

ENTROPY AND RENORMALIZED SOLUTIONS FOR THE $p(x)$ -LAPLACIAN EQUATION WITH MEASURE DATA

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Abstract

In this paper we prove the existence and uniqueness of both entropy solutions and renormalized solutions for the $p(x)$ -Laplacian equation with variable exponents and a signed measure in $L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$. Moreover, we obtain the equivalence of entropy solutions and renormalized solutions.

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1. Introduction

Let Ω be a bounded open domain in \mathbb{R}^N ($N \geq 2$) with Lipschitz boundary $\partial\Omega$. In this paper we study the nonlinear elliptic problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where the variable exponent $p: \overline{\Omega} \rightarrow (1, \infty)$ is a continuous function, and μ is a signed measure in $L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$.

The study of differential equations and variational problems with nonstandard growth conditions has aroused much interest owing to the development of elastic mechanics, electro-rheological fluid dynamics and image processing, and so on. We refer the reader to [13, 30, 31, 34] and the references therein. The $p(x)$ -growth conditions can be regarded as a very important class of nonstandard (p, q) -growth conditions. There are already numerous results for such kinds of problems (see [1–3, 5, 16, 17, 20]). The functional spaces used to deal with these problems are the generalized Lebesgue spaces $L^{p(x)}(\Omega)$ and the generalized Lebesgue–Sobolev spaces $W^{k,p(x)}(\Omega)$.

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Under our assumptions, it is reasonable to work with entropy solutions or renormalized solutions, which need less regularity than the usual weak solutions. The notion of renormalized solutions was first introduced by DiPerna and Lions [15] for the study of the Boltzmann equation. It was then adapted to the study of some nonlinear elliptic or parabolic problems and evolution problems in fluid mechanics. We refer the reader to [7–9, 12, 27] for details. At the same time the notion of entropy solutions was proposed by B enilan *et al.* [6] for nonlinear elliptic problems. This framework was extended to related problems with constant p in [4, 10, 11, 29]. The interesting and difficult cases are those of $1 < p \leq N$, since the variational methods of Leray–Lions (see [26]) can be easily applied for $p > N$, with the solution being bounded and its gradient being in $L^p(\Omega)$.

Recently, Sanch on and Urbano [32] studied a Dirichlet problem of the $p(x)$ -Laplace equation and obtained the existence and uniqueness of entropy solutions for L^1 data, as well as integrability results for the solution and its gradient. The proofs rely crucially on *a priori* estimates in Marcinkiewicz spaces with variable exponents.

The aim of this paper is to extend the results in [6, 11, 32], taking into account a signed measure μ in $L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$. As far as we know, there are few papers concerned with the nonlinear elliptic or parabolic equations involving measure data and multiple anisotropic exponents. Therefore, the study of problem (1.1) is a new and interesting topic. The uniqueness of the entropy solutions and renormalized solutions will strongly rely on the structure of the measure μ , that is, μ belongs to $L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$. If not, there is a counterexample in [11] showing why the definition of the entropy solution is not suitable for uniqueness if the measure μ is the Dirac mass.

Inspired by [6, 11, 25], we develop a refined method which is different from [32]. We first construct an approximate solution sequence for problem (1.1) and establish some *a priori* estimates. Next, we draw a subsequence to obtain a limit function, and prove this function is an entropy solution. The advantage of our method is that we can give the concrete expression of the limit function of the approximate solutions by means of truncation techniques and prove the strong convergence of the truncations of the approximate solutions. Based on this fact, we obtain that the entropy solution of problem (1.1) is also a renormalized solution. It is worth pointing out that we generalize the definition of renormalized solutions from L^1 functions to measurable functions. This allows us to drop some redundant conditions imposed on the exponent $p(x)$ in the previous work. Even for the constant exponent case, to the best of our knowledge, there seems to be no result about the existence and uniqueness of renormalized solutions to such elliptic equations with a signed measure μ in $L^1(\Omega) + W^{-1,p'}(\Omega)$. By choosing suitable test functions and employing the decomposition for the region of integration according to the different contributions corresponding to different integration sets, we prove the uniqueness of the entropy solutions and renormalized solutions, and thus the equivalence of the entropy solutions and renormalized solutions.

For the convenience of the reader, we recall some definitions and basic properties of the generalized Lebesgue spaces $L^{p(x)}(\Omega)$ and the generalized Lebesgue–Sobolev spaces $W^{k,p(x)}(\Omega)$.

Set $C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : \min_{x \in \overline{\Omega}} h(x) > 1\}$. For any $h \in C_+(\overline{\Omega})$ we define

$$h_+ = \sup_{x \in \Omega} h(x) \quad \text{and} \quad h_- = \inf_{x \in \Omega} h(x).$$

For any $p \in C_+(\overline{\Omega})$, we introduce the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions such that

$$\int_{\Omega} |u(x)|^{p(x)} dx < \infty,$$

endowed with the Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\},$$

which is a separable and reflexive Banach space. The dual space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. If $p(x)$ is a constant function, then the variable exponent Lebesgue space coincides with the classical Lebesgue space. The variable exponent Lebesgue space is a special case of the Orlicz–Musielak space treated by Musielak [28].

For any positive integer k , denote

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq k\},$$

where the norm is defined as

$$\|u\|_{W^{k,p(x)}(\Omega)} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{p(x)}.$$

Here $W^{k,p(x)}(\Omega)$ is also a Banach space. It is called a generalized Lebesgue–Sobolev space, which is a special generalized Orlicz–Sobolev space. An interesting feature of a generalized Lebesgue–Sobolev space is that smooth functions are not dense in this space without additional assumptions on the exponent $p(x)$. This was observed by Zhikov [33] in connection with the Lavrentiev phenomenon. However, when the exponent $p(x)$ is *log-Hölder* continuous, that is, there is a constant C such that

$$|p(x) - p(y)| \leq \frac{C}{-\log |x - y|} \quad (1.2)$$

for every $x, y \in \Omega$ with $|x - y| \leq 1/2$, then smooth functions are dense in variable exponent Sobolev spaces and there is no confusion in defining the Sobolev space with zero boundary values, $W_0^{1,p(\cdot)}(\Omega)$, as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{W^{1,p(\cdot)}(\Omega)}$ (see [21]).

Throughout this paper we assume that $p(x) \in C_+(\overline{\Omega})$ with $1 < p_- \leq p_+ < N$ satisfies the log-Hölder continuity condition (1.2), and μ is a signed measure in $L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$, that is,

$$\mu = f - \operatorname{div} F, \quad (1.3)$$

where $f \in L^1(\Omega)$ and $F \in (L^{p'(\cdot)}(\Omega))^N$. Let T_k denote the truncation function at height $k \geq 0$:

$$T_k(r) = \min\{k, \max\{r, -k\}\} = \begin{cases} k & \text{if } r \geq k, \\ r & \text{if } |r| < k, \\ -k & \text{if } r \leq -k. \end{cases}$$

We denote

$$\mathcal{T}_0^{1,p(\cdot)}(\Omega) = \{u : \Omega \rightarrow \mathbb{R} \text{ is measurable} \mid T_k(u) \in W_0^{1,p(\cdot)}(\Omega), \text{ for every } k > 0\}.$$

Next we define the very weak gradient of a measurable function $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$. The proof follows from [6, Lemma 2.1] due to the fact that $W_0^{1,p(\cdot)}(\Omega) \subset W_0^{1,p^-}(\Omega)$.

PROPOSITION 1.1. *For every measurable function $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$, there exists a unique measurable function $v : \Omega \rightarrow \mathbb{R}^N$, which we call the very weak gradient of u and denote $v = \nabla u$, such that*

$$\nabla T_k(u) = v \chi_{\{|u| < k\}} \quad \text{for almost every } x \in \Omega \text{ and for every } k > 0,$$

where χ_E denotes the characteristic function of a measurable set E . Moreover, if u belongs to $W_0^{1,1}(\Omega)$, then v coincides with the weak gradient of u .

The notion of the very weak gradient allows us to give the following definitions of entropy solutions and renormalized solutions for problem (1.1).

DEFINITION 1.2. A measurable function $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$ is an entropy solution to problem (1.1) if

$$\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla T_k(u - \phi) \, dx = \int_{\Omega} f T_k(u - \phi) \, dx + \int_{\Omega} F \cdot \nabla T_k(u - \phi) \, dx, \tag{1.4}$$

for all $\phi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

DEFINITION 1.3. A measurable function $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$ is a renormalized solution to problem (1.1) if the following conditions are satisfied:

- (i) $\lim_{n \rightarrow \infty} \int_{\{|n \leq |u| \leq n+1\}} |\nabla u|^{p(x)} \, dx = 0$;
- (ii) for every function $\varphi \in C_0^\infty(\Omega)$ and $S \in W^{1,\infty}(\mathbb{R})$ such that S has compact support,

$$\begin{aligned} & \int_{\Omega} [S(u) |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi + S'(u) |\nabla u|^{p(x)} \varphi] \, dx \\ &= \int_{\Omega} [f S(u) \varphi + F \cdot \nabla(S(u) \varphi)] \, dx \end{aligned} \tag{1.5}$$

holds.

Now we state our main results as follows.

THEOREM 1.4. *Assume that μ satisfies (1.3). Then there exists a unique entropy solution $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$ for problem (1.1).*

THEOREM 1.5. *Assume that μ satisfies (1.3). Then the entropy solution u in Theorem 1.4 is also a renormalized solution for problem (1.1). Moreover, the renormalized solution is unique.*

REMARK 1.6. The entropy solution for problem (1.1) is equivalent to the renormalized solution for problem (1.1).

Given two bounded measurable functions $p(\cdot), q(\cdot) : \Omega \rightarrow \mathbb{R}$, we write

$$q(\cdot) \ll p(\cdot) \quad \text{if } (p - q)_- > 0.$$

REMARK 1.7. If u is an entropy solution or renormalized solution of problem (1.1) and $p(\cdot) \gg 2 - 1/N$, then, following the arguments in [32] by some necessary changes and using some *a priori* estimates in Marcinkiewicz for u and ∇u , we can obtain that $u \in L^{q(\cdot)}(\Omega)$, for all $1 \ll q(\cdot) \ll q_0(\cdot)$, and $u \in W_0^{1,q(\cdot)}(\Omega)$, for all $1 \ll q(\cdot) \ll q_1(\cdot)$, where

$$q_0(\cdot) := \frac{p^*(\cdot)}{(p')_+} \quad \text{and} \quad q_1(\cdot) := \frac{q_0(\cdot)}{q_0(\cdot) + 1} p(\cdot).$$

The rest of this paper is organized as follows. In Section 2, we state some basic results that will be used later. We will prove the main results in Section 3. In the following sections C will represent a generic constant that may change from line to line even if in the same inequality.

2. Preliminaries

In this section, we first state some elementary results for the generalized Lebesgue spaces $L^{p(x)}(\Omega)$ and the generalized Lebesgue–Sobolev spaces $W^{k,p(x)}(\Omega)$.

LEMMA 2.1 [20, 24].

- (1) *The space $L^{p(\cdot)}(\Omega)$ is a separable, uniform convex Banach space, and its conjugate space is $L^{p'(\cdot)}(\Omega)$ where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$,*

$$\left| \int_{\Omega} uv \, dx \right| \leq \left(\frac{1}{p_-} + \frac{1}{(p')_-} \right) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)} \leq 2 \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)}.$$

- (2) *If $p_1, p_2 \in C_+(\overline{\Omega})$, $p_1(x) \leq p_2(x)$ for any $x \in \Omega$, then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$, whose norm does not exceed $|\Omega| + 1$.*

LEMMA 2.2 [20]. *If we denote*

$$\rho(u) = \int_{\Omega} |u|^{p(x)} dx, \quad \forall u \in L^{p(x)}(\Omega),$$

then

$$\min\{|u|_{p(x)}^{p_-}, |u|_{p(x)}^{p_+}\} \leq \rho(u) \leq \max\{|u|_{p(x)}^{p_-}, |u|_{p(x)}^{p_+}\}.$$

LEMMA 2.3 [20]. *Let $p \in C_+(\overline{\Omega})$. Then $W^{k,p(x)}(\Omega)$ is a separable and reflexive Banach space.*

LEMMA 2.4 [23, 24]. *Let $p \in C_+(\overline{\Omega})$. Then, for $u \in W_0^{1,p(\cdot)}(\Omega)$, the $p(\cdot)$ -Poincaré inequality*

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)}$$

holds, where the positive constant C depends on p and Ω .

LEMMA 2.5 [14, 19]. *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set with Lipschitz boundary and let $p(x) \in C_+(\overline{\Omega})$ with $1 < p_- \leq p_+ < N$ satisfy the log-Hölder continuity condition (1.2). If $q \in L^\infty(\Omega)$ with $q_- > 1$ satisfies*

$$q(x) \leq p^*(x) := \frac{Np(x)}{N - p(x)}, \quad \forall x \in \Omega,$$

then

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

and the imbedding is compact if $\inf_{x \in \Omega}(p^(x) - q(x)) > 0$.*

Now we begin to prove some *a priori* important estimates for entropy solutions of problem (1.1).

PROPOSITION 2.6. *If u is an entropy solution of problem (1.1), then there exists a positive constant C such that, for all $k > 1$,*

$$\text{meas}\{|u| > k\} \leq \frac{C(M + 1)^{(p^*)_- / p_-}}{k^{(p^*)_- - (1 - 1/p_-)}}$$

holds, where

$$M = \frac{p_-}{p_- - 1} \|f\|_{L^1(\Omega)} + \frac{p_-(p_+ - 1)}{p_+(p_- - 1)} \int_{\Omega} |F|^{p'(x)} dx.$$

PROOF. Choosing $\phi = 0$ in the entropy equality (1.4) and using Young’s inequality, we obtain

$$\begin{aligned} \int_{\Omega} |\nabla T_k(u)|^{p(x)} dx &= \int_{\{|u| \leq k\}} |\nabla u|^{p(x)} dx \\ &\leq \frac{kp_-}{p_- - 1} \|f\|_{L^1(\Omega)} + \frac{p_-(p_+ - 1)}{p_+(p_- - 1)} \int_{\Omega} |F|^{p'(x)} dx, \end{aligned}$$

which implies that, for all $k > 1$,

$$\frac{1}{k} \int_{\Omega} |\nabla T_k(u)|^{p(x)} dx \leq M_1 + \frac{M_2}{k} \leq M := M_1 + M_2, \tag{2.1}$$

where

$$M_1 = \frac{p_-}{p_- - 1} \|f\|_{L^1(\Omega)}, \quad M_2 = \frac{p_-(p_+ - 1)}{p_+(p_- - 1)} \int_{\Omega} |F|^{p'(x)} dx.$$

Recalling the Sobolev embedding theorem in Lemmas 2.5 and 2.1, we have the continuous embedding

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p^*(x)}(\Omega) \hookrightarrow L^{(p^*)_-}(\Omega),$$

where $p^*(x) = (Np(x)/N - p(x))$ and $(p^*)_- = (Np_-/N - p_-)$. It follows from Lemma 2.2 and (2.1) that, for every $k > 1$,

$$\begin{aligned} \|T_k(u)\|_{(p^*)_-} &= |T_k(u)|_{(p^*)_-} \leq C |\nabla T_k(u)|_{p(\cdot)} \\ &\leq C \left(\int_{\Omega} |\nabla T_k(u)|^{p(x)} dx \right)^{\beta} \leq C(Mk)^{\beta}, \end{aligned}$$

where

$$\beta = \begin{cases} \frac{1}{p_-} & \text{if } |\nabla T_k(u)|_{p(\cdot)} \geq 1, \\ \frac{1}{p_+} & \text{if } |\nabla T_k(u)|_{p(\cdot)} \leq 1. \end{cases}$$

Noting that $\{|u| \geq k\} = \{|T_k(u)| \geq k\}$,

$$\text{meas}\{|u| > k\} \leq \left(\frac{\|T_k(u)\|_{(p^*)_-}}{k} \right)^{(p^*)_-} \leq \frac{CM^{\beta(p^*)_-}}{k^{(p^*)_-(1-\beta)}} \leq \frac{C(M+1)^{(p^*)_-/p_-}}{k^{(p^*)_-(1-(1/p_-))}}.$$

This finishes the proof. □

PROPOSITION 2.7. *If u is an entropy solution of problem (1.1), then, for a given $a > 0$,*

$$\lim_{k \rightarrow \infty} \int_{\{k \leq |u| \leq k+a\}} |\nabla u|^{p(x)} dx = 0.$$

PROOF. For given $a, k > 0$, define the function $T_{k,a}(s) = T_a(s - T_k(s))$ as

$$T_{k,a}(s) = \begin{cases} s - k \text{ sign}(s) & \text{if } k \leq |s| < k + a, \\ a \text{ sign}(s) & \text{if } |s| \geq k + a, \\ 0 & \text{if } |s| \leq k. \end{cases}$$

Using $T_{k,a}(u)$ to take the place of $T_k(u - \phi)$ in (1.4), we find

$$\int_{\{k \leq |u| \leq k+a\}} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla u dx \leq \int_{\Omega} f T_{k,a}(u) dx + \int_{\{k \leq |u| \leq k+a\}} F \cdot \nabla u dx.$$

In view of Young’s inequality,

$$\begin{aligned} & \int_{\{k \leq |u| \leq k+a\}} |\nabla u|^{p(x)} dx \\ & \leq a \int_{\{|u| > k\}} |f| dx + \int_{\{k \leq |u| \leq k+a\}} \left(\frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{p(x) - 1}{p(x)} |F|^{p'(x)} \right) dx \\ & \leq a \int_{\{|u| > k\}} |f| dx + \frac{1}{p_-} \int_{\{k \leq |u| \leq k+a\}} |\nabla u|^{p(x)} dx \\ & \quad + \frac{p_+ - 1}{p_+} \int_{\{k \leq |u| \leq k+a\}} |F|^{p'(x)} dx, \end{aligned}$$

which implies that

$$\begin{aligned} & \int_{\{k \leq |u| \leq k+a\}} |\nabla u|^{p(x)} dx \\ & \leq \frac{ap_-}{p_- - 1} \int_{\{|u| > k\}} |f| dx + \frac{p_-(p_+ - 1)}{p_+(p_- - 1)} \int_{\{k \leq |u| \leq k+a\}} |F|^{p'(x)} dx. \end{aligned}$$

Therefore, we obtain that

$$\lim_{k \rightarrow \infty} \int_{\{k \leq |u| \leq k+a\}} |\nabla u|^{p(x)} dx = 0.$$

This finishes the proof. □

3. The proofs of main results

In this section we are ready to prove the main results. First we prove the existence and uniqueness of the entropy solutions for problem (1.1).

PROOF OF THEOREM 1.4. (1) *Existence of entropy solutions.* We first introduce the approximate problems. Find two sequences of $C_0^\infty(\Omega)$ functions $\{f_n\}$ and $\{F_n\}$ strongly converging respectively to f in $L^1(\Omega)$ and to F in $(L^{p'(\cdot)}(\Omega))^N$ such that

$$\|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}, \quad \int_{\Omega} |F_n|^{p'(x)} dx \leq \int_{\Omega} |F|^{p'(x)} dx. \tag{3.1}$$

Then we consider the approximate problems of (1.1),

$$\begin{cases} -\operatorname{div}(|\nabla u_n|^{p(x)-2} \nabla u_n) = f_n - \operatorname{div}(F_n) & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

By employing the variation methods and arguments in [18, Theorem 4.2] or [26], we easily find a unique weak solution $u_n \in W_0^{1,p(\cdot)}(\Omega)$ of problem (3.2), which is obviously an entropy solution. Our aim is to prove that a subsequence of these approximate solutions $\{u_n\}$ converges to a measurable function u , which is an entropy solution of problem (1.1). We will divide the proof into several steps. Some of the reasoning is based on the ideas developed in [6, 11, 25].

Choosing $T_k(u_n)$ as a test function in (3.2) and applying Young’s inequality on the right-hand side,

$$\int_{\Omega} |\nabla T_k(u_n)|^{p(x)} dx \leq k \|f_n\|_{L^1(\Omega)} + \int_{\Omega} \frac{1}{p(x)} |\nabla T_k(u_n)|^{p(x)} dx + \int_{\Omega} \frac{1}{p'(x)} |F_n|^{p'(x)} dx,$$

which implies that

$$\begin{aligned} \int_{\Omega} |\nabla T_k(u_n)|^{p(x)} dx &\leq \frac{kp_-}{p_- - 1} \|f_n\|_{L^1(\Omega)} + \frac{p_-(p_+ - 1)}{p_+(p_- - 1)} \int_{\Omega} |F_n|^{p'(x)} dx \\ &\leq \frac{kp_-}{p_- - 1} \|f\|_{L^1(\Omega)} + \frac{p_-(p_+ - 1)}{p_+(p_- - 1)} \int_{\Omega} |F|^{p'(x)} dx. \end{aligned} \tag{3.3}$$

Step 1. Construct a subsequence $\{u^n\}$ of $\{u_n\}$ and a limit function u such that, for every positive integer k ,

$$T_k u^n \rightarrow T_k u \quad \text{strongly in } L^q(\Omega),$$

where $q = (p_- + (p^*)_-)/2$.

As $\{T_1 u_n\}$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$ from (3.3), recalling Lemma 2.5, there exist a subsequence $\{u_n^1\}$ of $\{u_n\}$ and a function $v_1 \in L^q(\Omega)$ with $|v_1| \leq 1$ such that

$$T_1 u_n^1 \rightarrow v_1 \quad \text{strongly in } L^q(\Omega).$$

As $\{T_2 u_n^1\}$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$, there exist a subsequence $\{u_n^2\}$ of $\{u_n^1\}$ and a function $v_2 \in L^q(\Omega)$ with $|v_2| \leq 2$ such that

$$T_2 u_n^2 \rightarrow v_2 \quad \text{strongly in } L^q(\Omega).$$

By the same procedures, for every positive integer k , we find there exist a subsequence $\{u_n^k\}$ of $\{u_n^{k-1}\}$ and a function $v_k \in L^q(\Omega)$ with $|v_k| \leq k$ such that

$$T_k u_n^k \rightarrow v_k \quad \text{strongly in } L^q(\Omega).$$

Set $u^n = u_n^n, n = 1, 2, \dots$. Then we have, for every positive integer k ,

$$T_k u^n \rightarrow v_k \quad \text{strongly in } L^q(\Omega).$$

Let l, k be any two positive integers with $l < k$. Noting the equality

$$T_l(T_k u^n) = T_l u^n,$$

and sending $n \rightarrow \infty$, we conclude that

$$T_l v_k = v_l.$$

It follows from the construction of v_k that

$$\begin{aligned} \Omega &= \{0 \leq |v_1| < 1\} \cup \{|v_1| = 1\} \cup \{\text{a negligible set}\} \\ &= \{0 \leq |v_1| < 1\} \cup \{1 \leq |v_2| < 2\} \cup \{|v_2| = 2\} \cup \{\text{a negligible set}\} \\ &= \dots \\ &= \bigcup_{k=1} \{k - 1 \leq |v_k| < k\} \bigcup \{\text{a negligible set}\}. \end{aligned}$$

Then we define a measurable function u in Ω by

$$u(x) = \begin{cases} v_k(x) & \text{when } k - 1 \leq |v_k(x)| < k \text{ for } k = 1, 2, \dots \\ 0 & \text{a negligible set,} \end{cases}$$

which satisfies the condition

$$T_k u = v_k \quad \text{for } k = 1, 2, \dots$$

Step 2. Prove the convergence in measure of $\{u^n\}$ and find its subsequence which is almost everywhere (a.e.) convergent in Ω .

For every fixed $\epsilon > 0$, and every positive integer k , we know that

$$\{|u^n - u^m| > \epsilon\} \subset \{|u^n| > k\} \cup \{|u^m| > k\} \cup \{|T_k(u^n) - T_k(u^m)| > \epsilon\}.$$

Recalling the convergence of $\{T_k u^n\}$ in $L^q(\Omega)$ and Proposition 2.6, we conclude that

$$\limsup_{n,m \rightarrow \infty} \text{meas}\{|u^n - u^m| > \epsilon\} \leq C(p(\cdot), \|f\|_{L^1(\Omega)}, |F|_{p(\cdot)})k^{-\alpha},$$

where $\alpha = (p^*)_-(1 - (1/p_-)) > 0$.

Because of the arbitrariness of k , we prove that

$$\limsup_{n,m \rightarrow \infty} \text{meas}\{|u^n - u^m| > \epsilon\} = 0,$$

which implies the convergence in measure of $\{u^n\}$, and then we find an a.e. convergent subsequence (still denoted by $\{u^n\}$) in Ω such that

$$u^n \rightarrow u \quad \text{a.e. in } \Omega. \tag{3.4}$$

In the following we denote $\{u^n\}$ by $\{u_n\}$ for simplicity of notation.

Step 3. We prove that $T_k(u_n)$ strongly converges to $T_k(u)$ in $W_0^{1,p(\cdot)}(\Omega)$, for every $k > 0$.

Since the set $\{(k, h) : k, h = 1, 2, \dots\}$ is countable, using the boundedness of $T_k(u_n)$ and $T_{2k}(u_n - T_h(u_n))$ in $W_0^{1,p(\cdot)}(\Omega)$ and the diagonal procedures, we draw a subsequence (still denoted by $\{u_n\}$) from $\{u_n\}$ such that, for every pair of positive

integers $k, h,$

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,p(\cdot)}(\Omega), \tag{3.5}$$

$$T_{2k}(u_n - T_h(u_n)) \rightharpoonup T_{2k}(u - T_h(u)) \quad \text{weakly in } W_0^{1,p(\cdot)}(\Omega) \tag{3.6}$$

and

$$T_{2k}(u_n - T_h(u_n)) + T_k(u_n) - T_k(u) \rightharpoonup T_{2k}(u - T_h(u)) \quad \text{weakly in } W_0^{1,p(\cdot)}(\Omega). \tag{3.7}$$

First we prove that this subsequence $\{u_n\}$ has the property that $T_k u_n$ strongly converges to $T_k u$ in $W_0^{1,p(\cdot)}(\Omega)$ for every positive integer k .

Fix a positive integer k . Let h be a positive integer satisfying $h > k$. We choose

$$w_n = T_{2k}(u_n - T_h(u_n)) + T_k(u_n) - T_k(u)$$

as a test function in (3.2). If we set $M = 4k + h$, then it is easy to see that $\nabla w_n = 0$ where $|u_n| > M$. Therefore, we may write the weak form of (3.2) as

$$\int_{\Omega} |\nabla T_M(u_n)|^{p(x)-2} \nabla T_M(u_n) \cdot \nabla w_n \, dx = \int_{\Omega} f_n w_n \, dx + \int_{\Omega} F_n \cdot \nabla w_n \, dx.$$

Splitting the integral on the left-hand side on the sets where $|u_n| \leq k$ and where $|u_n| > k$, and discarding some nonnegative terms, we find

$$\begin{aligned} & \int_{\Omega} |\nabla T_M(u_n)|^{p(x)-2} \nabla T_M(u_n) \cdot \nabla T_{2k}(u_n - T_h(u_n)) + T_k(u_n) - T_k(u) \, dx \\ & \geq \int_{\Omega} |\nabla T_k(u_n)|^{p(x)-2} \nabla T_k(u_n) \cdot \nabla (T_k(u_n) - T_k(u)) \, dx \\ & \quad - \int_{\{|u_n|>k\}} \|\nabla T_M(u_n)\|^{p(x)-2} \|\nabla T_M(u_n)\| \|\nabla T_k(u)\| \, dx. \end{aligned}$$

It follows from the above inequality that

$$\begin{aligned} & \int_{\Omega} (|\nabla T_k(u_n)|^{p(x)-2} \nabla T_k(u_n) - |\nabla T_k(u)|^{p(x)-2} \nabla T_k(u)) \cdot \nabla (T_k(u_n) - T_k(u)) \, dx \\ & \leq \int_{\{|u_n|>k\}} \|\nabla T_M(u_n)\|^{p(x)-2} \|\nabla T_M(u_n)\| \|\nabla T_k(u)\| \, dx \\ & \quad + \int_{\Omega} f_n T_{2k}(u_n - T_h(u_n)) + T_k(u_n) - T_k(u) \, dx \\ & \quad + \int_{\Omega} F_n \cdot \nabla T_{2k}(u_n - T_h(u_n)) + T_k(u_n) - T_k(u) \, dx \\ & \quad - \int_{\Omega} |\nabla T_k(u)|^{p(x)-2} \nabla T_k(u) \cdot \nabla (T_k(u_n) - T_k(u)) \, dx \\ & := I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.8}$$

Using the properties of $L^{p(x)}(\Omega)$ and the similar estimates as in [25], we can show that the limits of I_1, I_2, I_3 and I_4 are zeros when n , and then h tend to infinity, respectively.

Therefore, passing to the limits in (3.8) as n , and then h tend to infinity, we deduce that

$$\lim_{n \rightarrow +\infty} E(n) = 0,$$

where

$$E(n) = \int_{\Omega} (|\nabla T_k(u_n)|^{p(x)-2} \nabla T_k(u_n) - |\nabla T_k(u)|^{p(x)-2} \nabla T_k(u)) \cdot \nabla(T_k(u_n) - T_k(u)) \, dx.$$

We recall the following well-known inequalities: for any two real vectors $a, b \in \mathbb{R}^N$,

$$(a|a|^{p-2} - b|b|^{p-2})(a - b) \geq c(p)|a - b|^p \quad \text{if } p \geq 2$$

and for every $\varepsilon \in (0, 1]$,

$$|a - b|^p \leq c(p)\varepsilon^{(p-2)/p}(a|a|^{p-2} - b|b|^{p-2})(a - b) + \varepsilon|b|^p \quad \text{if } 1 < p < 2,$$

where $c(p) = (2^{1-p}/p - 1)$ when $p \geq 2$ and $c(p) = (3^{2-p}/p - 1)$ when $1 < p < 2$.

Thus

$$\int_{\{x \in \Omega: p(x) \geq 2\}} |\nabla T_k(u_n) - \nabla T_k(u)|^{p(x)} \, dx \leq 2^{p_+-1}(p_+ - 1)E(n) \tag{3.9}$$

and

$$\begin{aligned} & \int_{\{x \in \Omega: 1 < p(x) < 2\}} |\nabla T_k(u_n) - \nabla T_k(u)|^{p(x)} \, dx \\ & \leq \frac{3^{2-p_-}}{p_- - 1} \cdot \varepsilon^{(p_- - 2)/p_-} E(n) + \varepsilon \int_{\Omega} |\nabla T_k(u)|^{p(x)} \, dx. \end{aligned} \tag{3.10}$$

Since $E(n) \rightarrow 0$ as $n \rightarrow +\infty$, then using the arbitrariness of ε and the fact that $\nabla T_k(u)$ is bounded in $(L^{p(\cdot)}(\Omega))^N$, we conclude that

$$\lim_{n \rightarrow +\infty} \int_{\Omega} |\nabla T_k(u_n) - \nabla T_k(u)|^{p(x)} \, dx = 0, \tag{3.11}$$

which implies that, for every positive integer k ,

$$T_k(u_n) \rightarrow T_k(u) \quad \text{strongly in } W_0^{1,p(\cdot)}(\Omega).$$

Moreover, for every positive real number r , there exists a positive integer k such that $r \leq k$. Recalling the fact that $T_r(T_k(\cdot)) = T_r(\cdot)$ and that

$$T_r(T_k(u_n)) \rightarrow T_r(T_k(u)) \quad \text{strongly in } W_0^{1,p(\cdot)}(\Omega),$$

we have, for every $r > 0$,

$$T_r(u_n) \rightarrow T_r(u) \quad \text{strongly in } W_0^{1,p(\cdot)}(\Omega), \tag{3.12}$$

which also implies that

$$|\nabla T_r(u_n)|^{p(x)-2} \nabla T_r(u_n) \rightarrow |\nabla T_r(u)|^{p(x)-2} \nabla T_r(u) \quad \text{in } (L^{p'(\cdot)}(\Omega))^N. \tag{3.13}$$

Step 4. Show that u is an entropy solution.

Now we choose $v_n = T_k(u_n - \phi)$ as a test function in (3.2) for $k > 0$ and $\phi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$. We note that, if $L = k + \|\phi\|_{L^\infty(\Omega)}$, then

$$\begin{aligned} & \int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla T_k(u_n - \phi) \, dx \\ &= \int_{\Omega} |\nabla T_L(u_n)|^{p(x)-2} \nabla T_L(u_n) \cdot \nabla T_k(u_n - \phi) \, dx \end{aligned}$$

and

$$\begin{aligned} & \int_{\Omega} |\nabla T_L(u_n)|^{p(x)-2} \nabla T_L(u_n) \cdot \nabla T_k(u_n - \phi) \, dx \\ &= \int_{\Omega} f_n T_k(u_n - \phi) \, dx + \int_{\Omega} F_n \cdot \nabla T_k(u_n - \phi) \, dx. \end{aligned}$$

Using (3.5) and (3.13), we pass to the limits as n tends to infinity to conclude that

$$\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla T_k(u - \phi) \, dx = \int_{\Omega} f T_k(u - \phi) \, dx + \int_{\Omega} F \cdot \nabla T_k(u - \phi) \, dx,$$

for every $k > 0$ and every $\phi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$. This finishes the proof of the existence of entropy solutions. \square

(2) *Uniqueness of entropy solutions.* Suppose that v is another entropy solution of problem (1.1); we will show that $u = v$ a.e. in Ω . We write the entropy equality (1.4) corresponding to solution u with test function $T_h(v)$ and v with test function $T_h(u)$. Adding up both equalities, we find that

$$\begin{aligned} & \int_{\{|u-T_h(v)| \leq k\}} [|\nabla u|^{p(x)-2} \nabla u - F] \cdot \nabla T_k(u - T_h(v)) \, dx \\ &+ \int_{\{|v-T_h(u)| \leq k\}} [|\nabla v|^{p(x)-2} \nabla v - F] \cdot \nabla T_k(v - T_h(u)) \, dx \\ &= \int_{\Omega} f [T_k(u - T_h(v)) + T_k(v - T_h(u))] \, dx. \end{aligned} \tag{3.14}$$

With similar arguments as in [11, 32] by some modifications, after discarding some nonnegative but uninteresting terms and passing to the limits in (3.14) as $h \rightarrow \infty$, we can reach the conclusion. Therefore, we complete the proof of Theorem 1.4. \square

REMARK 3.1. We note that it is not obvious that the entropy solution u of problem (1.1) can be a weak solution. However, if $\mu \in W^{-1,p'(\cdot)}(\Omega)$ or $\mu \in L^{m(\cdot)}(\Omega)$, $m(\cdot) = (p^*(\cdot))'$, then $u \in W_0^{1,p(\cdot)}(\Omega)$. Under the above assumptions, the existence and uniqueness of weak solutions for problem (1.1) is a straightforward generalization of [18].

Next, we prove that the entropy solution u is also a renormalized solution of problem (1.1) and the renormalized solution is unique.

PROOF OF THEOREM 1.5. (1) *The entropy solution is a renormalized solution.* First we observe that the entropy solution u in Theorem 1.4 satisfies the requirement that $T_k(u) \in W_0^{1,p(\cdot)}(\Omega)$, for given $k > 0$. Choosing $a = 1$ in Proposition 2.7, we obtain the renormalized condition, that is,

$$\lim_{k \rightarrow \infty} \int_{\{k \leq |u| \leq k+1\}} |\nabla u|^{p(x)} dx = 0.$$

Let $\{u_n\}$ be a sequence constructed in (3.2), which satisfies the requirement that $T_k(u_n)$ strongly converges to $T_k(u)$ in $W_0^{1,p(x)}(\Omega)$, for every $k > 0$.

Let $S \in W^{1,\infty}(\mathbb{R})$ be such that $\text{supp } S \subset [-M, M]$ for some $M > 0$. For every $\varphi \in C_0^\infty(\Omega)$, $S(u_n)\varphi \in W_0^{1,p(x)}(\Omega)$ is a test function in (3.2). It yields

$$\begin{aligned} & \int_{\Omega} [S(u_n)|\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla \varphi + S'(u_n)|\nabla u_n|^{p(x)} \varphi] dx \\ &= \int_{\Omega} [f_n S(u_n)\varphi + F_n \cdot \nabla(S(u_n)\varphi)] dx. \end{aligned} \tag{3.15}$$

For the left-hand side of (3.15), because $\text{supp } S' \subset [-M, M]$ we know that

$$S(u_n)|\nabla u_n|^{p(x)-2} \nabla u_n = S(u_n)|\nabla T_M(u_n)|^{p(x)-2} \nabla T_M(u_n)$$

and

$$S'(u_n)|\nabla u_n|^{p(x)} = S'(u_n)|\nabla T_M(u_n)|^{p(x)}.$$

Using (3.4), (3.12) and (3.13),

$$S(u_n)|\nabla T_M(u_n)|^{p(x)-2} \nabla T_M(u_n) \rightarrow S(u)|\nabla T_M(u)|^{p(x)-2} \nabla T_M(u) \quad \text{in } (L^{p'(\cdot)}(\Omega))^N$$

and

$$S'(u_n)|\nabla T_M(u_n)|^{p(x)} \rightarrow S'(u)|\nabla T_M(u)|^{p(x)} \quad \text{in } L^1(\Omega).$$

Noting that

$$\begin{aligned} S(u)|\nabla T_M(u)|^{p(x)-2} \nabla T_M(u) &= S(u)|\nabla u|^{p(x)-2} \nabla u, \\ S'(u)|\nabla T_M(u)|^{p(x)} &= S'(u)|\nabla u|^{p(x)}, \end{aligned}$$

we deduce that

$$S(u_n)|\nabla u_n|^{p(x)-2}\nabla u_n \rightarrow S(u)|\nabla u|^{p(x)-2}\nabla u \quad \text{in } (L^{p'(\cdot)}(\Omega))^N$$

and

$$S'(u_n)|\nabla u_n|^{p(x)} \rightarrow S'(u)|\nabla u|^{p(x)} \quad \text{in } L^1(\Omega).$$

For the right-hand side of (3.15), since $\nabla[S(u_n)\varphi] = \nabla[S(T_M(u_n))\varphi]$, thanks to the strong convergence of $T_M(u_n)$, f_n and F_n , it is easy to pass to the limits in the terms on the right-hand side. Therefore, we obtain

$$\begin{aligned} & \int_{\Omega} [S(u)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi + S'(u)|\nabla u|^{p(x)}\varphi] dx \\ &= \int_{\Omega} [fS(u)\varphi + F \cdot \nabla(S(u)\varphi)] dx, \end{aligned}$$

which is (ii) in Definition 1.3. This completes the proof of the existence of renormalized solutions.

(2) *Uniqueness of renormalized solutions.* Now we prove the uniqueness of renormalized solutions for problem (1.1) by choosing an appropriate test function. Let u and v be two renormalized solutions for problem (1.1). Fix a positive number k . For $\sigma > 0$, let S_{σ} be the function defined by

$$\begin{cases} S_{\sigma}(r) = r & \text{if } |r| < \sigma, \\ S_{\sigma}(r) = (\sigma + \frac{1}{2}) \mp \frac{1}{2}(r \mp (\sigma + 1))^2 & \text{if } \sigma \leq \pm r \leq \sigma + 1, \\ S_{\sigma}(r) = \pm(\sigma + \frac{1}{2}) & \text{if } \pm r > \sigma + 1. \end{cases} \quad (3.16)$$

It is obvious that

$$\begin{cases} S'_{\sigma}(r) = 1 & \text{if } |r| < \sigma, \\ S'_{\sigma}(r) = \sigma + 1 - |r| & \text{if } \sigma \leq |r| \leq \sigma + 1, \\ S'_{\sigma}(r) = 0 & \text{if } |r| > \sigma + 1. \end{cases}$$

It is easy to check $S_{\sigma} \in W^{2,\infty}(\mathbb{R})$ with $\text{supp } S'_{\sigma} \subset [-\sigma - 1, \sigma + 1]$ and $\text{supp } S''_{\sigma} \subset [\sigma, \sigma + 1] \cup [-\sigma - 1, -\sigma]$. Therefore, we may take $S = S'_{\sigma}$ in (1.5) to have

$$\begin{aligned} & \int_{\Omega} [S'_{\sigma}(u)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi + S''_{\sigma}(u)|\nabla u|^{p(x)}\varphi] dx \\ &= \int_{\Omega} [fS'_{\sigma}(u)\varphi + F \cdot \nabla(S'_{\sigma}(u)\varphi)] dx \end{aligned}$$

and

$$\begin{aligned} & \int_{\Omega} [S'_{\sigma}(v)|\nabla v|^{p(x)-2}\nabla v \cdot \nabla\varphi + S''_{\sigma}(v)|\nabla v|^{p(x)}\varphi] dx \\ &= \int_{\Omega} [fS'_{\sigma}(v)\varphi + F \cdot \nabla(S'_{\sigma}(v)\varphi)] dx. \end{aligned}$$

As $T_k(S_\sigma(u) - S_\sigma(v)) \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, we plug $\varphi = T_k(S_\sigma(u) - S_\sigma(v))$ as a test function in the above equalities and subtract them to deduce that

$$J_1 + J_2 = J_3 + J_4 + J_5, \tag{3.17}$$

where

$$\begin{aligned} J_1 &= \int_{\Omega} (S'_\sigma(u)|\nabla u|^{p(x)-2}\nabla u - S'_\sigma(v)|\nabla v|^{p(x)-2}\nabla v) \cdot \nabla T_k(S_\sigma(u) - S_\sigma(v)) \, dx, \\ J_2 &= \int_{\Omega} [S''_\sigma(u)|\nabla u|^{p(x)} - S''_\sigma(v)|\nabla v|^{p(x)}]T_k(S_\sigma(u) - S_\sigma(v)) \, dx, \\ J_3 &= \int_{\Omega} f(S'_\sigma(u) - S'_\sigma(v))T_k(S_\sigma(u) - S_\sigma(v)) \, dx, \\ J_4 &= \int_{\Omega} T_k(S_\sigma(u) - S_\sigma(v))F \cdot \nabla(S'_\sigma(u) - S'_\sigma(v)) \, dx, \\ J_5 &= \int_{\Omega} F(S'_\sigma(u) - S'_\sigma(v)) \cdot \nabla T_k(S_\sigma(u) - S_\sigma(v)) \, dx. \end{aligned}$$

We estimate J_1, J_2 and J_3 one by one. Writing

$$\begin{aligned} J_1 &= \int_{\Omega} [|\nabla S_\sigma(u)|^{p(x)-2}\nabla S_\sigma(u) - |\nabla S_\sigma(v)|^{p(x)-2}\nabla S_\sigma(v)] \cdot \nabla T_k(S_\sigma(u) - S_\sigma(v)) \, dx \\ &\quad + \int_{\Omega} [S'_\sigma(u) - S'_\sigma(v)]|S'_\sigma(u)|^{p(x)-2}|\nabla u|^{p(x)-2}\nabla u \cdot \nabla T_k(S_\sigma(u) - S_\sigma(v)) \, dx \\ &\quad - \int_{\Omega} [S'_\sigma(v) - S'_\sigma(v)]|S'_\sigma(v)|^{p(x)-2}|\nabla v|^{p(x)-2}\nabla v \cdot \nabla T_k(S_\sigma(u) - S_\sigma(v)) \, dx \\ &:= J_1^1 + J_1^2 + J_1^3, \end{aligned}$$

and setting $\sigma \geq k$,

$$J_1^1 \geq \int_{\{|u-v|\leq k\} \cap \{|u|, |v|\leq k\}} (|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v) \cdot \nabla(u - v) \, dx. \tag{3.18}$$

Recalling that $\text{supp } S'_\sigma \subset [-\sigma - 1, \sigma + 1]$ and $\text{supp } S''_\sigma \subset [\sigma, \sigma + 1] \cup [-\sigma - 1, -\sigma]$, we obtain

$$\begin{aligned} |J_1^2| &\leq \int_{\{\sigma \leq |u| \leq \sigma+1\}} |\nabla u|^{p(x)} \, dx \\ &\quad + \int_{\{\sigma \leq |u| \leq \sigma+1\} \cap \{|v| \leq \sigma+1\} \cap \{|S_\sigma(u) - S_\sigma(v)| \leq k\}} |\nabla u|^{p(x)-1} |\nabla v| \, dx \end{aligned}$$

$$\begin{aligned} &\leq 2 \left(\int_{\{\sigma \leq |u| \leq \sigma+1\}} |\nabla u|^{p(x)} dx \right. \\ &\quad \left. + \int_{\{\sigma \leq |u| \leq \sigma+1\} \cap \{\sigma-k \leq |v| \leq \sigma+1\}} |\nabla u|^{p(x)-1} |\nabla v| dx \right) \\ &\leq C \left(\int_{\{\sigma \leq |u| \leq \sigma+1\}} |\nabla u|^{p(x)} dx + \int_{\{\sigma-k \leq |v| \leq \sigma+1\}} |\nabla v|^{p(x)} dx \right). \end{aligned}$$

Moreover, we may get the similar estimate for J_1^3 . Furthermore,

$$|J_2| \leq C \left(\int_{\{\sigma \leq |u| \leq \sigma+1\}} |\nabla u|^{p(x)} dx + \int_{\{\sigma \leq |v| \leq \sigma+1\}} |\nabla v|^{p(x)} dx \right).$$

From the above estimates and Definition 1.3(i), we obtain

$$\lim_{\sigma \rightarrow \infty} (|J_1^2| + |J_1^3| + |J_2|) = 0.$$

Observing that

$$f(S'_\sigma(u) - S'_\sigma(v)) \rightarrow 0 \quad \text{strongly in } L^1(\Omega)$$

as $\sigma \rightarrow \infty$ and using the Lebesgue dominated convergence theorem, we deduce that

$$\lim_{\sigma \rightarrow \infty} |J_3| = 0.$$

From Lemma 2.1 and $S_\sigma \in W^{2,\infty}(\mathbb{R})$,

$$\begin{aligned} |J_4| &\leq k \int_{\Omega} |F| (|S''_\sigma(u) \nabla u| + |S''_\sigma(v) \nabla v|) dx \\ &\leq k \int_{\{\sigma \leq |u| \leq \sigma+1\}} |F| \cdot |\nabla u| dx + \int_{\{\sigma \leq |v| \leq \sigma+1\}} |F| \cdot |\nabla v| dx \\ &\leq 2k |F|_{p'(\cdot)} (|\nabla u|_{p(\cdot), \{\sigma \leq |u| \leq \sigma+1\}} + |\nabla v|_{p(\cdot), \{\sigma \leq |v| \leq \sigma+1\}}). \end{aligned}$$

By means of Lemma 2.2 and Definition 1.3(i), we obtain

$$\lim_{\sigma \rightarrow \infty} |J_4| = 0.$$

Next we show that

$$\lim_{\sigma \rightarrow \infty} |J_5| = 0.$$

Write

$$\begin{aligned} |J_5| &\leq \int_{\Omega} |F| |S'_\sigma(u) - S'_\sigma(v)| T'_k(S_\sigma(u) - S_\sigma(v)) |S'_\sigma(u) \nabla u - S'_\sigma(v) \nabla v| dx \\ &= \int_{\Omega} G(u, v) dx. \end{aligned} \tag{3.19}$$

We divide the estimate of J_5 into several cases according to the different integration sets. Denote

$$\begin{aligned}
 E_1 &= \{|u| \leq \sigma, |v| \leq \sigma\} \cup \{|u| \geq \sigma + 1, |v| \geq \sigma + 1\}, \\
 E_2 &= \{|u| \leq \sigma, |v| \geq \sigma + 1\} \cup \{|v| \leq \sigma, |u| \geq \sigma + 1\} = E_{21} \cup E_{22}, \\
 E_3 &= \{|u| \leq \sigma, \sigma \leq |v| \leq \sigma + 1\} \cup \{|v| \leq \sigma, \sigma \leq |u| \leq \sigma + 1\} = E_{31} \cup E_{32}, \\
 E_4 &= \{\sigma \leq |u| \leq \sigma + 1, \sigma \leq |v| \leq \sigma + 1\}, \\
 E_5 &= \{\sigma \leq |u| \leq \sigma + 1, |v| \geq \sigma + 1\} \cup \{\sigma \leq |v| \leq \sigma + 1, |u| \geq \sigma + 1\}.
 \end{aligned}$$

It is obvious that $\Omega = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$.

From the definition of S_σ and $\text{supp } S'_\sigma \subset [-\sigma - 1, \sigma + 1]$, we deduce the following estimates.

Estimate on E_1 .

$$\int_{E_1} G(u, v) \, dx = 0.$$

Estimate on E_2 .

$$\begin{aligned}
 \int_{E_{21}} G(u, v) \, dx &\leq \int_{\{|S_\sigma(u) - S_\sigma(v)| \leq k\} \cap \{|u| \leq \sigma\} \cap \{|v| \geq \sigma + 1\}} |F| |\nabla u| \, dx \\
 &\leq \int_{\{\sigma - k \leq |u| \leq \sigma\}} |F| |\nabla u| \, dx
 \end{aligned}$$

and

$$\int_{E_{22}} G(u, v) \, dx \leq \int_{\{\sigma - k \leq |v| \leq \sigma\}} |F| |\nabla v| \, dx.$$

Estimate on E_3 .

$$\begin{aligned}
 \int_{E_{31}} G(u, v) \, dx &\leq \int_{\{|S_\sigma(u) - S_\sigma(v)| \leq k\} \cap \{|u| \leq \sigma\} \cap \{\sigma \leq |v| \leq \sigma + 1\}} G(u, v) \, dx \\
 &\leq 2 \left(\int_{\{\sigma - k \leq |u| \leq \sigma\}} |F| |\nabla u| \, dx + \int_{\{\sigma - k \leq |v| \leq \sigma + 1\}} |F| |\nabla v| \, dx \right)
 \end{aligned}$$

and

$$\int_{E_{32}} G(u, v) \, dx \leq 2 \left(\int_{\{\sigma - k \leq |u| \leq \sigma + 1\}} |F| |\nabla u| \, dx + \int_{\{\sigma - k \leq |v| \leq \sigma\}} |F| |\nabla v| \, dx \right).$$

Estimate on E_4 .

$$\int_{E_4} G(u, v) \, dx \leq 2 \left(\int_{\{\sigma \leq |u| \leq \sigma + 1\}} |F| |\nabla u| \, dx + \int_{\{\sigma \leq |v| \leq \sigma + 1\}} |F| |\nabla v| \, dx \right).$$

Estimate on E_5 .

$$\int_{E_5} G(u, v) \, dx \leq \int_{\{\sigma \leq |u| \leq \sigma + 1\}} |F| |\nabla u| \, dx + \int_{\{\sigma \leq |v| \leq \sigma + 1\}} |F| |\nabla v| \, dx.$$

Summing up the above estimates,

$$|J_5| \leq \int_{\Omega} G(u, v) \, dx \leq C \left(\int_{\{\sigma-k \leq |u| \leq \sigma+1\}} |F| |\nabla u| \, dx + \int_{\{\sigma-k \leq |v| \leq \sigma+1\}} |F| |\nabla v| \, dx \right),$$

which implies from Lemmas 2.1, 2.2 and Definition 1.3(i) that

$$\lim_{\sigma \rightarrow \infty} |J_5| = 0.$$

Therefore, sending $\sigma \rightarrow \infty$ in (3.17) and recalling (3.18),

$$\int_{\{|u| \leq k/2, |v| \leq k/2\}} (|\nabla u|^{p(x)-2} - |\nabla v|^{p(x)-2} \nabla v) \cdot \nabla(u - v) \, dx = 0,$$

which implies $\nabla u = \nabla v$ a.e. on the set $\{|u| \leq k/2, |v| \leq k/2\}$. Since k is arbitrary, we conclude that $\nabla u = \nabla v$ a.e. in Ω .

Moreover, it follows from Lemma 2.4 that

$$|T_k(u - v)|_{p(x)} \leq C |\nabla T_k(u - v)|_{p(x)} = 0 \quad \text{for all } k > 0,$$

and hence $u = v$ a.e. in Ω . Therefore, we obtain the uniqueness of renormalized solutions. This completes the proof of Theorem 1.5. □

4. Extensions

The existence results above of the entropy solutions and renormalized solutions can be extended to the general problem

$$\begin{cases} -\operatorname{div}(a(x, u, \nabla u) + \Phi(u)) = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where μ is a signed measure in $L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$, $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function such that, for every $s \in \mathbb{R}$, we have $\xi, \eta \in \mathbb{R}^N$ and, for almost every $x \in \Omega$,

$$\begin{aligned} a(x, s, \xi) \cdot \xi &\geq \lambda |\xi|^{p(x)}, \\ |a(x, s, \xi)| &\leq b(|s|)(c(x) + d|\xi|^{p(x)-1}), \\ (a(x, s, \xi) - a(x, s, \eta)) \cdot (\xi - \eta) &> 0, \quad \text{where } \xi \neq \eta. \end{aligned}$$

Herein, $p_- > 1$, $\lambda > 0$, $d \in \mathbb{R}$, $c(x) \in L^{p'(\cdot)}(\Omega)$ and $b : [0, +\infty) \rightarrow (0, +\infty)$, $\Phi : \mathbb{R} \rightarrow \mathbb{R}^N$ are continuous functions.

When $p(x)$ is a constant, we know from [11] that $\mu \in L^1(\Omega) + W^{-1,p'}(\Omega)$ if and only if $\mu \in \mathcal{M}_b^p(\Omega)$, that is, every signed measure that is zero on the sets of zero

p -capacity can be decomposed into the sum of a function in $L^1(\Omega)$ and an element in $W^{-1,p'}(\Omega)$, and conversely, every signed measure in $L^1(\Omega) + W^{-1,p'}(\Omega)$ has zero measure for the sets of zero p -capacity. For the variable exponent case, using the same arguments as in [11], we feel that the similar decomposition result should be true by the properties of $L^{p(\cdot)}(\Omega)$ and the relative $p(\cdot)$ -capacity (see [22]), but this extension remains open.

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