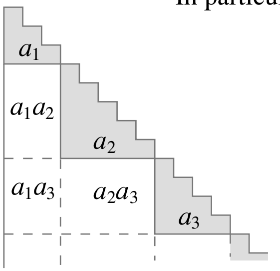
108.37 A parent figure for a family of triangle number identities

$$T_{a_1 + a_2 + a_3 + \ldots + a_n} = T_{a_1} + T_{a_2} + T_{a_2} + \ldots + T_{a_n} + \sum_{\substack{1 \leqslant i,j \leqslant n \\ i \neq j}} a_i a_j$$



In particular:

$$T_{ab} = aT_b + b^2T_{a-1} = bT_a + a^2T_{b-1} \quad [1, \text{p. } 102]$$

$$= aT_b + (T_{b-1} + T_b)T_{a-1}$$

$$= (a + T_{a-1})T_b + T_{a-1}T_{b-1}$$

$$= T_{a-1}T_{b-1} + T_aT_b \quad [1, \text{p. } 101]$$

$$T_{a^2} = T_{a-1}^2 + T_a^2 \quad [1, \text{p. } 99]$$

Reference

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108.38 A number whose square root is the sum of its digits

The number 81 has the property that its square root is equal to the sum of its digits: $\sqrt{81} = 8 + 1$.

Are there other numbers with the same property?

To answer this question, we, first of all, prove the following Lemma.

Lemma: If $n \geqslant 5$, then $10^{(n-1)/2} > 9n$.

Proof: We use mathematical induction.

For $n = 5$, (1) is true, since $10^{(5-1)/2} = 100 > 9 \times 5 = 45$.

Now we assume that (1) holds for $n = k$, so that $10^{(k-1)/2} > 9k$. Then

$$10^{k/2} = \sqrt{10} \times 10^{(k-1)/2} > 3 \times 9k = 27k = 9(k+1) + 9(2k-1). \qquad (1)$$

But $k \geqslant 5$, so $10^{k/2} > 9(k + 1)$, whence (1) holds for $n = k + 1$.

Now we prove the main Theorem.

Theorem: If A is a positive integer whose square root equals the sum of its digits, then $A = 1$ or $A = 81$.