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RIESZ TRANSFORM ESTIMATES IN THE ABSENCE OF A PRESERVATION CONDITION AND APPLICATIONS TO THE DIRICHLET LAPLACIAN

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It has been asked (see [8]) whether the $\mathcal{L}^p$ boundedness of the Riesz transform observed on $\mathbb{R}^n$ could be extended to a reasonable class of noncompact manifolds. Many partial answers have been given. In particular, we mention [3] for the case $1 < p \leq 2$ and [1] for the case $p \geq 2$. In [1], the $\mathcal{L}^p$ boundedness of the Riesz transform was tied to the $\mathcal{L}^p$ boundedness of the Gaffney inequality. The result held on noncompact manifolds satisfying doubling and Poincaré conditions, along with a stochastic completeness or *preservation condition*.

The main result of this thesis is the extension of the theorems of [1] to prove sufficient conditions for $\mathcal{L}^p$ bounds:

$$\||\nabla L^{-1/2} f|\|_{\mathcal{L}^p} \lesssim \|f\|_{\mathcal{L}^p},$$

with $p > 2$, for generalised Riesz transforms $\nabla L^{-1/2}$ in cases where a preservation condition does not hold. To say that a preservation condition does not hold is to say that

$$e^{-tL} 1 \neq 1.$$

To compensate for the lack of a preservation condition, two new conditions are required. These are a Hardy inequality and a localised norm bound on the associated heat semigroup gradient. A good-λ method is employed in the proof. This type of method is common in the literature (see for example [1] or [2]). The results are general enough to apply in a large number of circumstances. Further to this main result, the thesis presents two significant extensions.

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The first extension is to Riesz transforms on nondoubling domains. Here both the global doubling and the preservation conditions do not hold. This extension applies specifically to the circumstance of a manifold with boundary and Dirichlet boundary conditions. Strict upper bounds relating to heat semigroup decay are required to take the place of standard doubling estimates. Once again the result gives sufficient conditions for $\mathcal{L}^p$ boundedness of the Riesz transform in such a case. An added benefit of this nondoubling extension is that the Poincaré inequality is no longer required near the boundary.

The second extension shows that the weighted $\mathcal{L}^p$ boundedness of the Riesz transform observed on $\mathbb{R}^n$ can also be extended in some degree to a reasonable class of noncompact manifolds. Once more the result is a list of sufficient conditions, now ensuring weighted $\mathcal{L}^p$ Riesz transform bounds in the absence of a preservation condition. This second extension includes generalised deriving of weight classes associated to skewed maximal functions and other operators. These weight classes are a variation on the traditional Muckenhoupt A_p weight classes, adapted for the chosen domain.

Applications are to the case of Dirichlet Laplacians on various subsets of $\mathbb{R}^n$, primarily exterior and halfspace-like domains. These applications are particularly motivated by recent results in [5]. Heat kernel bounds from [6, 7] and [9] are used in the proofs of the various conditions. Inner uniform domains and their properties from [4] also feature in the applications.

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