HARMONIC AND ANALYTIC FUNCTIONS OF SEVERAL VARIABLES AND THE MAXIMAL THEOREM OF HARDY AND LITTLEWOOD

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1. Introduction and principal theorems. The present paper, an edited excerpt from my dissertation,[1] arose from the suggestion of S. Bochner that I try to extend the maximal theorem of Hardy and Littlewood **(2)** to functions analytic in the solid unit hypersphere

$$S_{2n}\colon\ r^2 \equiv |z_1|^2 + \ldots + |z_n|^2 < 1.$$

If one writes the analytic function of n complex variables, $f(z_1, \ldots, z_n)$, as $f(r, P)$ where

$$P \in S_{2n-1}\colon\ |z_1|^2 + \ldots + |z_n|^2 = 1,$$

then the theorem in question and its generalization are contained in

THEOREM 1. *If, for some $\lambda > 0$, f satisfies*

$$\int_{S_{2n-1}} |f(r, P)|^\lambda dV_P \leqslant C^\lambda,\ r < 1, \tag{1}$$

where dV_P is the volume element on S_{2n-1} at P and C^λ is a constant, then for the same λ

$$\int_{S_{2n-1}} (\sup_{0 \leqslant r < 1} |f(r, P)|)^\lambda dV_P \leqslant \alpha_n C^\lambda, \tag{2}$$

α_n being independent of f.

From Theorem 1 one can deduce a generalization of a classical theorem due to the brothers Riesz **(7**, Chap. VII**)**:

THEOREM 2. *Under the same general hypotheses in and preceding Theorem* 1, *and assuming* (1), *there exists a function $f(P)$ of class L^λ on S_{2n-1} such that*

$$\lim_{r \to 1} \int_{S_{2n-1}} |f(r, P) - f(P)|^\lambda dV_P = 0. \tag{3}$$

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[1]Princeton, 1947. Abstracts of the results appeared as **(3)** and **(4)**. The decision to publish in full, after so long a delay, is motivated by repeated requests of other workers in the field and by overlapping with published material obtained independently by Zygmund, Calderón, and others. A particular impetus is the preceding paper by K. T. Smith **(5)**, in which a substantial part of the underlying methods and results of my paper are obtained independently from a point of view not too different from mine.

As Zygmund remarked **(9)**, Theorem 2 follows immediately from Theorem 1 and the theorem of Calderón and Zygmund to the effect that, under the same hypotheses, $f(r, P)$ has a point-wise limit, $f(P)$, almost everywhere. For the latter, convergence would be majorized according to (2) and would, therefore, imply mean convergence. However, in the less delicate sense $\lambda \geqslant 1$ (3) follows directly from (2) without the intervention of the theorem on point-wise convergence, as will be seen in §5.

Now the proof of Theorem 1, to be found in §5, can be reduced by means of a sequence of theorems on analytic, harmonic, and subharmonic functions (§§4 and 5) exactly as in **(2)** to the proof of a theorem of purely real-variable nature:

THEOREM 3. *Let $f(P)$ belong to L^p, $p > 1$, on*

$$S_{n-1}: \quad x_1^2 + \ldots + x_n^2 = 1,$$

and let $\sigma_r(P)$ be the spherical cap of radius r (measured on S_{n-1}) about P on S_{n-1} and $V(r)$ its volume as measured on S_{n-1}. Define $f^*(P)$ by

$$f^*(P) = \sup_{0 \leqslant r \leqslant \pi} \frac{1}{V(r)} \int_{\sigma_r(P)} |f(P')| \, dV_{P'}. \tag{4}$$

Then $f^*(P)$ satisfies

$$\int_{S_{n-1}} \{f^*(P)\}^p dV_P \leqslant C_{n,p} \int_{S_{n-1}} |f(P)|^p dV_P, \tag{5}$$

where $C_{n,p}$ depends only[2] *on n and p. If $p = 1$ this is no longer true; however, if $|f(P)| \log^+ |f(P)|$ is integrable, then*

$$\int_{S_{n-1}} f^*(P) \, dV_P \leqslant B_n \int_{S_{n-1}} |f(P)| \log^+ |f(P)| \, dV_P + C_n. \tag{6}$$

The proof of Theorem 3 is the essence of the matter, and the method of analysis was supplied by Wiener in a profound paper **(6)**. There he shows, by a reasoning which is closely related to F. Riesz's proof of the case $n = 1$ of Theorem 3, but simpler and more powerful, that both Birkhoff's ergodic theorem and the Hardy-Littlewood theorem for $n = 1$ have a common source and that both can be extended by the same method, the former to a theorem on averages over an n-parameter abelian group, the latter to a theorem on averages over Euclidean n-space.

Now, Wiener in a lucid fashion reduces everything to a simple measure-theoretic lemma, which he calls "of Vitali type" although it is much more elementary. In studying his paper I noticed that this lemma, although formulated for sets in ordinary n-space, in fact applied to a more general situation from which, in particular, Theorem 3 would follow by Wiener's arguments.

A diagnosis of the elements needed explicitly or implicitly in extending

[2] Wiener's method does not deliver the best constants. Smith's paper (**5**) does.

Wiener's argument to, say, the surface of the hypersphere leads one to describe a metric space with a metric M and an outer measure m as having Euclidean character or *Property* A if, without regard to logical niceties, it is such that (i) spheres of equal radius in M have equal measure in m and vice versa (this very restrictive condition of *homogeneity* may be replaced by a much weaker one of a sort of uniformity in important cases); (ii) countable sets are null-sets; and, most important, (iii) the measure of the set γ covered by a sphere σ and all spheres overlapping σ and having smaller or equal radius satisfies $m(\gamma) \leqslant Cm(\sigma)$ where C depends only on M and m.
Then one has

THEOREM A. *In a space possessing Property* A *let a set S of outer measure $m(S)$ be such that every $P \in S$ is the center of one member, $\sigma(P)$, of a certain family of spheres. Then given $\epsilon > 0$ there is a finite number of mutually disjoint members, σ_i, of the family such that*

$$\Sigma_i\, m(\sigma_i) \geqslant C^{-1}\, m(S) - \epsilon \tag{7}$$

where C is the constant of Property A.

For $(n - 1)$-space and S_{n-1} (as will be seen) $C = 3^{n-1}$. The proof of Theorem A will occupy §2, and the deduction of Theorem 3, §3.

The generality of Theorem A permits the immediate extension of Wiener's generalization of Birkhoff's ergodic theorem to those groups of measure-preserving transformations of a set which admit an invariant metric possessing Property A and which may well be non-commutative. This application is in my dissertation;[3] but I do not reproduce it here since there is already a surfeit of related ergodic theorems on the market.

Besides the hypersphere there are other generalizations of the unit circle, notably the *polycylinder*: $|z_1| = r_1 < 1, \ldots, |z_n| = r_n < 1$ whose boundary is the multitorus,

$$T_n\colon\ |z_1| = 1, \ldots, |z_n| = 1.$$

The analogue of Theorem 1 for this domain was derived independently and announced almost simultaneously by Zygmund **(8)** in stronger form and me **(3)**. I prove it again here not merely because the proof is different but because the technique of proof will serve to demonstrate a more interesting generalization **(4)**:

THEOREM 4. *Let $f(z_1, \ldots, z_n)$ be analytic in $|z_1| < 1, \ldots, |z_n| < 1$ and satisfy*

$$\int_0^{2\pi} \cdots \int_0^{2\pi} |f(r_1e^{i\theta_1}, \ldots, r_ne^{i\theta_n})|^\lambda d\theta_1 \ldots d\theta_n \leqslant C^\lambda;\ r_1, \ldots, r_n < 1;\ \lambda > 0, \tag{8}$$

then

[3]The reference at the end of **(4)** to ergodic theorems for compact groups is erroneous or at least misleading. The theorems actually meant are analogous to ergodic theorems **(6)** but deal with averages over sets tending to zero (like derivatives).

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \{\sup_{\Delta} |f(r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n})|\}^\lambda d\theta_1 \ldots d\theta_n \leqslant \alpha_n C^\lambda \tag{9}$$

where α_n, C^λ have the same meanings as before and Δ is the region, for fixed $\theta_1, \ldots, \theta_n$, described by

$$0 \leqslant \frac{1 - r_i}{1 - r_j} \leqslant K; \qquad i, j = 1, \ldots, n, i \neq j, \tag{10}$$

K being any positive constant.

The necessity for (10) is related to the fact that the values of a function harmonic in each $|z_i| < 1$ are determined solely by its boundary values on T_n, which is thus a "distinguished boundary surface" in Bergman's terminology.

If in Theorem 3 and Theorem A one observes that when dealing with T_n, spheres may be replaced by hyper-cubes, while $C = 3^n$, then one will see that Theorem 4 follows from them as Theorem 3 does—provided, that is, that one proves the more delicate version of the connecting link between Theorems 1 and 3 (§4).

2. Proof of Theorem A. Consider any point P of S whose $\sigma(P)$ overlaps only a finite number of $\sigma(P)$. This certainly implies that we can find some sphere (not a $\sigma(P)$!) with P as center such that within this sphere there is no other point of S. This, by familiar reasoning, implies that the set of such points P is denumerable, hence of measure 0. Let us, therefore, discard this set and the $\sigma(P)$ belonging to it and ignore them in further reasoning.

Let B_1 be the least upper bound of the radii of $\sigma(P)$. Obviously one may assume $B_1 < \infty$. Otherwise, the theorem is trivial.

After this remark I shall, in fact, prove (7) with S replaced by the set consisting of the union σ of all $\sigma(P)$, and $m(S)$ replaced by $m(\sigma) - \epsilon$ for any ϵ. This, of course, will prove (7), since σ contains S.

Let $V(B_1)$ be the volume of a sphere of radius B_1. By the definition of B_1, one can choose $\sigma(P_1)$ such that its volume $V_1 \geqslant V(B_1) - \frac{1}{2}\epsilon$ (obviously $V_1 \leqslant V(B_1)$). Let σ_1 be the set consisting of the union of $\sigma(P_1)$ and all adjoining $\sigma(P)$. The volume of σ_1 is not greater than $CV(B_1)$. In fact, let $\bar{\sigma}$ be a sphere of radius B_1 about P_1 (not a $\sigma(P)$!) and hence of volume $V(B_1)$. Now σ_1 is certainly contained in the union of $\bar{\sigma}$ and all spheres adjoining it. But these latter are certainly of radius $\leqslant B_1$ by the definition of B_1. Hence Property A implies that the union in question has volume $\leqslant CV(B_1)$. Let $V(B_2)$ be the least upper bound of the volumes of those $\sigma(P)$ not in σ_1. Choose such a $\sigma(P_2)$ whose volume $V_2 \geqslant V(B_2) - \frac{1}{2}\epsilon$. Let σ_2 be the union of $\sigma(P_2)$ and all adjoining $\sigma(P)$ not in σ_1. As before, the volume of σ_2 is $\leqslant CV(B_2)$.

Continue this process inductively. One obtains a sequence $\sigma(P_k)$, obviously disjoint, with volumes V_k subject to these inequalities, where the $V(B_k)$ are defined similarly,

$$\sum_{k=1}^{\infty} \left(V(B_k) - \frac{\epsilon}{2^k} \right) \leqslant \sum_{k=1}^{\infty} V_k \leqslant m(\sigma) .$$

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Since $\sum_k V(B_k)$ is convergent $V(B_k) \to 0$. This implies that the union of σ_k where σ_k is defined inductively as above, exhausts all $\sigma(P)$; for, if it did not, but omitted, say, one $\sigma(P')$, then from some k' on $V(B_k)$ would equal the volume of $\sigma(P')$.

Summing up, one has $\sigma = \Sigma\sigma_k$. Therefore,

$$\frac{1}{C} m(\sigma) \leqslant \Sigma\, V(B_k)$$

but

$$\Sigma\, V_k \geqslant \Sigma\, V(B_k) - \epsilon \geqslant \frac{1}{C} m(\sigma) - \epsilon.$$

Now one chooses K so that

$$\sum_{k=1}^{K} V_k \geqslant \sum_{1}^{\infty} V_k - \epsilon$$

and one has finally

$$\frac{1}{C} m(\sigma) - 2\,\epsilon \leqslant \sum_{k=1}^{K} V_k.$$

LEMMA 1. *Theorem* A *applies to* S_{n-1} *with spherical caps as the* $\sigma(P)$ *and to* T_n *with hypercubes,* $-\phi \leqslant \theta_i \leqslant \phi$, $(i = 1, \ldots, n)$, *as* $\sigma(P)$, *where* $C = 3^{n-1}$ *in the first and* $C = 3^n$ *in the second case.*

Proof. The first part is obvious as is the very last statement. In dealing with S_{n-1} I remark that

$$V_r = C_{n-1} \int_0^r \sin^{n-2}\theta\, d\theta.$$

Now the volume of a sphere of radius r plus those adjoining it of smaller r radius is certainly less than or equal to that of a sphere of radius $3r$. Since $3r \leqslant \pi$,

$$\sin 3r = 3 \sin r - 4 \sin^3 r \leqslant 3 \sin r$$

so that $\sin^{n-2} 3r \leqslant 3^{n-2} \sin^{n-1} r$. Therefore

$$\int_0^{3r} \sin^{n-2}\theta\, d\theta = 3 \int_0^r \sin^{n-2} 3\,\theta' d\theta' \leqslant 3^{n-1} \int_0^r \sin^{n-2}\theta d\theta\,.$$

3. Proof of Theorem 3.

The key to Theorem 3 is the important

THEOREM 5. *The measure of the set* S_α *of points* P *for which* $f^*(P) > \alpha$ *does not exceed*

$$\frac{3^{n-1}}{\alpha} \int_{S_{n-1}} |f(P)|\, dV_P$$

It also does not exceed

$$\frac{2.3^{n-1}}{\alpha} \int_{|f(P)|>\frac{1}{2}\alpha} |f(P)|\, dV_P.$$

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Proof. For each $P \in S_\alpha$ by definition one can find an r_P such that

$$\int_{\sigma_P(r_P)} |f(P')|\, dV_{P'} \geqslant V(r_P)\alpha.$$

By Lemma 1 one can find a finite number of the $\sigma_P(r_P)$ whose total measure exceeds $3^{-(n-1)}\, m(S_\alpha) - \epsilon$. One has then

$$\int_{S_{n-1}} |f(P)|\, dV_P \geqslant \int_\Sigma |f(P)|\, dV_P \geqslant \frac{1}{3^{n-1}}\, m(S_\alpha)\, \alpha - \epsilon$$

where Σ is the finite set of $\sigma_P(r_P)$. The last statement is proved as follows: Let $h(P) = |f(P)|$ when $|f(P)| \geqslant \frac{1}{2}\alpha$, otherwise zero. Let $h^*(P)$ be defined in the same manner as $f^*(P)$. Obviously, we have $f^*(P) \leqslant h^*(P) + \frac{1}{2}\alpha$. Consequently $m(S_\alpha) \leqslant$ the measure of the set of P for which $h^*(P) > \frac{1}{2}\alpha$, which by the preceding part of the theorem is

$$\leqslant \frac{2.3^{n-1}}{\alpha} \int_{S_{n-1}} h(P)\, dV_P = \frac{2.3^{n-1}}{\alpha} \int_{|f(P)| \geqslant \frac{1}{2}\alpha} |f(P)|\, dV_P.$$

A similar proof yields

THEOREM 6. *Let $f(P)$ belong to L on the multitorus T_n. Let $\gamma(P)$ be the "cube"*

$$\theta_{i_p} - \phi \leqslant \theta_i \leqslant \theta_{i_p} + \phi$$

with P as center and side 2ϕ. Then the measure $m(S_\alpha)$ of the set of points P where

$$f^*(P) \equiv \underset{0 \leqslant \phi \leqslant \pi}{\text{l.u.b.}} \frac{1}{2^n \phi^n} \int_{\gamma(P)} f(P')\, dV_{P'} > \alpha$$

is

$$\leqslant \frac{3^n}{\alpha} \int_{T_n} |f(P)|\, dV_P,$$

where dV_P is the volume element on T_n. It also does not exceed

$$\frac{2.3^n}{\alpha} \int_{|f(P)| \geqslant \frac{1}{2}\alpha} |f(P)|\, dV_P.$$

Proof of Theorem 3. Let $m(x)$ be the measure of the set of points where $|f(P)| > x$, and $m^*(x)$ be the measure of the set of points where $f^*(P) > x$. If $s(x)$ is any non-negative increasing function of x then

$$\int_{S_{n-1}} s(f(P))\, dV_P = -\int_0^\infty s(x)\, dm(x)$$

$$\int_{S_{n-1}} s(f^*(P))\, dV_P = -\int_0^\infty s(x)\, dm^*(x)$$

(**7**, p. 242).

Since, from Theorem 5,

$$m^*(x) \leqslant \frac{2.3^{n-1}}{x}\int_{|f(P)|\geqslant\frac{1}{2}\alpha}|f(P)|\,dV_P = -\frac{2.3^{n-1}}{x}\int_{\frac{1}{2}x}^{\infty} y\,dm(y),$$

by formal substitution and interchange of integrations we have

$$\int_0^\infty m^*(x)\,x^{p-1}dx \leqslant -\,2.3^{n-1}\int_0^\infty x^{p-1}dx\int_{\frac{1}{2}x}^\infty y\,dm(y)$$

$$= -\,2.3^{n-1}\int_0^\infty ydm\,(y)\int_0^{2y}x^{p-2}dx$$

$$= \frac{-2^p.3^{n-1}}{(p-1)}\int_0^\infty y^p dm(y).$$

But this latter

$$= \frac{2^p.3^{n-1}}{(p-1)}\int_{S_{n-1}}|f(P)|^p dV_P$$

and is, therefore, finite. As a consequence

$$\lim_{\xi\to\infty}\int_\xi^{2\xi} m^*(x)\,x^{p-1}dx = 0;$$

however, since $m^*(x)$ is a decreasing function

$$m^*(2\xi)\,\xi^p\frac{2^p-1}{p} = m^*(2\xi)\int_\xi^{2\xi}x^{p-1}dx \leqslant \int_\xi^{2\xi}m^*(x)\,x^{p-1}dx;$$

therefore, $\lim m^*(\xi)\xi^p = 0$, and we can integrate by parts, getting

$$\int_{S_{n-1}}\{f^*(P)\}^p dV_p = -\int_0^\infty x^p dm^*(x) \leqslant \frac{2^p.3^{n-1}}{(p-1)}\int_{S_{n-1}}|f(P)|^p dV_P,$$

which is (5) with $C_{n,p} = 2^p.3^{n-1}/(p-1)$.

The second statement has been proved by Hardy-Littlewood **(2)** for $n = 2$. The third statement has a similar proof. This time we notice that for the same reasons

$$\int_1^\infty m^*(x)\,dx \leqslant -\,2.3^{n-1}\int_1^\infty ydm\,(y)\int_1^{2y}\frac{dx}{x} = -\,2.3^{n-1}\int_1^\infty y\log 2y\,dm(y)$$

$$\leqslant 2.3^{n-1}\int_{S_{n-1}}|f(P)|\log^+|f(P)|\,dV_P + 2.3^{n-1}\log 2\int_{S_{n-1}}|f(P)|\,dV_P\,.$$

Integrating by parts and noting that

$$|f| \leqslant e + |f|\log^+|f|$$

and

$$\int_0^1 m^*(x)dx \leqslant V(S_{n-1})$$

we have (6). These constants are not as good as those of Hardy-Littlewood in the original case, $n = 2$.

THEOREM 7. *Let $f(P)$ belong to L^p, $p > 1$ on T_n. Then if $f^*(P)$ is defined as in Theorem 7*

$$\int_{T_n} \{f^*(P)\}^p dV_P \leqslant C_{n,p} \int_{T_n} |f(P)|^p dV_P$$

where $C_{n,p}$ depends only on n and p. This is no longer true for $p = 1$; however, if $|f(P)| \log^+|f(P)|$ is integrable on T_n, then

$$\int_{T_n} f^*(P)\, dV_P \leqslant B_n \int_{T_n} |f(P)| \log^+|f(P)|\, dV_P + C_n$$

where B_n depends only on n.

The proof is exactly like that of Theorem 3 with appropriate changes.

It will now be seen immediately that Theorem 3 can be extended to an arbitrary space with Property A, where 3^{n-1} is replaced by C.

4. Theorems which relate radial suprema to averages.

THEOREM 8. *Let $f(P)$ belong to L on S_{n-1}. Let $u(r, P)$ be the harmonic function in S_n which takes on the values $f(P)$ on S_{n-1}. If*

$$U(P) = \sup_{0 \leqslant r < 1} |u(r, P)|$$

then $U(P) \leqslant A_n f^*(P)$, where $f^*(P)$ is defined as in Theorem 3 and A_n is a constant depending only on n.

Proof. Define polar coordinates in n-space:

$$\begin{aligned}
x_1 &= r \cos \theta_1,\\
x_2 &= r \sin \theta_1 \cos \theta_2,\\
&\dots\\
x_{n-1} &= r \sin \theta_1 \dots \sin \theta_{n-2} \cos \theta_{n-1},\\
x_n &= r \sin \theta_1 \dots \qquad \sin \theta_{n-1}.
\end{aligned}$$

For fixed P we may assume that P is the point $x_1 = 1, x_i = 0; i > 1$, in which case θ_1 becomes the geodesic distance of any other point on S_{n-1} from P. We have then

$$u(r, P) = \frac{1}{\omega_n} \int_0^\pi d\,\theta_1 \dots \int_0^\pi d\,\theta_{n-2} \int_{-\pi}^{\pi} P_n(r, \theta_1) \sin^{n-2}\theta_1 f(Q)\omega \cdot d\,\theta_{n-1}$$

where

$$\omega_n = \frac{2(\sqrt{\pi})^n}{\Gamma(\frac{1}{2}n)} \text{ and } P_n(r, \theta)$$

is the Poisson kernel for the sphere,

$$\frac{1 - r^2}{(1 - 2r\cos\theta + r^2)^{\frac{1}{2}n}}, \; \omega = \sin^{n-3}\theta_2 \dots \sin\theta_{n-2},$$

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and Q has coordinates $(\theta_1, \ldots, \theta_{n-1})$ (**1**, Chap. IV). Now let us observe that in order to prove the lemma, it is sufficient to prove $|u(r, P)| < A_n f^*(P)$ where A_n is fixed and independent of r. We integrate by parts, then, with respect to θ_1 and obtain

$$u(r, P) = \frac{1}{\omega_n}\left\{P_n(r, \pi) \int_0^\pi d\theta_1 \ldots \int_{-\pi}^\pi f(Q) \sin^{n-2}\theta_1 \omega d\theta_{n-1}\right.$$
$$\left. - \int_0^\pi d\theta_1 \cdot \frac{dP_n(r, \theta_1)}{d\theta_1} \cdot \int_0^{\theta_1} \int_0^\pi d\theta_2 \ldots \int_{-\pi}^\pi d\theta_{n-1} f(Q') \sin^{n-2}\theta \omega d\theta\right\}$$

where the coordinates of Q' are $(\theta, \theta_2, \ldots, \theta_{n-1})$. Recalling the definition of $f^*(P)$, more explicitly

$$f^*(P) = \underset{0 \leqslant \theta \leqslant \pi}{\text{l.u.b.}}\left(C_n \int_0^\theta \sin^{n-2}\theta\, d\theta\right)^{-1} \int_0^\theta \left[\int_0^\pi d\theta_2 \ldots \int_{-\pi}^\pi d\theta_{n-1} |f(Q)| \cdot \sin^{n-2}\theta \cdot \omega\right] d\theta$$

where

$$C_n = \int_0^\pi d\theta_2 \ldots \int_{-\pi}^\pi \omega d\theta_{n-1} = \omega_{n-1};$$

therefore

$$|u(r, P)| \leqslant K_n f^*(P) + C_n f^*(P) \int_0^\pi \left|\frac{dP_n}{d\theta_1}(r, \theta_1)\right| \cdot \left|\int_0^{\theta_1} \sin^{n-2}\theta\, d\theta\right| d\theta_1$$
$$\leqslant f^*(P)\left[K_n + C_n \int_{-\pi}^\pi \left|\frac{dP_n}{d\theta_1}(r, \theta_1)\right| \cdot |\theta_1| \cdot d\theta_1\right]$$

where K_n is a constant depending only on n. The last expression (**2**, p. 107) in brackets is $\leqslant D_n$. This completes the proof.

THEOREM 9. *Let $f(P)$ belong to L on T_n. Let $u(r_1, \ldots, r_n, P)$ be the function which is harmonic in the polycylinder,*

$$P_n\colon\ |z_i| < 1 \ (i = 1, \ldots, n)$$

and which assumes the values $f(P)$ on T_n. Then, if $f^*(P)$ is defined as in Theorem 7, we have

$$\sup_{(r_1, \ldots, r_n) \in \Delta} |u(r_1, \ldots, r_n, P)| \leqslant A_\Delta f^*(P)$$

where Δ is the region of $0 \leqslant r_i < 1$, $(i = 1, \ldots, n)$, described by (10). *A_Δ depends only on Δ (i.e., K) and n.*

Proof. I remark that the latter restriction seems quite essential as is evidenced not only in the proof but by the implications of the lemma (cf. the next section). I observe again that one can assume that P is the point $\theta_i = 0$ $(i = 1, \ldots, n)$ on T_n. Because of a few complications it will be easier to present here the proof for $n = 2$ only. The extension to arbitrary n is straightforward.

First,

$$u(r_1, r_2, P) = \frac{1}{\pi 4^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} P(r_1, \theta_1)\, P(r_2, \theta_2)\, f(\theta_1, \theta_2)\, d\theta_1 d\theta_2$$

where $P(r, \theta) = (1 - r^2)/(1 - 2\cos\theta + r^2)$ is the usual Poisson kernel. Repeated integration by parts and interchange of integration gives, after taking absolute values

$$\begin{aligned}
|u(r_1, r_2, P)| \leqslant \frac{1}{\pi 4^2}\Bigg[&\frac{1 - r_1}{1 + r_1}\cdot\frac{1 - r_2}{1 + r_2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(\theta_1, \theta_2)|\, d\theta_1 d\theta_2 \\
&+ \frac{1 - r_2}{1 + r_2} \int_{-\pi}^{\pi} \left|\frac{d}{d\theta_1} P(r_1, \theta_1)\right| d\theta_1 \int_0^{\theta_1} \int_{-\pi}^{\pi} |f(\theta, \theta')|\, d\theta d\theta' \\
&+ \frac{1 - r_1}{1 + r_1} \int_{-\pi}^{\pi} \left|\frac{d}{d\theta_2} P(r_2, \theta_2)\right| d\theta_2 \int_0^{\bar{k}_2} \int_{-\pi}^{\pi} |f(\theta, \theta')|\, d\theta d\theta' \\
&+ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \Bigg\{\left|\frac{d}{d\theta_2} P(r_2, \theta_2)\, \frac{d}{d\theta_1} P(r_1, \theta_1)\right| \\
&\qquad (\operatorname{sgn}\theta_1 \cdot \operatorname{sgn}\theta_2) \int_0^{\theta_1} \int_0^{\theta_2} |f(\theta, \theta')|\, d\theta d\theta'\Bigg\}\, d\theta_1 d\theta_2\Bigg] \\
= \frac{1}{4\pi^2}[I_1 + I_2 + I_3 + I_4].&
\end{aligned}$$

Recalling the definition of $f^*(P)$ one has $I_1 \leqslant 4\pi^2 f^*(P)$. To get an inequality for I_2 (and I_3) one observes that the inner double integral is less than or equal to

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(\theta_1, \theta_2)|\, d\theta_1 d\theta_2 \leqslant 4\pi^2 f^*(P).$$

Furthermore,

$$\begin{aligned}
&\frac{1 - r_2}{1 + r_2} \int_{-\pi}^{\pi} \left|\frac{d}{d\theta_1} P(r_1, \theta_1)\right| d\theta_1 \\
&= \frac{1 - r_2}{1 + r_2}\left[- \int_0^{\pi} \frac{d}{d\theta_1} P(r_1, \theta_1)\, d\theta_1 + \int_{-\pi}^{0} \frac{d}{d\theta_1} P(r_1, \theta_1)\, d\theta_1\right] \\
&= \frac{1 - r_2}{1 + r_2}\left[+ 2\left(\frac{1 + r_1}{1 - r_1}\right) - 2\left(\frac{1 - r_1}{1 + r_1}\right)\right] \leqslant 4K.
\end{aligned}$$

Therefore I_2 (and I_3) $\leqslant 4\pi^2 \,.\, 4Kf^*(P)$. Next, in I_4 the inner double integral is less than or equal to

$$\operatorname{sgn}\theta_1 \cdot \operatorname{sgn}\theta_2 \int_0^{\theta_2} \int_0^{(\operatorname{sgn}\theta_1)|\theta_2|} \leqslant \begin{cases} \displaystyle\int_{-\theta_2}^{\theta_2} \int_{-\theta_2}^{\theta_2}, & |\theta_1| \leqslant |\theta_2|, \\ \displaystyle\int_{-\theta_1}^{\theta_1} \int_{-\theta_1}^{\theta_1}, & |\theta_1| > |\theta_2|. \end{cases}$$

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We split up I_4,

$$I_4 = \int\int_{|\theta_1| \leqslant |\theta_2|} + \int\int_{|\theta_1| > |\theta_2|}$$

which with the previous remark gives us

$$I_4 \leqslant 4f^*(P)\Big[\int\int_{|\theta_1|>|\theta_2|} \frac{d}{d\theta_1} P(r_1, \theta_1) \cdot \frac{d}{d\theta_2} P(r_2, \theta_2)| \cdot |\theta_1|^2 d\theta_1 d\theta_2$$

$$+ \int\int_{|\theta_1|\leqslant|\theta_2|} \left|\frac{d}{d\theta_1} P(r_1, \theta_1) \cdot \frac{d}{d\theta_2} P(r_2, \theta_2)\right| \cdot |\theta_2|^2 d\theta_1 d\theta_2\Big],$$

the first integral in brackets is less than

$$2\pi r_1(1 - r_1^2) \int_{-\pi}^{\pi} \left|\frac{\theta \sin \theta}{(1 - 2r_1 \cos \theta + r_1^2)^2}\right| d\theta \cdot \int_{-\pi}^{\pi} \left|\frac{d}{d\theta} P(r_2, \theta)\right| d\theta$$

$$\leqslant 2r_1 \pi (1 + r_1)(1 - r_1)\left[2\frac{(1 + r_2)}{1 - r_2}\right] \cdot \int_{-\pi}^{\pi} \left|\frac{\theta \sin \theta}{(1 - 2r_1 \cos \theta + r_1^2)^2}\right| d\theta.$$

The integral on the right is less than or equal to a constant C (the reasoning is the same as in the Hardy-Littlewood reference in the proof of Lemma 3). Therefore, this first integral in brackets (and similarly the other) is less than or equal to $16\pi K\, C$. Setting

$$A_\Delta = (1 + 8K + 32KC/\pi),$$

one completes the proof.

5. Proofs of Theorems 1, 2, and 4 and related theorems.

THEOREM 10. *Let $f(P)$ belong to L^p, $p > 1$ on S_{n-1}. Let $u(r, P)$ be the function harmonic in S_n with boundary values $f(P)$. If $U(P) = \sup_{0 \leqslant r < 1} |u(r, P)|$, then*

$$\frac{1}{V_1} \int_{S_{n-1}} \{U(P)\}^p dV_P \leqslant C_{n,p} \frac{1}{V_1} \int_{S_{n-1}} |f(P)|^p dV_P$$

where $C_{n,p}$ is a constant, depending only on n and p. For $p = 1$ this is not true. However, if $|f(P)| \log^+|f(P)|$ is integrable on S_{n-1}, then

$$\frac{1}{V_1} \int_{S_{n-1}} U(P)\, dV_P \leqslant \frac{B_n}{V_1} \int_{S_{n-1}} |f(P)| \log^+|f(P)|\, dV_P + \frac{C_n}{V_1}$$

where B_n and C_n depend only on n.

Proof. The first and third statements are corollaries of Theorem 3 and Theorem 9. The second statement has been proved in **(2)** for $n = 2$. As a corollary of Theorem 10 we have

THEOREM 11. *Let $u(r, P)$ be harmonic in S_n and let it be such that*

$$(11) \qquad \frac{1}{V_1} \int_{S_{n-1}} |u(r, P)|^p dV_P \leqslant C^p$$

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for all $r < 1$, *and fixed* $p > 1$. *Then if* $U(P)$ *is defined as in Theorem* 10

$$\frac{1}{V_1}\int_{S_{n-1}}\{U(P)\}^p dV_P \leqslant C_{n,p}C^p \tag{12}$$

Proof. Let

$$U_R(P) = \sup_{0\leqslant r<R<1}|u(r,P)|;$$

then

$$\frac{1}{V_1}\int_{S_{n-1}}\{U_R(P)\}^p dV_P \leqslant \frac{C_{n,p}}{V_1}\int_{S_{n-1}}|u(R,P)|^p dV_P \leqslant C_{n,p}C^p,$$

by Theorem 10. Now $U_R(P) \uparrow U(P)$ as $R \to 1$. Hence, Lebesgue's monotone convergence theorem completes the proof.

THEOREM 12. *Let* $u(r, P)$ *satisfy* (11). *Then there exists a function* $u(P) \in L^p$ *on* S_{n-1} *such that*

$$\lim_{r\to 1}\int |u(r,P) - u(P)|^p dV_P = 0 \tag{13}$$

Proof. Hölder's inequality implies

$$\frac{1}{V_1}\int_{S_{n-1}}|u(r,P)|\, dV_P \leqslant \left(\frac{1}{V_1}\int_{S_{n-1}}|u(r,P)|^p dV_P\right)^{1/p} \leqslant (C_{n,p})^{1/p}.$$

As a result

$$F(r,S) = \int_S u(r,P)\, dV_P,$$

where S is a measurable subset of S_{n-1}, constitute a set of absolutely continuous set-functions on S_{n-1} which are uniformly bounded. According to Radon's theory of integration one can form, for any Φ continuous on S_{n-1}, the Radon-Stieltjes integral

$$\int_{S_{n-1}}\Phi\, dF(r,S).$$

Now it is a classic theorem of Radon that from the uniformly bounded set of set-functions $F(r, S)$ on S_{n-1} one can extract a sequence $F(r_m, S)$ and find another bounded set-function $F(S)$ such that, for any continuous Φ,

$$\lim_{m\to\infty}\int_{S_{n-1}}\Phi\, dF(r_m,S) = \int_{S_{n-1}}\Phi\, dF(S) \tag{14}$$

where the $r_m \to 1$, otherwise the theorem is trivial.
Now

$$|F(r,S)| \leqslant \int_S |u(r,P)|\, dV_P \leqslant \int_S U(P)\, dV_P \tag{15}$$

so that the $F(r, S)$ are uniformly absolutely continuous, since $U(P)$ by (12) and Holder's inequality belongs to L. Therefore, by choosing Φ in (14) to be

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the characteristic function (rounded-off) of S one sees that $F(S)$ is also absolutely continuous, and, therefore, the integral of a point function $u(P)$ of class L. Accordingly, if one picks Φ in (14) to be the Poisson kernel one finds

$$u(r, P) = \int_{S_{n-1}} P_n(r, Q)\, u(Q)\, dV_Q. \tag{16}$$

One also sees from (15), by applying Lebesgue's differentiation theorem, that $|u(r, P)| \leqslant U(P)$ almost everywhere so that $u(P)$ is in L^p by (12).

The reasoning in (**7**, p. 85) using (16) completes the proof of (13).

To prove (2), I observe first that a subharmonic function $w(r, p)$ satisfying (11) also satisfies the analogue of (12). The proof of this is reduced to (12) by the device of the harmonic majorant of $w(r, P)$ and is to be found in (**2**, p. 113, footnote 1) which carries over word for word to several variables.

Next, following Hardy and Littlewood, I set $w(r, P) = |f|^{\frac{1}{2}\lambda}$ and observe that (1) implies that $w(r, P)$ satisfies (11) with $p = 2 > 1$. That $|f|^{\frac{1}{2}\lambda}$ is subharmonic is a simple consequence of the mean-value theorem for the function $f^{\frac{1}{2}\lambda}$ which is analytic in the neighborhood of any point where $f \neq 0$. Then (2) follows immediately.

The proof of (9) follows from Theorems 6 and 9 through the intermediary of theorems analogous to 10 and 11 exactly as in the proof of (2).

Finally (3), Theorem 2, follows from Theorem 12 when $\lambda > 1$ since $|Rf| \leqslant |f|$ and $|If| \leqslant |f|$ and the convergence of f follows from that of Rf and If by Minkowski's inequality. When $\lambda = 1$ the Hardy-Littlewood inequality is still valid for f, unlike the harmonic function; therefore the reason, ing of Theorem 12 may be repeated to account for this case.

Obviously, a similar theorem may be deduced for the polycylinder (cf. also **10**).

References

1. R. Courant and D. Hilbert, *Methoden der Mathematischen Physik*, vol. II (Berlin, 1937).

2. G. H. Hardy and J. E. Littlewood, *A maximal theorem with function-theoretic applications*, Acta Math., *54* (1930), 81–116.

3. H. E. Rauch. *Generalisation d'une proposition de Hardy et Littlewood et de théorèmes ergodiques qui s'y rattachent*, Comptes Rendus, *227* (1948), 887–889.

4. ———, *A Poisson formula and the Hardy and Littlewood theorem for matrix spaces*, Bull. Amer. Math. Soc., *55* (1949), Abstract 250.

5. K. T. Smith, *A generalization of an inequality of Hardy and Littlewood*, Can. J. Math., *8* (1956), 157-170.

6. N. Wiener, *The ergodic theorem*, Duke Math. J., *5* (1939), 1–18.

7. A. Zygmund, *Trigonometrical Series* (Warsaw, 1935).

8. ———, *On the existence of boundary values for regular functions of several complex variables*, Bull. Amer. Math. Soc., *54* (1948), Abstract 502t.

9. ———, *A remark on functions of several complex variables*, Acta Szeged, *12* (1950), 66–68.

10. ———, *On the boundary values of functions of several complex variables*, Fund. Math., *36* (1949), 207–305.

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