THE SUBNORMAL SUBGROUP STRUCTURE OF
THE INFINITE SYMPLECTIC GROUP

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1. Introduction

Let R be a ring with identity and let Ω be a totally ordered set. Let Ω' be a totally ordered set which is disjoint from and equipotent to Ω with $':\Omega \rightarrow \Omega'$ an order preserving bijection. Define $\Omega_1 = \Omega \cup \Omega'$ and let Ω_1 be totally ordered by inheriting the order from Ω and Ω' and with $\omega < \lambda'$, for all $\omega \in \Omega$ and $\lambda' \in \Omega'$. Let M be the free R-module $R^{(\Omega_1)}$. We define the alternate bilinear form $(*, *)$ on M by

$$
\left.\begin{aligned}
(e_\omega, e_{\omega'}) &= 1 \\
(e_{\omega'}, e_\omega) &= -1
\end{aligned}\right\} \quad \text{for all } \omega \in \Omega,
$$

$$
\left.\begin{aligned}
(e_\alpha, e_\beta) &= 0 \\
(e_{\alpha'}, e_{\beta'}) &= 0
\end{aligned}\right\} \quad \text{for all } \alpha, \beta \in \Omega,
$$

$$
\left.\begin{aligned}
(e_{\alpha'}, e_\beta) &= 0 \\
(e_\alpha, e_{\beta'}) &= 0
\end{aligned}\right\} \quad \text{for all } \alpha, \beta \in \Omega, \quad \alpha \neq \beta.
$$

The symplectic group $Sp(\Omega, R)$ is the group of R-automorphisms X of M for which $(X(x), X(y)) = (x, y)$, for all $x, y \in M$. We see that $Sp(\Omega, R)$ is a subgroup of $GL(\Omega_1, R)$ and, since Ω_1 is totally ordered, we can think of any $X \in Sp(\Omega, R)$ as an invertible $\Omega_1 \times \Omega_1$ matrix over R partitioned as

$$
\begin{array}{c} \Omega \\ \Omega' \end{array}
\left[\begin{array}{c|c} & \\ \hline & \end{array} \right] \\
\quad \Omega \quad \Omega'
$$

We shall call $X \in Sp(\Omega, R)$ a *symplectic transformation* or *matrix*. If we let J denote the $\Omega_1 \times \Omega_1$ matrix

$$
\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},
$$

where I is the $\Omega \times \Omega$ identity matrix then   clear that X is a symplectic matrix if and only if $X'JX = J$. (X' denotes the transpose matrix of X.)

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The structure of $Sp(\Omega, R)$ has been the subject of many investigations: when Ω is finite and R is a division ring, Dieudonné studied $Sp(\Omega, R)$ in (4) and (5); when Ω is infinite, Spiegel in (8) examined $Sp(\Omega, R)$ when R is a division ring and Maxwell in (6) classified the normal subgroups of $Sp(\Omega, R)$ when R is a commutative ring in which 2 is a unit. We continue this investigation to give a classification of the subnormal subgroups of $Sp(\Omega, R)$ when Ω is infinite and R is a commutative ring with identity. The classification we shall obtain is similar to the "sandwiching" results of Bass (3) and Wilson (9) in which the congruence subgroups play a major rôle.

2. Preliminaries

For the rest of this paper Ω shall always denote an infinite set and all rings R will have identity. Let $a \in R$, $\alpha, \beta \in \Omega_1$; we denote the $\Omega_1 \times \Omega_1$ matrix whose only non-zero entry is in the (α, β)th position and is a by $ae_{\alpha\beta}$. We let 1 denote the $\Omega_1 \times \Omega_1$ identity matrix. Following Robertson (7), we define the $\Omega_1 \times \Omega_1$ matrices $t_{\lambda\mu}(a)$, $r_{\lambda\mu}(a)$, $r_{\lambda\lambda}(a)$, $s_{\lambda\mu}(a)$ and $s_{\lambda\lambda}(a)$, for $a \in R$ and $\lambda, \mu \in \Omega$, $\lambda \neq \mu$, by

$$t_{\lambda\mu}(a) = 1 + a(e_{\lambda\mu} - e_{\mu'\lambda'}),$$

$$r_{\lambda\mu}(a) = 1 + a(e_{\lambda\mu'} + e_{\mu\lambda'}),$$

$$r_{\lambda\lambda}(a) = 1 + ae_{\lambda\lambda'},$$

$$s_{\lambda\mu}(a) = 1 + a(e_{\lambda'\mu} + e_{\mu'\lambda}),$$

$$s_{\lambda\lambda}(a) = 1 + ae_{\lambda'\lambda}.$$

We also quote from (7) the following lemma which is a list of commutator identities.

Lemma 1. *Let R be a commutative ring, $a, b \in R$ and $\lambda, \mu, \alpha, \beta \in \Omega$.*

(i)	$[r_{\lambda\mu}(b), s_{\alpha\beta}(a)] = 1,$	λ, μ *distinct from α, β,*
(ii)	$[r_{\lambda\alpha}(b), s_{\alpha\beta}(a)] = t_{\lambda\beta}(ba)$	α, β, λ *distinct,*
(iii)	$[r_{\lambda\alpha}(b), s_{\alpha\alpha}(a)] = t_{\lambda\alpha}(ba)r_{\lambda\lambda}(b^2 a)$	$\lambda \neq \alpha,$
(iv)	$[r_{\lambda\lambda}(b), s_{\alpha\lambda}(a)] = t_{\lambda\alpha}(ba)s_{\alpha\alpha}(-ba^2)$	$\lambda \neq \alpha,$
(v)	$[s_{\lambda\mu}(a), t_{\lambda\beta}(b)] = s_{\mu\beta}(ab)$	λ, μ, β *distinct,*
(vi)	$[s_{\lambda\mu}(a), t_{\alpha\mu}(b)] = 1$	$\alpha \neq \lambda, \mu,$
(vii)	$[s_{\lambda\mu}(a), t_{\lambda\mu}(b)] = s_{\mu\mu}(2ab)$	$\lambda \neq \mu,$
(viii)	$[s_{\lambda\lambda}(a), t_{\lambda\mu}(b)] = s_{\lambda\mu}(ab)s_{\mu\mu}(ab^2)$	$\lambda \neq \mu,$
(ix)	$[s_{\lambda\mu}(a), t_{\alpha\beta}(b)] = 1$	λ, μ *distinct from α, β,*
(x)	$[r_{\lambda\mu}(a), t_{\alpha\beta}(b)] = 1$	λ, μ *distinct from α, β,*
(xi)	$[r_{\lambda\mu}(a), t_{\alpha\mu}(b)] = r_{\alpha\lambda}(-ab)$	λ, μ, α *distinct,*
(xii)	$[r_{\lambda\mu}(a), t_{\lambda\beta}(b)] = 1$	$\beta \neq \lambda, \mu,$
(xiii)	$[r_{\lambda\mu}(a), t_{\lambda\mu}(b)] = r_{\lambda\lambda}(-2ab)$	$\lambda \neq \mu,$
(xiv)	$[r_{\lambda\lambda}(a), t_{\alpha\lambda}(b)] = r_{\alpha\lambda}(-ab)r_{\alpha\alpha}(ab^2)$	$\lambda \neq \alpha.$

We define $ESp(\Omega, R)$ to be the subgroup of $Sp(\Omega, R)$ generated by $\{r_{\lambda\mu}(a), s_{\lambda\mu}(a) : \lambda, \mu \in \Omega, a \in R\}$. For any two-sided ideal $\mathfrak{p}$ of R, $ESp(\Omega, \mathfrak{p})$ will denote the normal closure of

$\{r_{\lambda\mu}(a), s_{\lambda\mu}(a): \lambda, \mu \in \Omega, a \in \mathfrak{p}\}$ in $ESp(\Omega, R)$, $Sp(\Omega, \mathfrak{p})$ will denote the kernel of the group homomorphism $Sp(\Omega, R) \to Sp(\Omega, R/\mathfrak{p})$ induced by the projection $R \to R/\mathfrak{p}$ and $Sp'(\Omega, \mathfrak{p})$ will denote the inverse image of the centre of $Sp(\Omega, R/\mathfrak{p})$. Clearly

$$ESp(\Omega, \mathfrak{p}) \leq Sp(\Omega, \mathfrak{p}) \leq Sp'(\Omega, \mathfrak{p}).$$

It is easy to show that the groups $ESp(\Omega, R)$ and $ESp(\Omega, \mathfrak{p})$ coincide with $ESp(R)$ and $ESp(\mathfrak{p})$ of Maxwell (**6**). It follows from (**6**) that when R is commutative $ESp(\Omega, R)$ and $ESp(\Omega, \mathfrak{p})$ are normal subgroups of $Sp(\Omega, R)$. However, the proofs of 2.2–2.5 of (**6**) do not depend upon the commutativity of R and thus we may state

Lemma 2. *For any two-sided ideal* $\mathfrak{p}$ *of a ring* R, $ESp(\Omega, R)$ *and* $ESp(\Omega, \mathfrak{p})$ *are normal subgroups of* $Sp(\Omega, R)$ *and* $ESp(\Omega, \mathfrak{p}) = [ESp(\Omega, R), Sp'(\Omega, \mathfrak{p})]$.

Simple matrix arguments show that the centre of $Sp(\Omega, R)$ comprises all matrices of the form

$$\begin{bmatrix} rI & 0 \\ 0 & rI \end{bmatrix},$$

for all central units r of R with $r^2 = 1$. Since all matrices in $ESp(\Omega, R)$ differ from the identity matrix in only finitely many rows, we see that $ESp(\Omega, R)$ has trivial centre. From these remarks we are able to deduce

Lemma 3. *Let* R *be a commutative ring. If* H *is a subgroup of* $Sp(\Omega, R)$ *that is normalised by* $ESp(\Omega, R)$ *then the following assertions are equivalent.*

  (i) $H \leq Z(Sp(\Omega, R))$.
  (ii) $[H, ESp(\Omega, R)] = 1$.
  (iii) $H \cap ESp(\Omega, R) = 1$.

(For any group G we use $Z(G)$ to denote the centre of G.)

For any symplectic matrix X we let $J(X)$ denote the two-sided ideal generated by the matrix entries $X_{\alpha\beta}$, $X_{\alpha\alpha} - X_{\beta\beta}$, for all $\alpha, \beta \in \Omega_1$, $\alpha \neq \beta$ (c.f. (**9**)). We shall call $J(X)$ the *level* of X and the entries $X_{\alpha\beta}$ and $X_{\alpha\alpha} - X_{\beta\beta}$ the *generators* of $J(X)$. For any subgroup H of $Sp(\Omega, R)$ we denote by $J(H)$ the ideal which is the sum of the ideals $J(X)$, for all $X \in H$, and by $K(H)$ the ideal which is the sum of the ideals $J(X)$, for all $X \in H \cap ESp(\Omega, R)$, (c.f. (**2**)). For any commutative ring R and any $X \in Sp(\Omega, R)$ it is clear that $J(X)$ is the least ideal $\mathfrak{p}$ of R such that $X \in Sp'(\Omega, \mathfrak{p})$ and that $X \in Z(Sp(\Omega, R))$ if and only if $J(X) = 0$.

We now complete this section by recalling some general group theoretic definitions. We say that a subgroup H of a group G is a *subnormal* subgroup of G if there exists a normal series of subgroups

$$H = H_d \lhd H_{d-1} \lhd \cdots \lhd H_0 = G.$$

We shall write $H \lhd^d G$. The least integer d such that $H \lhd^d G$ is called the *defect* of H in G. We define the terms $\gamma_i(G)$ of the *lower central series* of G by $\gamma_1(G) = G$,

$\gamma_i(G) = [G, \gamma_{i-1}(G)]$, $i = 2, 3, \ldots$. A group G is called *nilpotent* if $\gamma_m(G) = 1$, for some integer m. If $c + 1$ is the least value of m satisfying this condition then c is called the *class* of G.

3. Statement and discussion of results

We shall prove

Theorem 1. *Let R be a commutative ring with identity and let Ω be an infinite linearly ordered set. Whenever G is a subgroup of $Sp(\Omega, R)$ that contains $ESp(\Omega, R)$ and H is a subnormal subgroup of G of defect d, then there exists an ideal $\mathfrak{p}$ of R such that*

$$ESp(\Omega, (4\mathfrak{p})^{f(d)}) \leq H \leq Sp'(\Omega, \mathfrak{p})$$

where $f(d) = (6^d - 1)/5$, for each integer $d \geq 1$.

Notice that if H is a normal subgroup of $Sp(\Omega, R)$ then we can take $G = Sp(\Omega, R)$ and $d = 1$. If we assume that 2 is a unit in R then we obtain the inclusions $ESp(\Omega, \mathfrak{p}) \leq H \leq Sp'(\Omega, \mathfrak{p})$. Conversely, if we assume that such inclusions hold for a subgroup H of $Sp(\Omega, R)$ then it follows from Lemma 2 that H is normalised by $ESp(\Omega, R)$. Thus, Theorem 1 reduces to the main theorem of (6) when applied to normal subgroups of $Sp(\Omega, R)$. In fact, our next result shows more clearly how Theorem 1 extends the work of (6).

Theorem 2. *Let R be a commutative ring in which 2 is a unit and let H be a subgroup of $ESp(\Omega, R)$. The following assertions are equivalent.*

  (i) *H is a subnormal subgroup of $ESp(\Omega, R)$.*
  (ii) *For some ideal $\mathfrak{p}$ of R and some integer m*

$$ESp(\Omega, \mathfrak{p}^m) \leq H \leq Sp'(\Omega, \mathfrak{p}).$$

We shall see that the ideal $\mathfrak{p}$ of Theorems 1 and 2 is $J(H)$, the level of H. We have remarked that the sandwiching

$$ESp(\Omega, J(H)) \leq H \leq Sp'(\Omega, J(H))$$

of a subgroup H of $ESp(\Omega, R)$ is a sufficient condition for the normality of H. However, the following example shows that, when 2 is not a unit in R, the sandwiching

$$ESp(\Omega, 4J(H)) \leq H \leq Sp'(\Omega, J(H))$$

of Theorem 1 is not in general a sufficient condition for the normality of H.

Example. If we let N denote the set of natural numbers $\{1, 2, 3, \ldots\}$ and let H be the subgroup of $ESp(N, Z)$ generated by $ESp(N, 4Z)$ and $\{r_{ij}(n) : i, j \in N, n \in Z\}$ then we see

that $ESp(N, 4Z) \leq H \leq Sp'(N, Z)$. However, H is not a normal subgroup of $ESp(N, Z)$ since $t_{12}(1) \notin H$ while $t_{12}(1) = [r_{13}(1), s_{23}(1)] \in [H, ESp(N, Z)]$.

4. Technical lemmas

We begin with

Lemma 4. *If H is a subgroup of $Sp(\Omega, R)$ that is normalised by $ESp(\Omega, \mathfrak{p})$, for some two-sided ideal $\mathfrak{p}$ of R then $K(H)$ contains yx, for all generators y of $J(H)$ and all $x \in \mathfrak{p}$.*

Proof. Let $x \in \mathfrak{p}$. We must consider $y = A_{\mu\rho}$ or $y = A_{\mu\mu} - A_{\rho\rho}$, for some $\mu, \rho \in \Omega_1$, $\mu \neq \rho$ and $A \in H$. We examine only the case $\mu, \rho \in \Omega$ in detail; the other cases can be proved using similar arguments. First suppose $y = A_{\mu\rho}$. Pick $\lambda \in \Omega$, $\lambda \neq \rho, \mu$ and put $t = t_{\rho\lambda}(x)$. Then $[t, A^{-1}] \in ESp(\Omega, R) \cap H$, by hypothesis and so for all $\psi \in \Omega_1$, $\psi \neq \mu$, $[t, A^{-1}]_{\mu\psi} \in K(H)$ and $[t, A^{-1}]_{\mu\mu} - 1 \in K(H)$, since elements of $ESp(\Omega, R)$ differ from the identity matrix in only finitely many rows. But

$$[t, A^{-1}]_{\mu\psi} = A_{\mu\rho} x A_{\lambda\psi}^{-1} - A_{\mu\lambda'} x A_{\rho'\psi}^{-1}, \quad \psi \neq \mu$$

and

$$[t, A^{-1}]_{\mu\mu} - 1 = A_{\mu\rho} x A_{\lambda\mu}^{-1} - A_{\mu\lambda'} x A_{\rho'\mu}^{-1}.$$

But

$$A_{\mu\rho} x = ([t, A^{-1}]_{\mu\mu} - 1) A_{\mu\lambda} + \sum_{\psi \neq \mu} [t, A^{-1}]_{\mu\psi} A_{\psi\lambda}$$

and hence $A_{\mu\rho} x \in K(H)$.

It remains to show that $K(H)$ contains $(A_{\mu\mu} - A_{\rho\rho})x$. If we examine $[t, A^{-1}]_{\rho\psi}$ and $[t, A^{-1}]_{\rho\rho} - 1$, applying the same technique as above, we see that $A_{\rho\rho} x - x A_{\lambda\lambda} \in K(H)$ where λ is chosen as above. Similarly we see that $K(H)$ contains $A_{\mu\mu} x - x A_{\lambda\lambda}$ and hence $K(H)$ contains $(A_{\mu\mu} - A_{\rho\rho})x$.

Lemma 4 allows us to deduce

Corollary 1. *If H is a subgroup of $Sp(\Omega, R)$ that is normalised by $ESp(\Omega, \mathfrak{p})$, for some two-sided ideal $\mathfrak{p}$ of R, then $J(H)\mathfrak{p} \leq K(H)$.*

Next we prove

Lemma 5. *Let R be a commutative ring and let H be a subgroup of $Sp(\Omega, R)$ that is normalised by $ESp(\Omega, \mathfrak{p})$, for some ideal $\mathfrak{p}$ of R. Let $A \in H \cap ESp(\Omega, R)$ and let x be any generator of $J(A)$. Let $u_i \in \mathfrak{p}$, $i = 1, \ldots, 5$, let $y = 4u_5 u_4 u_3 u_2 u_1 x$ and let $\lambda, \mu \in \Omega$, $\lambda \neq \mu$. H contains $r_{\lambda\mu}(y)$, $r_{\lambda\lambda}(y)$, $s_{\lambda\mu}(y)$ and $s_{\lambda\lambda}(y)$.*

Proof. We may suppose that $x \neq 0$, for otherwise there is nothing to prove. We must consider $x = A_{\alpha\beta}$ or $x = A_{\alpha\alpha} - 1$, for some $\alpha, \beta \in \Omega_1$, $\alpha \neq \beta$. We examine only the case $\alpha, \beta \in \Omega$ in detail; the other cases can be proved using similar arguments.

First suppose that $x = A_{\alpha\beta}$. Let $\phi \in \Omega$, $\phi \neq \lambda$, μ, α, β, be such that the ϕth column and the ϕ'th row of A are trivial. (A row (column) is said to be trivial if it is equal to the corresponding row (column) of the identity matrix.) From Lemma 1 we see that

$$[r_{\phi\beta}(-u_2), [t_{\phi\alpha}(u_1), A]] = r_{\phi\beta}(2u_2u_1x) \tag{1}$$

for any $u_1, u_2 \in \mathfrak{p}$. Let $\psi \in \Omega$, $\psi \neq \phi$, λ, μ, α, β be such that the ψ'th row and the ψth column of A are trivial:

$$[r_{\phi\phi}(y), t_{\psi\phi}(u)] = r_{\psi\phi}(-uy)r_{\psi\psi}(u^2y)$$

shows that H contains $r_{\psi\phi}(2u_3u_2u_1x)$, since $r_{\psi\psi}(u^2y) \in H$ by (1). Finally the identities

$$r_{\lambda\mu}(4u_5u_4u_3u_2u_1x) = [r_{\psi\phi}(2u_3u_2u_1x), t_{\mu\phi}(u_4), t_{\lambda\psi}(2u_5)]$$

$$r_{\lambda\lambda}(4u_5u_4u_3u_2u_1x) = [r_{\psi\phi}(2u_3u_2u_1x), t_{\lambda\phi}(u_4), t_{\lambda\psi}(u_5)]$$

$$s_{\lambda\mu}(4u_5u_4u_3u_2u_1x) = [s_{\phi\lambda}(2u_5), [s_{\psi\mu}(-u_4), r_{\psi\phi}(2u_3u_2u_1x)]]$$

$$S_{\lambda\lambda}(4u_5u_4u_3u_2u_1x) = [s_{\phi\lambda}(u_5), [s_{\psi\lambda}(-u_4), r_{\psi\phi}(2u_3u_2u_1x)]]$$

show that H contains the required matrices when $x = A_{\alpha\beta}$.

Now suppose that $x = A_{\alpha\alpha} - 1$. If we let ϕ be as above then Lemma 1 shows that

$$r_{\phi\phi}(2u_2u_1x) = [r_{\phi\alpha}(-u_2), [t_{\phi\alpha}(u_1), A]].$$

The proof of this case is now completed using repeated applications of Lemma 1, as above.

We are now able to deduce

Corollary 2. *If R is a commutative ring and if H is a subgroup of $Sp(\Omega, R)$ that is normalised by $ESp(\Omega, \mathfrak{p})$, for some ideal $\mathfrak{p}$ of R, then H contains $ESp(\Omega, 4\mathfrak{p}^6J(H))$.*

5. The proof of the theorems

We begin with the proof of Theorem 1. Let H, G, d and f be as in the statement of the theorem. Whenever $\mathfrak{p} \leq \mathfrak{q}$, $Sp'(\Omega, \mathfrak{p}) \leq Sp'(\Omega, \mathfrak{q})$. Thus, since $X \in Sp'(\Omega, J(X))$, from the remarks following Lemma 3, we see that $X \in Sp'(\Omega, J(H))$, for all $X \in H$. Hence $H \leq Sp'(\Omega, J(H))$. It remains to prove the first inclusion and to do this we shall use an inductive argument. If $d = 1$ then H is normalised by $ESp(\Omega, R)$ so that H contains

$$ESp(\Omega, 4R^6J(H)) = ESp(\Omega, 4J(H)).$$

Now suppose that $ESp(\Omega, (4J(K))^{f(k)}) \leqq K$, for all subgroups K with subnormal chains of length $k < d$. If we put $J_0 = J(H_{d-1})$, where

$$H = H_d \lhd H_{d-1} \lhd^{d-1} G,$$

then we see that H_{d-1} contains $ESp(\Omega, (4J_0)^{f(d-1)})$. But $J = J(H) \leqq J_0$ so that H contains

$$ESp(\Omega, 4(4J)^{6f(d-1)}J) = ESp(\Omega, (4J)^{f(d)})$$

by Corollary 2. We conclude, by the principle of induction, that whenever H is as in the statement of Theorem 1, $ESp(\Omega, (4\mathfrak{p})^{f(d)}) \leqq H$, where $\mathfrak{p}$ is the level of H. This, together with our initial remarks, completes the proof of Theorem 1.

We now prove Theorem 2. The proof of (i) implies (ii) follows from Theorem 1. Suppose therefore that (ii) holds. Since $(ESp(\Omega, R) \cap Sp'(\Omega, \mathfrak{p}))/ESp(\Omega, \mathfrak{p}^m)$ is nilpotent there is a series

$$ESp(\Omega, \mathfrak{p}^m) \leqq \gamma_m \leqq \cdots \leqq \gamma_1 = ESp(\Omega, R) \cap Sp'(\Omega, \mathfrak{p})$$

where $\gamma_i = \gamma_i(ESp(\Omega, R) \cap Sp'(\Omega, \mathfrak{p}))$, $i = 1, \ldots, m$. Hence

$$H \leqq \gamma_m H \leqq \cdots \leqq \gamma_1 H \leqq ESp(\Omega, R)$$

is a normal series from H to $ESp(\Omega, R)$ of length $m+1$ and we conclude that H is a subnormal subgroup of $ESp(\Omega, R)$ of defect at most $m+1$.

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