THEOREMS RELATING HANKEL AND MEIJER'S BESSEL TRANSFORMS

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In this note a theorem, giving a relation between the Hankel transform of $f(x)$ and Meijer's Bessel function transform of $f(x)g(x)$, is proved. Some corollaries, obtained by specializing the function $g(x)$, are stated as theorems. These theorems are further illustrated by certain suitable examples in which certain integrals involving products of Bessel functions or of Gauss's hypergeometric function and Appell's hypergeometric function are evaluated. Throughout this note we use the following notations:

$$\mathfrak{L}\{f(x);\, p\} = \int_0^\infty e^{-px} f(x)\, dx, \tag{1}$$

$$\mathfrak{K}_\mu\{f(x);\, p\} = \int_0^\infty (px)^{\frac{1}{2}} K_\mu(px) f(x)\, dx, \tag{2}$$

$$\mathfrak{H}_\nu\{f(x);\, y\} = \int_0^\infty (xy)^{\frac{1}{2}} J_\nu(xy) f(x)\, dx, \tag{3}$$

$$\Gamma(a \pm b) = \Gamma(a+b)\Gamma(a-b),$$

and

$${}_2F_1\left(\begin{matrix} a \pm b \\ c \end{matrix};\, z\right) = {}_2F_1(a+b,\, a-b;\, c;\, z).$$

We call (2) a Meijer's Bessel function transform; it is a generalisation of (1), and

$$\mathfrak{K}_{\pm\frac{1}{2}}\{f(x);\, p\} = (\pi/2)^{\frac{1}{2}}\, \mathfrak{L}\{f(x);\, p\}.$$

THEOREM 1. *If* $f(x) \in L(0, \infty)$, $g(x) \in L(0, \infty)$, $\mathfrak{H}_\nu\{f(x);\, y\} \in L(0, \infty)$, $R(\frac{3}{2} \pm \mu) > 0$, $R(\frac{1}{2}+\nu) > 0$, $R(p) > 0$, *then*

$$\mathfrak{K}_\mu\{f(x)g(x);\, p\} = \int_0^\infty \mathfrak{H}_\nu\{f(x);\, y\}\, \mathfrak{H}_\nu\{(px)^{\frac{1}{2}} K_\mu(px) g(x);\, y\}\, dy. \tag{4}$$

Proof. Since [1, p. 5]

$$f(x) = \int_0^\infty (xy)^{\frac{1}{2}} J_\nu(xy)\, \mathfrak{H}_\nu\{f(x);\, y\}\, dy, \tag{5}$$

we have

$$\mathfrak{K}_\mu\{f(x)g(x);\, p\} = \int_0^\infty (px)^{\frac{1}{2}} K_\mu(px) g(x) \left[\int_0^\infty (xy)^{\frac{1}{2}} J_\nu(xy)\, \mathfrak{H}_\nu\{f(x);\, y\}\, dy\right] dx,$$

which on change of order of integration yields (4). Change of order of integration is admissible under the conditions given in the statement of the theorem, since

$$y^{\frac{1}{2}}J_\nu(xy)\mathfrak{H}_\nu\{f(x);\, y\} \sim A\cos(xy+\alpha)\mathfrak{H}_\nu\{f(x);\, y\} \quad \text{for large } y,$$

$$\sim By^{\frac{1}{2}+\nu}\mathfrak{H}_\nu\{f(x);\, y\} \quad \text{for small } y,$$

and

$$xK_\mu(px)J_\nu(xy)g(x) \sim e^{-px}\cos(xy+\alpha)g(x) \quad \text{for large } x,$$

$$\sim x^{1\pm\mu+\nu}g(x) \quad \text{for small } x.$$

If we take $\mu = \pm\frac{1}{2}$ in Theorem 1, we obtain

THEOREM 1(*a*). *If* $f(x)$ *and* $\mathfrak{H}_\nu\{f(x);\, y\} \in L(0,\, \infty)$, $g(x) \in L(0,\, \infty)$, and if $R(\frac{1}{2}+\nu) > 0$, $R(p) > 0$, *then*

$$\mathfrak{L}\{f(x)g(x);\, p\} = \int_0^\infty \mathfrak{H}_\nu\{f(x);\, y\}\,\mathfrak{L}\{(xy)^{\frac{1}{2}}J_\nu(xy)g(x);\, y\}\, dy. \tag{6}$$

When we take $g(x) = x^\lambda$ in Theorem 1, we obtain

THEOREM 1(*b*). *If* $f(x)$ *and* $\mathfrak{H}_\nu\{f(x);\, y\} \in L(0,\, \infty)$, *and if*

$$R(\tfrac{1}{2}+\nu) > 0,\; R(\tfrac{3}{2}\pm\mu) > 0,\; R(\lambda\pm\mu+\nu+2) > 0,\; R(p) > 0,$$

then

$$\mathfrak{K}_\mu\{x^\lambda f(x);\, p\} = \int_0^\infty \mathfrak{H}_\nu\{f(x);\, y\}\theta(p,\, y)\, dy, \tag{7}$$

where

$$\theta(p,\, y) = 2^\lambda p^{-\lambda-\nu-\frac{3}{2}} y^{\frac{1}{2}+\nu}\, \frac{\Gamma\{\frac{1}{2}(\lambda\pm\mu+\nu+2)\}}{\Gamma(\nu+1)}\; {}_2F_1[\tfrac{1}{2}(\lambda\pm\mu+\nu+2);\, 1+\nu;\, -y^2/p^2],$$

since [3, p. 52]

$$\int_0^\infty x^{\lambda+1}K_\mu(px)J_\nu(yx)\, dx = (py)^{-\frac{1}{2}}\theta(p,\, y),$$

for $R(p) > 0, \quad y > 0, \quad R(\lambda\pm\mu+\nu+2) > 0.$

If we put $\mu = \frac{1}{2}$, then replace $f(x)$ by $x^{\frac{1}{2}}f(x)$, λ by $\mu-\frac{1}{2}$ and write $y^{\frac{1}{2}}\mathfrak{H}_\nu\{f(x);\, y\}$ for $\mathfrak{H}_\nu\{x^{\frac{1}{2}}f(x);\, y\}$, we get the result given by Bhonsle [4], since [2, p. 129]

$${}_2F_1\left(-\frac{1}{2}\mu-\frac{1}{2}\nu,\, \frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu;\, 1-\mu;\, 1-\frac{1}{z^2}\right) = 2^{-\mu}(z^2-1)^{\frac{1}{2}\mu}z^{-\mu}\Gamma(1-\mu)P_\nu^\mu(z),$$

and

$$2^{2z-1}\Gamma(z)\Gamma(z+\tfrac{1}{2}) = \Gamma(\tfrac{1}{2})\Gamma(2z).$$

Similarly, if we take $g(x) = x^{\lambda-2}J_\rho(ax)$, we get

THEOREM 1(*c*). *If* $f(x)$ *and* $\mathfrak{H}_\nu\{f(x);\, y\}$ *both belong to* $L(0,\infty)$, *and if* $R(\tfrac{1}{2}+\nu)>0$, $R(\lambda\pm\mu+\nu+\rho)>0$, $R(p)>0$ *and a is real, then*

$$\mathfrak{K}_\mu\{x^{\lambda-2}J_\rho(ax)f(x);\, p\} = \int_0^\infty \mathfrak{H}_\nu\{f(x);\, y\}\theta(p, a;\, y)\, dy, \tag{8}$$

where

$$\theta(p, a;\, y) = \frac{\Gamma\{\tfrac{1}{2}(\lambda\pm\mu+\nu+\rho)\}}{\Gamma(1+\nu)\Gamma(1+\rho)}\, 2^{\lambda-2}a^\rho p^{\frac{1}{2}-\lambda-\nu-\rho}y^{\frac{1}{2}+\nu}$$

$$\times F_4\left[\tfrac{1}{2}(\lambda-\mu+\nu+\rho), \tfrac{1}{2}(\lambda+\mu+\nu+\rho);\, 1+\rho, 1+\nu;\, -\frac{a^2}{p^2}, -\frac{y^2}{p^2}\right],$$

F_4 being the fourth type of Appell's hypergeometric function of two variables.

The above result is readily obtained from (4) when we evaluate $\mathfrak{H}_\nu\{(px)^{\frac{1}{2}}K_\mu(px)g(x);\, y\}$ with the help of the result by Bailey [5, p. 38].

We now illustrate the above theorems by certain suitable examples and give below certain integrals involving products of Bessel functions.

Example 1. If we start with $f(x) = x^{\kappa-\lambda+\frac{1}{2}}K_\sigma(bx)$, then [3, p. 93]

$$\mathfrak{K}_\mu\{x^\lambda f(x);\, p\} = 2^{\kappa-1}b^\sigma p^{-\kappa-\sigma-\frac{1}{2}}\frac{\Gamma\{\tfrac{1}{2}(\kappa\pm\mu\pm\sigma+2)\}}{\Gamma(\kappa+2)}\, {}_2F_1\left(\begin{matrix}\tfrac{1}{2}(\kappa+\sigma\pm\mu+2) \\ \kappa+2\end{matrix};\, 1-\frac{b^2}{p^2}\right),$$

for $R(p+b)>0$, $R(\kappa\pm\mu\pm\sigma+2)>0$; and [3, p. 52]

$$\mathfrak{H}_\nu\{f(x);\, y\} = \frac{\Gamma\{\tfrac{1}{2}(\nu+\kappa-\lambda\pm\sigma+2)\}2^{\kappa-\lambda}y^{\frac{1}{2}+\nu}}{\Gamma(\nu+1)b^{\nu+\kappa-\lambda+2}}\, {}_2F_1\left[\begin{matrix}\tfrac{1}{2}(\nu+\kappa-\lambda\pm\sigma+2) \\ 1+\nu\end{matrix};\, -\frac{y^2}{b^2}\right],$$

for $R(b)>0$, $R(\nu+\kappa-\lambda\pm\sigma+2)>0$, $y>0$.

Therefore the result (7) gives us

$$\int_0^\infty y^{2c-1}\, {}_2F_1\left(\begin{matrix}\alpha, \beta \\ c\end{matrix};\, -\frac{y^2}{b^2}\right){}_2F_1\left(\begin{matrix}\gamma, \delta \\ c\end{matrix};\, -\frac{y^2}{p^2}\right)dy$$

$$= \tfrac{1}{2}\frac{\{\Gamma(c)\}^2\Gamma(\alpha+\gamma-c)\Gamma(\alpha+\delta-c)\Gamma(\beta+\gamma-c)\Gamma(\beta+\delta-c)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\delta)\Gamma(\alpha+\beta+\gamma+\delta-2c)}\, p^{2c-2\alpha}b^{2\alpha}$$

$$\times\, {}_2F_1\left(\alpha+\gamma-c, \alpha+\delta-c;\, \alpha+\beta+\gamma+\delta-2c;\, 1-\frac{b^2}{p^2}\right), \tag{9}$$

for $R(c)>0$, $R(\alpha+\gamma-c)>0$, $R(\alpha+\delta-c)>0$, $R(\beta+\gamma-c)>0$, $R(\beta+\delta-c)>0$, and the result

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(8) gives us

$$\int_0^\infty x^{\kappa-1}K_\mu(px)K_\sigma(bx)J_\rho(ax)\,dx$$

$$= \frac{\Gamma\{\frac{1}{2}(\nu+\kappa-\lambda\pm\sigma+2)\}\Gamma\{\frac{1}{2}(\lambda\pm\mu+\nu+\rho)\}}{\{\Gamma(\nu+1)\}^2\Gamma(1+\rho)p^{\lambda+\nu+\rho}}2^{\kappa-2}a^\rho \int_0^\infty y^{1+2\nu}{}_2F_1\left(\begin{matrix}\frac{1}{2}(\nu+\kappa-\lambda\pm\sigma+2)\\ \nu+1\end{matrix};-\frac{y^2}{b^2}\right)$$

$$\times F_4\left(\tfrac{1}{2}(\lambda-\mu+\nu+\rho), \tfrac{1}{2}(\lambda+\mu+\nu+\rho); 1+\rho, 1+\nu; -\frac{a^2}{p^2}, -\frac{y^2}{p^2}\right)dy. \qquad (10)$$

Since

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)}F_4(\alpha, \beta; \gamma, \delta; x, y) = \sum_{r=0}^\infty \frac{\Gamma(\alpha+r)\Gamma(\beta+r)}{r!\,\Gamma(\gamma+r)}x^r{}_2F_1(\alpha+r, \beta+r; \delta; y), \qquad (11)$$

the integral on the right of (10) can be evaluated with the help of (9). On evaluating it and then using the well known result

$$\frac{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)}{}_2F_1(a, b; c; 1-z)$$

$$= \Gamma(a)\Gamma(b)\Gamma(c-a-b){}_2F_1(a, b; a+b-c+1; z)$$

$$+\Gamma(c-a)\Gamma(c-b)\Gamma(a+b-c)z^{c-a-b}{}_2F_1(c-a, c-b; c-a-b+1; z), \qquad (12)$$

and interpreting it again with the help of (11), we get

$$\int_0^\infty x^{\kappa-1}J_\rho(ax)K_\sigma(bx)K_\mu(cx)\,dx = 2^{\kappa-3}\sum_{\sigma,-\sigma}\frac{a^\rho b^\sigma\Gamma(-\sigma)\Gamma\{\frac{1}{2}(\kappa+\sigma\pm\mu+\rho)\}}{\Gamma(1+\rho)c^{\kappa+\rho+\sigma}}$$

$$\times F_4\left[\tfrac{1}{2}(\kappa-\mu+\sigma+\rho), \tfrac{1}{2}(\kappa+\mu+\sigma+\rho) ; 1+\rho, 1+\sigma; -\frac{a^2}{c^2}, \frac{b^2}{c^2}\right], \qquad (13)$$

for $R(\kappa\pm\mu+\sigma+\rho) > 0$, $R(b+c) > 0$, $a > 0$.

Also we know that

$$K_\sigma(z) = \frac{\pi}{\sin(\sigma\pi)}\tfrac{1}{2}\{I_{-\sigma}(z)-I_\sigma(z)\} = \tfrac{1}{2}\sum_{\sigma,-\sigma}\Gamma(-\sigma)\Gamma(1+\sigma)I_\sigma(z), \qquad (14)$$

and therefore, from (13), we get

$$\int_0^\infty x^{\kappa-1}J_\rho(ax)I_\sigma(bx)K_\mu(cx)\,dx = 2^{\kappa-2}\frac{a^\rho b^\sigma\Gamma\{\frac{1}{2}(\kappa+\sigma\pm\mu+\rho)}{\Gamma(1+\rho)\Gamma(1+\sigma)c^{\kappa+\rho+\sigma}}$$

$$\times F_4\left[\tfrac{1}{2}(\kappa+\sigma-\mu+\rho), \tfrac{1}{2}(\kappa+\sigma+\mu+\rho); 1+\rho, 1+\sigma; -\frac{a^2}{c^2}, \frac{b^2}{c^2}\right], \qquad (15)$$

for $R(\kappa+\sigma\pm\mu+\rho) > 0$, $R(c-b) > 0$, $a > 0$.

When we compare (10) and (13), we get

$$\int_0^\infty y^{\gamma-1}{}_2F_1(\alpha, \beta; \gamma; -by)F_4(\xi, \eta; \zeta, \gamma; -ac, -cy)\,dy$$

$$= \sum_{\alpha,\beta} \frac{c^{\alpha-\gamma}\{\Gamma(\gamma)\}^2\Gamma(\beta-\alpha)\Gamma(\alpha+\xi-\gamma)\Gamma(\alpha+\eta-\gamma)}{b^\alpha\Gamma(\alpha)\Gamma(\beta)\Gamma(\xi)\Gamma(\eta)}$$

$$\times F_4\left(\alpha+\xi-\gamma, \alpha+\eta-\gamma; \zeta, \alpha-\beta+1; -ac, -\frac{c}{b}\right), \quad (16)$$

for $R(\alpha+\xi-\gamma) > 0$, $R(\alpha+\eta-\gamma) > 0$, $R(\beta+\xi-\gamma) > 0$, $R(\beta+\eta-\gamma) > 0$, $a, b, c > 0$.

Example 2. If we start with $f(x) = x^{\kappa-\lambda-\frac{3}{2}}K_\rho(ax)K_\sigma(bx)$ in (7), then, by virtue of (13), we get

$$\int_0^\infty x^{\kappa-1}K_\rho(ax)K_\sigma(bx)K_\mu(px)\,dx$$

$$= \sum_{\sigma,-\sigma} \frac{2^{\kappa-3}b^\sigma\Gamma(-\sigma)\Gamma\{\frac{1}{2}(\kappa-\lambda+\sigma\pm\rho+\nu)\}\Gamma\{\frac{1}{2}(\lambda\pm\mu+\nu+2)\}}{p^{\lambda+\nu+2}a^{\kappa-\lambda+\nu+\sigma}\{\Gamma(1+\nu)\}^2}$$

$$\times \int_0^\infty y^{1+2\nu}{}_2F_1\left[\begin{matrix}\frac{1}{2}(\lambda\pm\mu+\nu+2)\\ 1+\nu\end{matrix}; -\frac{y^2}{p^2}\right]F_4\left(\begin{matrix}\frac{1}{2}(\kappa-\lambda+\sigma-\rho+\nu), \frac{1}{2}(\kappa-\lambda+\sigma+\rho+\nu)\\ 1+\nu, 1+\sigma\end{matrix}; -\frac{y^2}{a^2}, \frac{b^2}{a^2}\right)dy$$

$$= \sum_{\sigma,-\sigma} \frac{2^{\kappa-3}b^\sigma\Gamma(-\sigma)\Gamma\{\frac{1}{2}(\kappa-\lambda+\sigma\pm\rho+\nu)\}\Gamma\{\frac{1}{2}(\lambda\pm\mu+\nu+2)\}}{p^{\lambda+\nu+2}a^{\kappa-\lambda+\nu+\sigma}\{\Gamma(1+\nu\}^2}$$

$$\times \int_0^\infty y^{1+2\nu}{}_2F_1\left[\begin{matrix}\frac{1}{2}(\lambda\pm\mu+\nu+2)\\ 1+\nu\end{matrix}; -\frac{y^2}{p^2}\right]$$

$$\times F_4\left(\begin{matrix}\frac{1}{2}(\kappa-\lambda+\sigma-\rho+\nu), \frac{1}{2}(\kappa-\lambda+\sigma+\rho+\nu)\\ 1+\sigma, 1+\nu\end{matrix}; \frac{b^2}{a^2}, -\frac{y^2}{a^2}\right)dy.$$

The integral on the right, when evaluated with the help of (16), gives us

$$\int_0^\infty x^{\kappa-1}K_\rho(ax)K_\sigma(bx)K_\mu(cx)\,dx$$

$$= \sum_{\sigma,-\sigma}\sum_{\mu,-\mu} \Gamma(-\sigma)\Gamma(-\mu)\Gamma\{\tfrac{1}{2}(\kappa+\mu+\sigma\pm\rho)2^{\kappa-4}a^{-\kappa-\mu-\sigma}b^\sigma c^\mu$$

$$\times F_4\left(\tfrac{1}{2}(\kappa+\mu+\sigma-\rho), \tfrac{1}{2}(\kappa+\mu+\sigma+\rho); 1+\sigma, 1+\mu; \frac{b^2}{a^2}, -\frac{c^2}{a^2}\right), \quad (17)$$

for $R(\kappa\pm\rho\pm\sigma\pm\mu) > 0$, $R(a+b+c) > 0$.

The above result can also be obtained by the application of (14) to the result given by Bailey [6].

The results (13) and (15) help us in obtaining two more particular cases of the Theorem 1, given below.

THEOREM 1(d). *If* $f(x)$, $\mathfrak{H}_\nu\{f(x); y\} \in L(0, \infty)$, *and if*

$$R(\lambda \pm \mu + \nu + \sigma) > 0, \quad R(\tfrac{1}{2} + \nu) > 0, \; R(p - a) > 0,$$

then

$$\mathfrak{K}_\mu\{x^{\lambda-2} I_\sigma(ax) f(x); p\} = 2^{\lambda-2} \frac{a^\sigma \Gamma\{\frac{1}{2}(\lambda + \sigma \pm \mu + \nu)\}}{p^{\lambda+\nu+\sigma-\frac{1}{2}} \Gamma(1+\sigma)\Gamma(1+\rho)}$$

$$\times \int_0^\infty \mathfrak{H}_\nu\{f(x); y\} y^{\frac{1}{2}+\nu} F_4\left(\tfrac{1}{2}(\lambda+\sigma-\mu+\nu), \tfrac{1}{2}(\lambda+\sigma+\mu+\nu); 1+\nu, 1+\sigma; -\frac{y^2}{p^2}, \frac{a^2}{p^2}\right) dy. \tag{18}$$

The above result is obtained when we take $g(x) = x^{\lambda-2} I_\sigma(ax)$ in (4) and use (15). Similarly, when we take $g(x) = x^{\lambda-2} K_\sigma(bx)$ in (4), then, by virtue of (13), we have

THEOREM 1(e). *If* $f(x)$, $\mathfrak{H}_\nu\{f(x); y\} \in L(0, \infty)$, *and if*

$$R(\lambda \pm \mu \pm \rho + \nu) > 0, \quad R(\tfrac{1}{2} + \nu) > 0, \; R(p + a) > 0,$$

then

$$\mathfrak{K}_\mu\{x^{\lambda-2} K_\sigma(ax) f(x); p\} = 2^{\lambda-3} \sum_{\sigma, -\sigma} \frac{a^\sigma \Gamma(-\sigma)\Gamma\{\frac{1}{2}(\lambda + \sigma \pm \mu + \nu)\}}{p^{\lambda+\nu+\sigma-\frac{1}{2}} \Gamma(1+\nu)}$$

$$\times \int_0^\infty y^{\frac{1}{2}+\nu} \mathfrak{H}_\nu\{f(x); y\} F_4\left(\tfrac{1}{2}(\lambda+\sigma-\mu+\nu), \tfrac{1}{2}(\lambda+\sigma+\mu+\nu); 1+\nu, 1+\sigma; -\frac{y^2}{p^2}, \frac{a^2}{p^2}\right) dy. \tag{19}$$

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