \mathrm{grad}\, g_{(s;t)})|\,dxdy \to 0 \qquad \text{as } n\to\infty$$

in $\boldsymbol{R}_+^{p+1}$ except for a set of $C_2^{(p+1)}$-capacity zero. Let $n\to\infty$ in (3).

$$\begin{aligned} u(s;t) = &\frac{2}{\sigma_{p+1}}\int u(x;\eta_k)\frac{t-\eta_k}{\sqrt{|s-x|^2+(t-\eta_k)^2}^{\,p+1}}dx \\ &-\frac{1}{\sigma_{p+1}(p-1)}\iint_{D_k}(\mathrm{grad}\, u, \mathrm{grad}\, g_{(s;t)})dxdy \end{aligned} \tag{5}$$

a.e. in D_{η_k}. Put $u_{n,\eta}(x) = u_n(x;\eta), u_\eta(x) = u(x;\eta)$ and

$$\begin{aligned} L &= \int_{\eta_k}^{2} y^\alpha dy\int|\mathrm{grad}\, u_n|^2\,dx \\ &= \int_{\eta_k}^{2} y^\alpha dy\int\left(|\mathrm{grad}\, u_{n,y}|^2 + \left|\frac{\partial u_n}{\partial y}(x;y)\right|^2\right)dx\,. \end{aligned}$$

Since

$$\int|\mathrm{grad}\, u_{n,y}|^2\,dx = \int|\xi|^2\,|\tilde{u}_{n,y}(\xi)|^2\,d\xi$$

and

$$\int\left|\frac{\partial u_n}{\partial y}(x\,;\,y)\right|^2 dx = \int\left|\frac{\partial \tilde{u}_{n,y}}{\partial y}(\xi)\right|^2 d\xi\ ,$$

we have

$$\begin{aligned} L &= \int_{\eta_k}^{2} y^\alpha dy \int\left(|\xi|^2\,|\tilde{u}_{n,y}(\xi)|^2 + \left|\frac{\partial \tilde{u}_{n,y}}{\partial y}(\xi)\right|^2\right)d\xi \\ &= \int d\xi \int_{\eta_k}^{2}\left(|\xi|^2\,|\tilde{u}_{n,y}(\xi)|^2 + \left|\frac{\partial \tilde{u}_{n,y}}{\partial y}(\xi)\right|^2\right)y^\alpha dy\,. \end{aligned}$$

By Lemma 2 (i),

$$L \geq c\int|\xi|^{1-\alpha}\,|\tilde{u}_{n,\eta_k}(\xi)|^2\,d\xi\ .$$

By the same argument,

$$\int_{\eta_k}^{2} y^\alpha dy\,|\mathrm{grad}\,(u_n - u_m)|^2\,dx \geq c\int|\xi|^{1-\alpha}\,|\tilde{u}_{n,\eta_k} - \tilde{u}_{m,\eta_k}|^2\,d\xi\ .$$

Since

$$\int_{\eta_k}^{2} y^\alpha dy\,|\mathrm{grad}\,(u_n - u_m)|^2\,dx \to 0 \qquad \text{as } n \to \infty,\ m \to \infty\ ,$$

there exists a locally integrable function u'_{η_k} in $\boldsymbol{R}^p$ such that

$$\int|\xi|^{1-\alpha}\,|\tilde{u}_{n,\eta_k} - \tilde{u}'_{\eta_k}|^2\,d\xi \to 0 \qquad \text{as } n \to \infty\ .$$

We have $u_{\eta_k} = u'_{\eta_k}$ a.e. in $\boldsymbol{R}^p$, because

$$\int|u_{n,\eta_k} - u_{\eta_k}|\,dx \to 0 \qquad \text{as } n \to \infty\ .$$

We obtain

$$(6)\qquad \int_{\eta_k}^{\eta_\ell} y^\alpha dy\,|\mathrm{grad}\,u_n|^2\,dx \geq c\int|\xi|^{1-\alpha}\,|\tilde{u}_{n,\eta_k} - \tilde{u}_{n,\eta_\ell}|^2\,d\xi \qquad (\ell < k)\ .$$

Let $n \to \infty$ in (6). We have

$$\int_{\eta_k}^{\eta_\ell} y^\alpha dy \int|\mathrm{grad}\,u|^2\,dx \geq c\int|\xi|^{1-\alpha}\,|\tilde{u}_{\eta_k} - \tilde{u}_{\eta_\ell}|^2\,d\xi \qquad (\ell < k)\ .$$

Since

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$$\int_{\eta_k}^{\eta_\ell} y^\alpha dy \int |\mathrm{grad}\, u|^2 \, dx \to 0 \qquad \text{as } \ell \to \infty,\ k \to \infty\ ,$$

there exists a locally integrable function u_0 in $\boldsymbol{R}^p$ such that

$$\int |\xi|^{1-\alpha} |\tilde{u}_{\eta_k} - \tilde{u}_0|^2 \, d\xi \to 0 \qquad \text{as } k \to \infty\ .$$

By the elementary formula

$$\frac{2}{\sigma_{p+1}} \int e^{-2\pi i \xi x} \frac{t - \eta_k}{\sqrt{|s - x|^2 + (t - \eta_k)^2}^{\,p+1}} dx \\ = e^{2\pi i \xi s} \cdot e^{-2\pi |\xi| (t - \eta_k)}$$

for $t > \eta_k$, we have

$$\left| \frac{2}{\sigma_{p+1}} \int u_{\eta_k} \frac{t - \eta_k}{\sqrt{|s - x|^2 + (t - \eta_k)^2}^{\,p+1}} dx - \frac{2}{\sigma_{p+1}} \int u_0 \frac{t}{\sqrt{|s - t|^2 + t^2}^{\,p+1}} dx \right| \\ = \left| \int \tilde{u}_{\eta_k} e^{-2\pi i \xi s} \cdot e^{-2\pi |\xi| (t - \eta_k)} d\xi - \int \tilde{u}_0 e^{-2\pi i \xi s} \cdot e^{-2\pi |\xi| t} d\xi \right| \\ \leq \int |\tilde{u}_{\eta_k}| \, |e^{-\pi |\xi| (t - \eta_k)} - e^{-2\pi |\xi| t}| \, d\xi + \int |\tilde{u}_{\eta_k} - \tilde{u}_0| \, e^{-2\pi |\xi| t} d\xi\ .$$

$$\left(\int |\tilde{u}_{\eta_k}| \, |e^{-2\pi |\xi| (t - \eta_k)} - e^{-2\pi |\xi| t}| \, d\xi \right)^2 \\ \leq \int |\xi|^{1-\alpha} |\tilde{u}_{\eta_k}|^2 \, d\xi \int \frac{1}{|\xi|^{1-\alpha}} e^{-4\pi |\xi| t} (e^{2\pi |\xi| \eta_k} - 1)^2 d\xi \to 0$$

as $k \to \infty$ and

$$\left(\int |\tilde{u}_{\eta_k} - \tilde{u}_0| \, e^{-2\pi |\xi| t} d\xi \right)^2 \\ \leq \int |\xi|^{1-\alpha} |\tilde{u}_{\eta_k} - \tilde{u}_0|^2 \, d\xi \cdot \int \frac{1}{|\xi|^{1-\alpha}} e^{-4 |\xi| t} d\xi \to 0$$

as $k \to \infty$, and hence

$$\frac{2}{\sigma_{p+1}} \int u_{\eta_k} \frac{t - \eta_k}{\sqrt{|s - x|^2 + (t - \eta_k)^2}^{\,p+1}} dx \\ + \frac{2}{\sigma_{p+1}} \int u_0 \frac{t}{\sqrt{|s - x|^2 + t^2}^{\,p+1}} dx$$

$(= h_{u_0}(s\,;t))$ as $k \to \infty$. Letting $k \to \infty$ in (5),

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$$u(s\,;t)=h_{u_0}(s\,;t)$$

(7)
$$-\lim_{k\to\infty}\frac{1}{\sigma_{p+1}(p-1)}\iint_{D_{\eta_k}}(\operatorname{grad}u,\operatorname{grad}g_{(s;t)})dxdy$$

a.e. in $|s|<2$ and $0<t<2$. Given $b>0$, put

$$E=\left\{s\in \boldsymbol{R}^p\,;\iint_{\boldsymbol{R}_+^{p+1}}|(\operatorname{grad}u,\operatorname{grad}g_{(s;t)})|\,dxdy>b\right\}.$$

For a compact set K in E, there exists a measurable function $\rho(s)$ on $\boldsymbol{R}^p$ satisfying

$$\iint|(\operatorname{grad}u,\operatorname{grad}g_{(s;\rho(s))})|\,dxdy>b$$

on K. By Lemma 1, we have, with constants c and c'

$$b^2<\iint d\mu_k(s)\left(\iint_{\boldsymbol{R}_+^{p+1}}|(\operatorname{grad}u,\operatorname{grad}g_{(s;\rho(s))}|\,dxdy\right)^2$$
$$\leq c\iint_{\boldsymbol{R}_+^{p+1}}|\operatorname{grad}u|^2\,y^\alpha dxdy\iint d\mu_k(s_1)d\mu_k(s_2)$$
$$\times\frac{1}{y^\alpha\sqrt{|s_1-x|^2+(\rho(s_1)-y)^2}^{\,p}\sqrt{|s_2-x|^2+(\rho(s^2)-y)^2}^{\,p}}dxdy$$
$$\leq c'\iint_{\boldsymbol{R}_+^{p+1}}|\operatorname{grad}u|^2\,y^\alpha dxdy\iint\frac{1}{|s_1-s_2|^{p-1+\alpha}}d\mu_k(s_1)d\mu_k(s_2)$$

where μ_k is the equilibrium measure on K of unit mass. Therefore

$$C_{1-\alpha}^{(p)}(K)\leq\frac{c'}{b^2}\iint_{\boldsymbol{R}_+^{p+1}}|\operatorname{grad}u|^2\,y^\alpha dxdy$$

and so the same inequality holds for E. Consequently,

$$C_{1-\alpha}^{(p)}\left(\left\{s\in R^p\,;\iint_{\boldsymbol{R}_+^{p+1}}|(\operatorname{grad}u,\operatorname{grad}g_{(s;t)})|\,dxdy=+\infty\right\}\right)=0$$

and hence

(8)
$$\lim_{k\to\infty}\int_{D_k}(\operatorname{grad}u,\operatorname{grad}g_{(s;t)})dxdy$$
$$=\iint(\operatorname{grad}u,\operatorname{grad}g_{(s;t)})dxdy$$

for s in $\boldsymbol{R}^p$ except for a set of $C_{1-\alpha}^{(p)}$-capacity zero. Hence the equality (8) holds for a.e. in $\boldsymbol{R}_+^{p+1}$. We obtain

$$u(s\,;\,t) = h_{u_0}(s\,;\,t) - \frac{1}{\sigma_{p+1}(p-1)} \iint (\operatorname{grad} u, \operatorname{grad} g_{(s;t)}) dxdy$$

a.e. in $|s| < 2$ and $0 < t < 2$. We have, in the same manner as the above, that

$$\lim_{r \to 0} \iint (\operatorname{grad} u, \operatorname{grad} g_{(s;t)}) dxdy$$

exists in $\boldsymbol{R}^p$ except for a set of $C_{1-\alpha}^{(p)}$-capacity zero. Put

$$v(s\,;\,t) = h_{u_0}(s\,;\,t) - \frac{1}{\sigma_{p+1}(p-1)} \iint (\operatorname{grad} u, \operatorname{grad} g_{(s;t)}) dxdy$$

in $|s| < 2$ and $0 < t < 2$, $v(s\,;\,t) = 0$ otherwise. Then $v = u$ a.e. in $\boldsymbol{R}_+^{p+1}$ and $\lim_{r \to 0} v(s\,;\,r)$ exists except for a set of $C_{1-\alpha}^{(p)}$-capacity zero. This completes the proof.

In particular, if u is finite continuous in $\boldsymbol{R}_+^{p+1}$, we can evidently choose $v = u$.

COROLLARY. *Let u be a locally integrable function in the open unit ball B_p with center 0 in $\boldsymbol{R}^p$ $(p \geq 3)$. If u satisfies the following condition:*

$$\iint_{B_p} |\operatorname{grad} u|^2 (1 - |x|)^\alpha dx < +\infty \qquad (0 \leq \alpha < 1)\,,$$

then there exists a locally integrable function v in B_p such that $u = v$ a.e. and $\lim_{r \to 1} v(r\xi)$ $(\xi \in \partial B_p)$ exists on ∂B_p except for a set of $C_{1-\alpha}^{(p-1)}$-capacity zero.

Proof. Similarly we may assume that u is supported by $\{x \in B_p\,;\, |x| > \frac{1}{2}, x_p > \frac{1}{2}\}$. Then by a suitable transformation from $\{x \in B_p\,;\, x_p > 0\}$ to $\boldsymbol{R}_+^p$, u is mapped to a function u' of the class in our theorem and hence this corollary is immediately followed.

3. The proof of Theorem 2.

We prepare five lemmas. In the following lemmas, we only consider the case of $p \geq 2$ except for in Lemma 8.

LEMMA 4. *For $\gamma \geq 1$ and $m > 0$, there exists a positive constant c_1 such that*

$$\sqrt{|s - x|^2 + y^2} \geq c_1 |s - s_0|^\gamma$$

for any $s, s_0 \in \boldsymbol{R}^p$ *with* $|s - s_0| \leq 1$ *and any* $(x; y) \in R(m, s_0, \gamma)$.

LEMMA 5. *For* $s, s_0 \in \boldsymbol{R}^p$ *and* $0 < \alpha < p$, *put* $u_{s_0}(s) = \dfrac{1}{|s - s_0|^{p-\alpha}}$. *There exists a constant* c_2 *depending on* p *and* α *such that*

$$h_{u_{s_0}}(x; y) \leq c_2 \frac{1}{(|s_0 - x|^2 + y^2)^{(p-\alpha)/2}}$$

for any $s_0 \in \boldsymbol{R}^p$ *and any* $(x; y) \in \boldsymbol{R}_+^{p+1}$.

Proof. Put

$$v_{s_0}(x; y) = \frac{1}{(|s_0 - x|^2 + y^2)^{(p-\alpha)/2}}\ .$$

It is sufficient to show that there exists a constant c_2 such that for any $(x; y) \in \boldsymbol{R}_+^{p+1}, h_{u_0}(x; y) \leq c_2 v_0(x; y)$. For $0 \leq m < +\infty$, put $\Gamma_m = \{(x; y);\ |x| = my\ y > 0\}$. For $m \in [0, 2]$ and $(x; y) \in \Gamma_m$, we obtain, with constants c_1' and c_2',

$$\begin{aligned} h_{u_0}(x; y) &= \frac{2}{\sigma_{p+1}} \int \frac{y}{(|s - x|^2 + y^2)^{(p+1)/2}} \frac{1}{|s|^{p-\alpha}} ds \\ &= \int e^{-2\pi i \xi x} \cdot e^{-2\pi |\xi| y} \cdot \frac{1}{|\xi|^{\alpha}} d\xi \\ &\leq c_1' \frac{1}{y^{p-\alpha}} \leq c_2' v_0(x; y)\ . \end{aligned}$$

For $m \in (2, \infty)$ and $(x; y) \in \Gamma_m$, put $x_0 = x/|x|$. We have, with constants c_3' and c_4',

$$\begin{aligned} h_{u_0}(x; y) &= \frac{2}{\sigma_{p+1}} \int \frac{y}{(|s - x|^2 + y^2)^{(p+1)/2}} \frac{1}{|s|^{p-\alpha}} ds \\ &= \frac{2}{\sigma_{p+1}} \frac{1}{|x|^{p-\alpha}} \int \frac{(1/m)}{(|s|^2 + (1/m)^2)^{(p+1)/2}} \frac{1}{|s - x_0|^{p-\alpha}} ds \\ &\leq c_3' \frac{1}{|x|^{p-\alpha}} \leq c_4' v_0(x; y)\ . \end{aligned}$$

Put $c_2 = \max(c_2', c_4')$. We obtain $h_{u_0}(x; y) \leq c_2 v_0(x; y)$ for any $(x; y) \in \boldsymbol{R}_+^{p+1}$. This completes the proof.

Let $L_{\text{loc}}^1(\boldsymbol{R}^p)$ be the usual Fréchet space of locally integrable functions in $\boldsymbol{R}^p$. We denote

$$P_\alpha^{(p)} = \left\{\omega \in L^1_{\mathrm{loc}}(\boldsymbol{R}^p)\,;\, \int |\xi|^\alpha |\tilde{\omega}(\xi)|^2\, d\xi < +\infty\right\} \qquad (0 < \alpha < p)\ .$$

Then $P_\alpha^{(p)}$ is a Banach space with norm $\|\omega\|_\alpha = \left(\int |\xi|^\alpha |\tilde{\omega}(\xi)|^2\, d\xi\right)^{1/2}$. The following lemma is essential in our proof.

LEMMA 6. *Let* $0<\beta\leq\alpha<p$ *and* $\omega \in P_\alpha^{(p)}$. *Then* h_ω *has* $T\left(\dfrac{p-(\beta/2)}{p-(\alpha/2)}\right)$-*limits on* $\boldsymbol{R}^p$ *except for a set of* $C_\beta^{(p)}$-*capacity zero.*

Proof. Put $\gamma = \dfrac{p-(\beta/2)}{p-(\alpha/2)}$. By [1], we can describe $\omega = U^\mu_{\alpha/2}$ in $L^1_{\mathrm{loc}}(\boldsymbol{R}^p)$, where μ is a square integrable function. Suppose $U^{|\mu|}_{\beta/2}(0) < +\infty$ and $S(\mu) \subset B(0,1)$. By Lemma 4 and Lemma 5, for any $(x\,;y) \in R(m,0,\gamma)$,

$$\begin{aligned} g_{U^{|\mu|}_{\alpha/2}}(x\,;\,y) &= \int |\mu(z)|\, dz\left(\frac{2}{\sigma_{p+1}}\int \frac{y}{(|s-x|^2+y^2)^{(p+1)/2}}\, \frac{1}{|s-z|^{p-(\alpha/2)}}ds\right) \\ &\leq c_2 \int \frac{1}{(|z-x|^2+y^2)^{(p-(\alpha/2))/2}}\,|\mu(z)|\, dz \\ &\leq c_1c_2\int \frac{1}{|z|^{p-(\beta/2)}}\,|\mu(z)|\, dz \leq c_1c_2 U^{|\mu|}_{\beta/2}(0)\ . \end{aligned}$$

It is well-known that $C_\beta^{(p)}(\{x \in \boldsymbol{R}^p\,;\, U^{|\mu|}_{\alpha/2}(x) = +\infty\}) = C_\beta^{(p)}(\{x \in \boldsymbol{R}^p\,;\, U^{|\mu|}_{\beta/2}(x) = +\infty\}) = 0$ for any square integrable function μ. We can prove Lemma 6 by the same way as in Lemma 3 and hence we omit the rest of our proof.

Let $H_\alpha^{(p+1)}(-1<\alpha<1)$ be the totality of harmonic functions in R_+^{p+1} with (1). $H_\alpha^{(p+1)}$ is a Banach space with norm $|||h|||_\alpha = \left(\iint |\mathrm{grad}\, h(x\,;\,y)|^2\, y^\alpha\, dxdy\right)^{1/2}$.

LEMMA 7. *Let* $-1<\alpha<1$ *and* $\omega \in P_{1-\alpha}^{(p)}$. *Then* $|||h_\omega|||_\alpha = c_\alpha \|\omega\|_{1-\alpha}$, *where* $c_\alpha = 2^{-\alpha-1}\pi^{-(\alpha+1)/2}(1+4\pi^2)^{1/2}\Gamma(\alpha+1)^{1/2}$.

Proof. If ω is a sufficiently smooth function with a compact support, we have

$$\begin{aligned} \int h_\omega(x\,;\,y) e^{-2\pi i\xi x} dx &= \int \omega(s)ds\left(\frac{2}{\sigma_{p+1}}\int \frac{y}{(|s-x|^2+y^2)^{(p+1)/2}} e^{-2\pi i\xi x}dx\right) \\ &= \int e^{-2\pi|\xi|y} e^{-2\pi i\xi s}\omega(s)ds = e^{-2\pi|\xi|y}\tilde{\omega}(\xi)\ . \end{aligned}$$

Put $\omega_y(x) = h_\omega(x\,;y)$. Then $\tilde{\omega}_y(\xi) = e^{-2\pi|\xi|y}\tilde{\omega}(\xi)$ and $\partial\tilde{\omega}_y/\partial y(\xi) = -2\pi|\xi|e^{-2\pi|\xi|y} \cdot\tilde{\omega}(\xi)$. Hence we obtain

$$\begin{aligned}|||h_\omega|||_\alpha^2 &= \int_0^\infty y^\alpha dy \int\Big(|\xi|^2|\tilde{\omega}_y(\xi)|^2 + \Big|\frac{\partial\tilde{\omega}_y}{\partial y}(\xi)\Big|^2\Big)d\xi \\ &= (1+4\pi^2)\int|\xi|^2|\tilde{\omega}(\xi)|^2\,d\xi\int_0^\infty e^{-4\pi|\xi|y}y^\alpha dy \\ &= c_\alpha^2\,\|\omega\|_{1-\alpha}^2\,.\end{aligned}$$

By the limit process, we obtain the equality of Lemma 7 for any $\omega\in P_{1-\alpha}^{(p)}$. This completes the proof.

LEMMA 8. *Let h be harmonic in $\boldsymbol{R}^p$ $(p\ge 3)$. If h satisfies the following condition*:

$$\int_{|x|>1}|\mathrm{grad}\,h|^2\frac{1}{|x|^\alpha}dx < +\infty \qquad (0\le\alpha\le 1)\,,$$

then h is constant.

Proof. Put $u = |\mathrm{grad}\,h|^2$. Then u is subharmonic. Assume $u(0) > 0$. Then

$$\int_1^\infty\frac{1}{r^\eta}dr\int_{\partial B(0,r)}u\,d\sigma \ge \sigma_p\Big(\int_1^\infty r^{p-1-\eta}\,dr\Big)u(0) = +\infty\,.$$

This is contradiction. Hence $u(0) = 0$. By the same argument, $u\equiv 0$. This completes the proof.

We are going to show Theorem 2. Let $0\le\alpha<1$. For any $\eta>0$, there exists a distribution T_η with finite Newton energy and a constant c_6 which is independent on η such that $h(x\,;y) = U_2^{T_\eta}(x\,;y) + c_6$ in $t\ge\eta$. We may assume $c_6 = 0$. Put $\omega_\eta(x) = h(x\,;\eta)$. By the same argument as in Theorem 1, we have $\omega_\eta\in P_{1-\alpha}^{(p)}\cap P_1^{(p)}$ and $\|\omega_\eta - \omega_{\eta'}\|_{1-\alpha}\to 0$ as $\eta,\eta'\to 0$. There exists $\omega\in P_{1-\alpha}^{(p)}$ such that $\|\omega_\eta-\omega\|_{1-\alpha}\to 0$ as $\eta\to 0$. Since $h_{\omega_\eta}(x\,;0) = h(x\,;\eta)$ and $h_{\omega_\eta}(x\,;y) - h(x\,;y+\eta)\in H_0^{(p+1)}$, we have $h_{\omega_\eta}(x\,;y) = h(x\,;y+\eta)$. Letting $\eta\to 0$, we have $h_\omega(x\,;y) = h(x\,;y)$. By lemma 6, the assertion of this theorem holds in the $0\le\alpha<1$ case.

Let $-1<\alpha<0$. Since for almost all y,

$$\int|\mathrm{grad}\,h(x\,;y)|^2\,dx < +\infty\,. \tag{9}$$

We can assume that (9) holds for $y = 1/n$ $(n = 1,2,\cdots)$. There exists a sequence of numbers $(d_n)_{n=1}^{\infty}$ such that $\omega_n = h(x; 1/n) + d_n \in P_2^{(\alpha)}$. Then

$$\|\omega_n\|_2^2 \leq \int |\text{grad } h(x; 1/n)|^2 \, dx < +\infty .$$

By Lemma 8, we have $h(x; y + 1/n) = h_{\omega_n}(x; y) + d_n$. Let $\ell(t) = t^{-\alpha}$ in $t < 1$ and $h(t) = t^2$ on $t \geq 1$. There exists a constant c_7 such that

$$\begin{aligned} \iint_{R_+^{p+1}} |\text{grad } h_{\omega_n}|^2 \frac{1}{1 + y^{-\alpha}} dxdy &= \iint_{R_+^{p+1}} |\text{grad } h(x; y)|^2 \frac{1}{1 + y^{-\alpha}} dxdy \\ &\leq c_7 \iint_{R_+^{p+1}} |\text{grad } h|^2 \frac{1}{1 + y^{-\alpha}} dxdy < +\infty . \end{aligned}$$

Since for $t < 1$,

$$\begin{aligned} \int_0^\infty \frac{1}{1 + t^{-\alpha}} e^{-ty} dy &= t^{-\alpha-1} \int_0^\infty \frac{1}{t^{-\alpha} + y^{-\alpha}} e^{-y} dy \\ &\geq t^{-\alpha-1} \int_0^\infty \frac{1}{1 + y^{-\alpha}} e^{-y} dy . \end{aligned}$$

We have, with a constant c_8,

$$\begin{aligned} \iint_{R_+^{p+1}} |\text{grad } h_{\omega_n}|^2 \frac{1}{1 + y^{-\alpha}} dxdy &= \int |\xi|^2 |\tilde{\omega}_n|^2 \, d\xi \int_0^\infty \frac{1}{1 + y^{-\alpha}} e^{-4\pi|\xi|y} dy \\ &\geq c_8 \int \ell(|\xi|) |\tilde{\omega}_n|^2 \, d\xi . \end{aligned}$$

Moreover

$$\begin{aligned} &\int \ell(|\xi|) \left(\frac{2}{\sigma_p} \int \frac{1}{(|x - z|^2 + y^2)^{(p+1)/2}} e^{-2\pi i \xi z} dz \right)^2 dz \\ &\leq \int e^{-2\pi|\xi|y} d\xi < +\infty . \end{aligned}$$

By the same argument as in the $0 \leq \alpha < 1$ case, there exists a $\omega_0 \in L_{\text{loc}}^1(R^p)$ such that $h_{\omega_0}(x; y) = h(x; y)$. By Lemma (2) (ii), we have $\omega_0 \in P_{1-\alpha}^{(p)}$. By Lemma 6, the assertion of this theorem holds in the $-1 < \alpha < 0$ case. This completes the proof.

4. The proof of Theorem 3.

Let E be the set in Theorem 3. For positive integers n, m, put $E_{n,m} = E \cap B(0, n) \cap \{s \in R^p ; S_\alpha(s; 2) \leq m\}$. Then $E = \bigcup E_{n,m}$. Since $S_\alpha(s; 2)$ is lower semi-continuous, $\{s \in \boldsymbol{R}_p ; S_\alpha(s; 2) \leq m\}$ is closed. Therefore it is sufficient to show the following

THEOREM 3′. *Let* $p \geq 2$, $-1 < \alpha < 1$, E *be a compact set in* $B(0,1)$ *and* h *harmonic in* $\boldsymbol{R}_+^{p+1}$. *If* $S_\alpha(s;2)$ *is bounded on* E, *then* $\lim_{r\to 0} h(s;r)$ *exists on* E *except for a set of* $C_{1-\alpha}^{(p)}$*-capacity zero.*

Put $U_1 = \bigcup_{s\in E} \Gamma_a(s;3/2)$ $(a<1)$, $U_2 = \{(x;y);y>1\}$ and $R = U_1 \cap U_2$. Then the following lemma holds.

LEMMA 9. ([12]) *There exists a sequence* $(R_n)_{n=1}^\infty$ *of domains in* $\boldsymbol{R}_+^{p+1}$ *satisfying the following four conditions:*

(i) $R_n \subset R$,

(ii) $R_{n_2} \subset R_{n_1}$ *for* $n_1 > n_2$,

(iii) $\mathrm{dis}\,(\partial R_n, \partial R) \to 0$ *as* $n \to \infty$,

(iv) $\partial R_n = \{(x;y);y = \delta_n(x)\}$,

where $\delta_n(x)$ *is an infinitely differential function such that* $0 < \delta_n(x) < 3/2$ *on* $\boldsymbol{R}^p$, $\delta_n(x) = 1$ *on* $|x| \geq 2$ *and* $|\partial\delta_n/\partial x_i| \leq 1/a$ $(i = 1, \cdots, p)$.

LEMMA 10. *Let* $S_\alpha(x;2)^{1/2} \leq M_1$ *on* E. *There exists a constant* c_0 *depending on* p, α *and* a *such that*

$$\iint_{U_1\cap R_n} |\mathrm{grad}\, h(s;y)|^2 (y-\delta_n(x))^\alpha dxdy \leq c_0 M_1^2 \,.$$

Proof. Let $E_0 = \bigcup_{s\in E} \{x \in \boldsymbol{R}^p; |x-s| < (3/2)a\}$. For $x \in E_0$, define $y(x) = \inf\{y;(x;y)\in U_1\}$. Evidently, $y(x)$ is measurable on E_0 and $0 \leq y(x) < 3/2$. We define a vector valued measurable function $s(x) = (s_1(x), \cdots, s_p(x))$ such that $(x;y(x)) \in \partial\Gamma_a(s(x);3/2)$. There exists a constant $k(0<k<\frac{1}{2})$ depending only on a such that for any $(x;y) \in U_1$, the open ball $B_k(x;y)$ with center $(x;y)$ and radius ky is containted in $\Gamma_1(s(x);2)$. Since $|\mathrm{grad}\, h(x;y)|^2$ is subharmonic, we have, with constants c_1 and c_2 (depending on p, α and a),

$$\begin{aligned} |\mathrm{grad}\, h(x;y)|^2 &\leq \frac{(p+1)}{\sigma_{p+1}(ky)^{p+1}} \iint_{B_k(x;y)} |\mathrm{grad}\, h(s;t)|^2 dsdt \\ &\leq c_1 \iint_{B_k(x;y)} |\mathrm{grad}\, h(s;t)|^2 \frac{1}{t^{p+1}} dsdt \\ &\leq c_2 \iint_{\substack{\Gamma_1(s(x);2)\\ t\geq y/2}} |\mathrm{grad}\, h(s;t)|^2 \frac{1}{t^{p+1}} dsdt \,. \end{aligned}$$

Hence we have

$$\begin{aligned} &|\mathrm{grad}\, h(x;y)|^2 (y-\delta_n(x))^\alpha \\ &\leq 2^\alpha c_2 \left(\frac{y-\delta_n(x)}{2}\right)^\alpha \iint_{\substack{\Gamma_1(s(x);2)\\ t\geq y/2}} |\mathrm{grad}\, h(s;t)|^2 \frac{1}{t^{p+1}} dsdt \,. \end{aligned}$$

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Then

$$\begin{aligned}
&\iint_{U_1 \cap R_n} |\mathrm{grad}\, h(x\,;y)|^2 (y-\delta_n(x))^\alpha dxdy \\
&\quad = \int_{E_0} dx \int_{\delta_n(x)}^{3/2} |\mathrm{grad}\, h(x\,;y)|^2 (y-\delta_n(x))^\alpha dy \\
&\quad \le 2^{\alpha+1} c_2 \int_{E_0} dx \int_0^\infty r^\alpha dr \iint_{\substack{\Gamma_1(s(x);2) \\ t \ge r}} |\mathrm{grad}\, h(s\,;t)|^2 \frac{1}{t^{p+1}} dsdt \\
&\quad = \frac{2^{\alpha+1}}{\alpha+1} c_2 \int_{E_0} dx \iint_{\Gamma_1(s(x);2)} |\mathrm{grad}\, h(s\,;t)|^2 \frac{1}{t^{p-\alpha}} dsdt \le c_0 M_1^2 \,.
\end{aligned}$$

LEMMA 11. *There exists a constant c_3 depending on p, α and a such that for any $(x\,;y) \in U_1$,*

$$y^{(1+\alpha)/2} |\mathrm{grad}\, h(x\,;y)|^2 \le c_3 M_1 \,.$$

By the elementary calculation, we can show this inequality.

We are going to prove Theorem 3′. By Lemma 10, there exists a continuously differentiable function $u(x\,;y)$ on $\boldsymbol{R}^{p+1}$ such that $u(x\,;y) = h(x\,;y)$ on $|x| \le 2$, $0 < y \le 2$, $u(x\,;y) = 0$ on $|x| \ge 3$ or $y \ge 3$ and

$$\iint_{R_n} |\mathrm{grad}\, u(x\,;y)|^2 (y-\delta_n(x))^\alpha dxdy \le c_0 M_1^2 + 1 \,.$$

Define $u_n(x\,;y) = u(x\,;y+\delta_n(x))$ and $\omega_n(x) = u_n(x\,;0)$. Then we have, with constant c_0' depending on α,

$$\iint_{\boldsymbol{R}_+^{p+1}} |\mathrm{grad}\, u_n(x\,;y)|^2 y^\alpha dxdy \le c_0'(c_0 M_1^2 + 1) \,.$$

By Lemma 2 (ii), there exists a constant c_4 depending on p and α such that

$$\int |\xi|^{1-\alpha} |\tilde{\omega}_n(\xi)|^2 d\xi \le c_4 \iint_{\boldsymbol{R}_+^{p+1}} |\mathrm{grad}\, u_n(x\,;y)|^2 y^\alpha dxdy (\le c_4 c_0'(c_0 M_1^2 + 1)) \,.$$

Hence we may assume that there exists $\omega \in P_{1-\alpha}^{(p)}$ such that $\omega_n \to \omega$ weakly in $P_{1-\alpha}^{(p)}$ as $n \to \infty$. By [1], we can describe $\omega = U_{(1-\alpha)/2}^{\mu}$ a.e., where μ is a square integrable function. Assume that $s_0 \in E$ and $U_{(1-\alpha)/2}^{|\mu|}(s_0) < +\infty$. Put $U_{(1-\alpha)/2}^{\mu}(s_0) = b$. We show $\lim_{r\to 0} h(s_0\,;r) = b$. Set $D = \{(x\,;y)\,; |x| < 2, 0 < y < 2\}$, $\partial D_1 = \partial D \cap \{(x\,;y)\,; y > 1\}$, $\partial D_2 = \partial D - \partial D_1$, $D_0 = R \cap D$, $D_n = R_n \cap D$, $\partial D_{n,1} = \partial R_n \cap U_1$, $\partial D_{n,2} = (\partial R_n - \partial D_{n,1}) \cap \{(x\,;y)\,; |x| < 2\}$ and $\partial D_{n,3} = \partial D_n - (\partial D_{n,1} \cup \partial D_{n,2})$. Put $T = \sup\{h(x\,;y)\,; (x\,;y) \in \partial D_1\}$ and

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$$M_2 = \sup\{y^{(1+\alpha)/2}|\text{grad}\, h(x;y)|^2; (x;y)\in\{(s;t); |s|<3, \tfrac{1}{2}<t<2)\}\,.$$

Let $H(x;y)$ be the harmonic function in D whose boundary values equal to 1 on ∂D_1 and 0 on ∂D_2. For $(x;y)\in\partial D_{n,1}$, $B_k(x;y)$ is contained in $\Gamma_1(s(x);2)$. For any $(s;t)\in B_k(x;y)$, we have, with a constant ℓ_1 depending on p, α and a,

$$\begin{aligned}&|h(x;y)-h(s;t)|\\&\qquad\le ky\sup\{|\text{grad}\, h(x';y')|; (x';y')\in B_k(x;y)\}\le \ell_1 M_1 y^{(1-\alpha)/2}\,.\end{aligned}$$

In particular, for $(s;t)\in\partial R_n\cap B_k(x;y)$, we have,

$$|h(x:y)-\omega_n(s)|\le \ell_1 M_1 y^{(1-\alpha)/2}$$

and hence

$$\begin{aligned}\omega_n(s)-b-\ell_1 M_1 y^{(1-\alpha)/2}&\le h(x;y)-b\\&\le \omega_n(s)-b+\ell_1 M_1 y^{(1-\alpha)/2}\,.\end{aligned}\tag{10}$$

Since $|\partial\delta_n/\partial x_i|\le 1/a$ $(i=1,\cdots,p)$, the inequality (10) holds for $s\in\boldsymbol{R}^p$ as long as $|x-s|\le\sqrt{1+(p/a^2)}$ $(=\ell_2)$. Let ℓ_3 be a number with $p/\sigma_p(\ell_2)^p<\ell_3<2^p\cdot p/\sigma_p(\ell_2)^p$ and $\psi(s)$ an infinitely differentiable function such that $\psi(s)=\ell_3$ on $|s|\le\ell_2/2$, $\psi(s)=0$ on $|s|\ge\ell_2$, $0\le\psi(s)\le\ell_3$ and $\int\psi(s)ds=1$. Put $\psi_y(s)=(1/y^p)\psi(s/y)$ and $\tau_x\psi_y(s)=\psi_y(s-x)$. By (10),

$$\begin{aligned}\int(\omega_n-b)\tau_x\psi_y ds-\ell_1 M_1 y^{(1-\alpha)/2}&\le h(x;y)-b\\&\le\int(\omega_n-b)\tau_x\psi_y ds+\ell_1 M_1 y^{(1-\alpha)/2}\,.\end{aligned}$$

Letting $n\to\infty$, we have

$$\begin{aligned}\int(\omega-b)\tau_x\psi_y ds-\ell_1 M_1 y^{(1-\alpha)/2}&\le h(x;y)-b\\&\le\int(\omega-b)\tau_x\psi_y ds+\ell_1 M_1 y^{(1-\alpha)/2}\,.\end{aligned}\tag{11}$$

We see, with a positive constant ℓ_0 depending on p, α and a,

$$\frac{2}{\sigma_{p+1}}\frac{y}{((s-x)^2+y^2)^{(p+1)/2}}\ge\frac{1}{\ell_0}\tau_x\psi_y(s)$$

for $|s-x|\le\ell_2 y$. Hence by (11),

$$|h(s;y)-b|\le\ell_0 h_{|\omega-b|}(x;y)+\ell_1 M_1 y^{(1-\alpha)/2}\,.$$

By the same argument, there exists constants ℓ_0' and ℓ_1' depending on p, α and a such that for any $(x; y) \in \partial D_{n,2}$,

$$|h(x; y) - b| \leq \ell_0' h_{|\omega - b|}(x; y) + \ell_1' M_2 y^{(1-\alpha)/2} .$$

Since $\partial D_{n,3} = \partial D_1$, we see

$$|h(x; y) - b| \leq (T + |b|) H(x; y)$$

for any $(x; y) \in \partial D_{n,3}$. Put $\ell_0' = \max(\ell_0, \ell_0''), \ell_1'' = \max(\ell_1, \ell_1')$ and $M = \max(M_1, M_2)$, we have

$$|h(x; y) - b| \leq \ell_0'' h_{|\omega - b|}(x; y) + (T + |b|) H(x; y) + \ell_1'' M y^{(1-\alpha)/2}$$

for any $(x; y) \in \partial D_n$ and hence this inequality holds in D_n. Letting $n \to \infty$, we obtain that this inequality holds in D. We see $\lim_{r \to 0} h_{|\omega - b|} \cdot (s_0; r) = 0$. Hence we obtain $\lim_{r \to 0} h(s_0; r) = b$. Since $C_{1-\alpha}(\{s \in E; U^{|\mu|}_{(1-\alpha)/2}(s) = +\infty\}) = 0$, this completes the proof.

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