NON-EXISTENCE OF ESTIMATES OF PRESCRIBED ACCURACY IN FIXED SAMPLE SIZE

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1. Let X be a random variable whose density (or distribution if discrete) $f(x; \theta)$ depends on an unknown parameter θ, real or vector-valued. By making observations on X we want to know whether there exist estimates of prescribed accuracy for the real-valued parametric function $g(\theta)$. By an estimate of prescribed accuracy for $g(\theta)$ we mean a confidence interval of prescribed length and confidence coefficient or a point estimate with prescribed expected loss W. In the following our loss functions W will always satisfy the requirement that $W(\delta, \theta) = V(|\delta - \theta|)$, where V is a strictly increasing function of its argument. The class of such loss functions includes among others the squared error loss.

In this note we prove the non-existence of estimates of prescribed accuracy in fixed sample size procedures for

(A) σ^s, where $s \geq 1$ is real and σ is the scale parameter in a density function of the form $(1/\sigma)\ f(x/\sigma)$.

(B) λ^s, where $s \geq 1$ is real and λ is the Poisson parameter.

(C) the parametric function $g(\theta) = \psi(\mu, \sigma)$ where μ, σ are respectively the location and scale parameters in a given density function f of the form $(1/\sigma)\ f((x - \mu)/\sigma)$. The function ψ is real-valued and, for a given σ, is not bounded in μ. The result of Zacks [4] is a special case of ours when f is the normal density with mean μ and variance σ^2 and $\psi(\mu, \sigma) = \mu + \phi(\sigma)$, ϕ real-valued.

(D) the parametric function $g(\theta) = \psi(\mu, \sigma)$ where $-\infty < \mu < \infty$ and $\sigma > 0$ are the parameters in the log-normal density and ψ is as defined in (C) above.

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For other examples of interest see [1].

In [1] it is shown that if $g(\theta)$ has a bounded length confidence interval based on one-stage [m-stage or sequential] sampling plans then g is uniformly continuous [continuous] on the metric space $(\mathcal{P}, d)$ where $\mathcal{P} = \{f(x;\theta) : \theta \varepsilon \Omega$, the parameter space$\}$ and d is the usual absolute variational distance on $\mathcal{P}$ defined by

$$d(f_{\theta_1}, f_{\theta_2}) = \frac{1}{2}\int |f_{\theta_1} - f_{\theta_2}|\, dx, \qquad \theta_1, \theta_2 \varepsilon \Omega .$$

In view of this result, to prove the non-existence of bounded length confidence intervals for $g(\theta)$ based on one-stage sampling plans, it is enough to show that g is not uniformly continuous on $(\mathcal{P}, d)$. That this in turn implies the non-existence of estimates of prescribed accuracy for $g(\theta)$ in one-stage sampling plans follows from a simple application of Chebichev's inequality.

2. In this section we consider the examples listed above individually to show that in each case the parametric function $g(\theta)$ is not uniformly continuous on the corresponding metric space $(\mathcal{P}, d)$.

A. Let $\mathcal{P} = \{P_\sigma = \frac{1}{\sigma} f(\frac{x}{\sigma});\ \sigma > 0,\ x > 0\}$ be the class of all density functions involving a scale parameter σ for a given f and let $g(P_\sigma) = \sigma^s$, $s \geq 1$ or $s \leq -1$ being real. On setting $y = \log x$, $\theta = \log \sigma$, $y - \theta_1 = z$ and $e^z f(e^z) = \psi(z)$ we find that

$$d(P_{\sigma_1}, P_{\sigma_2}) = \frac{1}{2}\int_0^\infty \left|\frac{1}{\sigma_1} f(\frac{x}{\sigma_1}) - \frac{1}{\sigma_2} f(\frac{x}{\sigma_2})\right| dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} |\psi(z) - \psi(z - (\theta_2 - \theta_1))|\, dz ,$$

which, by Scheffe's theorem [2], tends to zero as $\theta_2 - \theta_1 \to 0$. It follows, therefore, that $d(P_{\theta_1}, P_{\theta_2})$ can be made arbitrarily small by choosing σ_1, σ_2 so that $\theta_2 - \theta_1 = \log(\sigma_2/\sigma_1) \to 0$ or equivalently $\frac{\sigma_2}{\sigma_1} \to 1$.

Let $\sigma_2 = \sigma_1 + 1$ so that $\frac{\sigma_2}{\sigma_1} \to 1$ as $\sigma_1 \to \infty$. It is then easily seen that for $s \geq 1$, $|\sigma_2^s - \sigma_1^s| \geq 1$ proving thereby that the parametric function $g(P_\sigma) = \sigma^s$, $s \geq 1$, is not uniformly continuous on $(\mathbb{P}, d)$. That this is also the case when $s \leq -1$ can be seen by taking $\sigma_2 = \sigma_1 + \sigma_1^2$. This establishes the non-existence of estimates of prescribed accuracy in fixed sample size for σ^s, $s \geq 1$ or $s \leq -1$ being any real number.

B. Let $\mathbb{P} = \{P_\lambda = e^{-\lambda}\lambda^x/x!\}$ be the class of all Poisson distributions with parameter λ. Here $g(P_\lambda) = \lambda^s$, $s \geq 1$ any real number. Suppose $\lambda_2 = \lambda_1 + c$ where c is a positive real number. Then

$$d(P_{\lambda_1}, P_{\lambda_2}) = \frac{1}{2} \sum_{x=0}^{\infty} |e^{-\lambda_1}\lambda_1^x - e^{-\lambda_2}\lambda_2^x|/x!$$

$$= F(x_o; \lambda_1) - F(x_o; \lambda_2)$$

where $f(x; \theta)$ is the Poisson cummulative distribution function with parameter θ and $x_o = (\lambda_2 - \lambda_1)/(\log \lambda_2 - \log \lambda_1)$. Using Mean Value theorem it follows that

$$d(P_{\lambda_1}, P_{\lambda_2}) = c\, e^{-\lambda}\lambda^{\lambda*}/\lambda*!$$

for some λ, $\lambda*$ lying between λ_1 and λ_2. On using Stirlings' formula for factorials, it is easily seen that $d(P_{\lambda_1}, P_{\lambda_2}) \to 0$ as $\lambda_1 \to \infty$. On the other hand $|\lambda_2^s - \lambda_1^s| \geq c^s$ for $s \geq 1$ proving thereby our assertion that g is not uniformly continuous on $(\mathbb{P}, d)$.

C. Let $\mathbb{P}$ be the class of distributions with density functions $(1/\sigma)\, f((x-\mu)/\sigma)$, where μ is real, $\sigma > 0$ and f is a given density function. Here the density function f is known but the location and scale parameters are unknown. Let $\theta = (\mu, \sigma)$, and $g(\theta) = \psi(\mu, \sigma)$ where ψ is as defined above in section 1. If $\theta_i = (\mu_i, \sigma_i)$, $i = 1, 2$, then

$$(*)\ldots\ d(P_{\theta_1}, P_{\theta_2}) = \frac{1}{2}\int_{-\infty}^{\infty} |f(x;\theta_1) - f(x;\theta_2)|\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} |(1/\sigma_1) f((x-\mu_1)/\sigma_1) - (1/\sigma_2) f((x-\mu_2)/\sigma_2)|\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} |f(y) - (\sigma_1/\sigma_2) f[(\sigma_1/\sigma_2)y + (\mu_1-\mu_2)/\sigma_2]|\,dy$$

where $y = (x-\mu_1)/\sigma_1$. If in $(*)$ we let $\sigma_2 \to \infty$ such that $(\sigma_1/\sigma_2) \to 1$, it follows from Scheffe's theorem [2] that $d(P_{\theta_1}, P_{\theta_2}) \to 0$. On the other hand for $\sigma_1 = 1 + \sigma_2$, $(\sigma_1/\sigma_2) \to 1$ as $\sigma_2 \to \infty$, but

$$|g(\theta_2) - g(\theta_1)| = |\psi(\mu_1, \sigma_1) - \psi(\mu_2, \sigma_2)|$$

$$\geq \big|\, |\psi(\mu_1, 1+\sigma_2)| - |\psi(\mu_2, \sigma_2)| \,\big|$$

where the R.H.S. is, according to the definition of ψ, unbounded. This proves our assertion about the non-uniform continuity of $g(\theta)$ on $(\mathbb{P}, d)$.

D. In the log-normal case where the density f is given by

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}}\,(1/x\sigma)\, e^{-\frac{1}{2\sigma^2}[\log x - \mu]^2}$$

if $x > 0$ and equal to zero otherwise, the d-distance between the two distributions with parameters $\theta_1 = (\mu_1, \sigma_1)$ and $\theta_2 = (\mu_2, \sigma_2)$ is

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$$d(P_{\theta_1}, P_{\theta_2}) = \frac{1}{2} \int_0^\infty |f(x;\theta_1) - f(x;\theta_2)|\,dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} |(1/\sigma_1)\, n((y-\mu_1)/\sigma_1) - (1/\sigma_2)\, n((y-\mu_2)/\sigma_2)|\,dy$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} |n(z) - (\sigma_1/\sigma_2)\, n[(\sigma_1/\sigma_2)\, z + (\mu_1 - \mu_2)/\sigma_2]|\,dz.$$

Here $n(x)$ is the standard normal $(0, 1)$ density, $y = \log x$ and $z = (y - \mu_1)/\sigma_1$. Proceeding as in (C) we prove that $g(\theta) = \psi(\mu, \sigma)$ is not uniformly continuous on $(\mathbb{P}, d)$. This, in particular, implies that the mean ν of the log normal density,

$$\nu = e^{\mu + \frac{1}{2}\sigma^2}$$

cannot have estimates of prescribed accuracy in fixed sample size procedures.

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