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ON A C-PLANE OF ORDER 25

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Rao, Rodabaugh, Wilke and Zemmer [J. Combin. Theory Ser. A. 11(1971),72-92] constructed a number of new VW systems called C-systems from the exceptional near-fields and established that they coordinatize translation planes not isomorphic to generalized André planes. In this paper the translation complement of the plane coordinatized by the C-system I-1 has been found. This plane has the interesting property that its translation complement divides the ideal points into two orbits of lengths 10 and 16. Further, the translation complement contains a subgroup isomorphic to SL(2,5) and therefore one of the exceptional Walker's planes of order 25 [H. Luneberg, Translation Planes, Springer-Verlag (1980), pp.235-244] is indeed the C-plane corresponding to the C-system I-1, which was discovered in 1969.

1. Introduction

Rao, Rodabaugh, Wilke and Zemmer have discovered a new class of translation planes [7] called C-planes and proved that they were not isomorphic to generalized André planes. In [3] Kenneth Lueder considered the full collineation groups of planes corresponding to C-systems which are derivations. The aim of this paper is to determine the full collineation group of the C-plane corresponding to the C-system I-1 [7,p.80] which is not a derivation. This plane has the following interesting properties:

(1) the translation complement divides the ideal points into two orbits of lengths 10 and 16 and it is of order 7680;

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(2) the translation complement contains a subgroup isomorphic to SL(2,5) and hence one of the exceptional Walker's planes of order 25 [4,p,244] is indeed the C-plane corresponding to the C-system I-1 discovered in 1969. Further the translation complement also contains subgroups isomorphic to the quarternion group.

We note that the properties (1) and (2) above overlap with some properties deduced in a paper of Ostrom [10], but the spirit of this paper is entirely different from that of Ostrom [10]. The scheme of this paper is such that no result of this paper can be omitted, if we want to prove Theorems 3.8 and 3.9, which were not covered by Ostrom [10].

2. Description of the C-plane corresponding to the C-system I-1

In [5,p.391] Marshall Hall describes the structure of the exceptional near-field numbered I of order 25 by giving the generators of the multiplicative group in the form of matrices. These are $\begin{pmatrix}0&1\\4&0\end{pmatrix}$ and $\begin{pmatrix}1&3\\4&3\end{pmatrix}$, which are 2×2 matrices over GF(5). An examination of the structure of the matrices shows that the matrices are of the following forms:

$$\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}, \begin{pmatrix}0 & a\\ 4a^{-1} & 0\end{pmatrix} \text{ and } \begin{pmatrix}a & b\\ 2b^{-1} & 3a^{-1}\end{pmatrix}$$

where $a,b = 1,2,3$ and 4. The collection of these matrices together with the zero matrix is a 1-spread set [1,p.220] over GF(5) which defines the exceptional near-field $(F,+,\cdot)$, where $F = \{(x,y) \mid x,y \in GF(5)\}$. Addition on F is defined component-wise and the multiplication $\cdot$ on F is defined by:

$$(x,y) \cdot (a,b) = (a,b)M(x,y)$$

where $M(x,y)$ is the unique matrix in the above 1-spread set with the first row (x,y).

Let T be the linear transformation given by $\begin{pmatrix}1&0\\0&4\end{pmatrix}$ on F. Let $G = \langle xT \cdot x^{-1} \mid x \in F' \rangle$ where $F' = F - \{(0,0)\}$. Following the procedure of [7], the 1-spread set C over GF(5) for the C-system I-1 is

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given by

$$\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}a & 0\\0 & a^{-1}\end{pmatrix}, \begin{pmatrix}0 & a\\4a^{-1} & 0\end{pmatrix} \text{ and } \begin{pmatrix}a & b\\3b^{-1} & 2a^{-1}\end{pmatrix}$$

where $a,b = 1,2,3$ and 4. The matrices of C together with their corresponding characteristic polynomials are listed in table 1. The entry a,b under the head C.P. of M_i in table 1, denotes that the matrix M_i has the characteristic polynomial $\lambda^2 + a\lambda + b$. In what follows the C-system means the C-system I-1 and the C-plane means the plane coordinatized by the C-system I-1.

TABLE 1

i.	M_i	C.P. of M_i a,b	i.	M_i	C.P. of M_i a,b	i.	M_i	C.P. of M_i a,b
0.	$\begin{pmatrix}00\\00\end{pmatrix}$	0,0	8.	$\begin{pmatrix}40\\04\end{pmatrix}$	2,1	16.	$\begin{pmatrix}44\\23\end{pmatrix}$	3,4
1.	$\begin{pmatrix}01\\40\end{pmatrix}$	0,1	9.	$\begin{pmatrix}11\\32\end{pmatrix}$	2,4	17.	$\begin{pmatrix}12\\42\end{pmatrix}$	2,4
2.	$\begin{pmatrix}02\\20\end{pmatrix}$	0,1	10.	$\begin{pmatrix}14\\22\end{pmatrix}$	2,4	18.	$\begin{pmatrix}13\\12\end{pmatrix}$	2,4
3.	$\begin{pmatrix}03\\30\end{pmatrix}$	0,1	11.	$\begin{pmatrix}22\\41\end{pmatrix}$	2,4	19.	$\begin{pmatrix}21\\31\end{pmatrix}$	2,4
4.	$\begin{pmatrix}04\\10\end{pmatrix}$	0,1	12.	$\begin{pmatrix}23\\11\end{pmatrix}$	2,4	20.	$\begin{pmatrix}24\\21\end{pmatrix}$	2,4
5.	$\begin{pmatrix}10\\01\end{pmatrix}$	3,1	13.	$\begin{pmatrix}32\\44\end{pmatrix}$	3,4	21.	$\begin{pmatrix}31\\34\end{pmatrix}$	3,4
6.	$\begin{pmatrix}20\\03\end{pmatrix}$	0,1	14.	$\begin{pmatrix}33\\14\end{pmatrix}$	3,4	22.	$\begin{pmatrix}34\\24\end{pmatrix}$	3,4
7.	$\begin{pmatrix}30\\02\end{pmatrix}$	0,1	15.	$\begin{pmatrix}41\\33\end{pmatrix}$	3,4	23.	$\begin{pmatrix}42\\43\end{pmatrix}$	3,4
						24.	$\begin{pmatrix}43\\13\end{pmatrix}$	3,4

The 1-spread set C has the following property which will be used in the sequel,

$$C = \{M_0\} \cup G \cup M_9 G \cup M_{17} G \qquad (2.1)$$

$$C = \{M_0\} \cup G \cup GM_9 \cup GM_{17} \qquad (2.2)$$

where $G = \langle M_1, M_6 \rangle$ which is isomorphic to the quaternion group. Note that G consists of matrices M_i, $1 \le i \le 8$ in table 1.

3. Some collineations of the C-plane

Let $V_i = \{(a,b,c,d) \mid (c,d) = (a,b)M_i,\ M_i \in C,\ a,b \in GF(5)\}$ for $0 \le i \le 24$ and $V_{25} = V_\infty = \{(0,0,p,q) \mid p,q \in GF(5)\}$ be subspaces of $V(4,5)$, where $V(4,5)$ is the 4-dimensional vector space over $GF(5)$. The incidence structure with V_i, $0 \le i \le 25$ and their cosets in the additive group of $V(4,5)$ as lines and vectors of $V(4,5)$ as points with inclusion as the incidence relation is the C-plane π whose collineations we seek to find.

Any non-singular linear transformation of $V(4,5)$ which permutes the subspaces V_i, $0 \le i \le 25$, among themselves induces a collineation belonging to the translation complement and conversely any collineation from the translation complement is induced by a non-singular linear transformation of $V(4,5)$ which permutes the subspaces V_i, $0 \le i \le 25$, among themselves.

It is easily verified that any non-singular linear transformation in the block matrix form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A, B, C and D are 2×2 matrices over $GF(5)$, induces a collineation of π if and only if the following conditions are satisfied. For any $M_i \in C$:

(a) if $A + M_i C$ is non-singular, then $(A+M_iC)^{-1}(B+M_iD) \in C$; if $A + M_i C$ is singular then $A + M_i C$ is the zero matrix and $B + M_i D$ is non-singular;

(b) if C is non-singular then $C^{-1}D \in C$; if C is singular then C is the zero matrix and D is non-singular.

We now exhibit some important non-singular linear mappings on $V(4,5)$ and show, through the following lemmas, that they actually induce collineations of π. In what follows collineation means a collineation from the translation complement and its action on the ideal points is denoted by the notation $(r,s,\ldots,t)$ meaning the ideal point corresponding to the line V_r is mapped onto the ideal point corresponding to the line V_s and so on, and finally the ideal point corresponding to the line V_t is mapped onto the ideal point corresponding to the line V_r.

LEMMA 3.1. *The group of all collineations that fix the ideal points* $0,25,5$ *and* 1 *is generated by*

$$\alpha = \begin{pmatrix} 22 & 00 \\ 32 & 00 \\ 00 & 22 \\ 00 & 32 \end{pmatrix} \quad \text{and} \quad \hat{\alpha} = \begin{pmatrix} 23 & 00 \\ 22 & 00 \\ 00 & 23 \\ 00 & 22 \end{pmatrix}$$

and is of order 16.

Proof. Any collineation which fixes the ideal points $0,25$ and 5 is of the form $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ for some $A \in GL(2,5)$ and satisfies

(i) $AM_1 = M_1A$;

(ii) for each matrix $M \in C$ there exists a matrix $N \in C$ such that $MA = AN$.

Taking $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a,b,c,d \in GF(5)$ and solving the simultaneous equations obtained from the relation (i), we get the general form of $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, $a,b \in GF(5)$. The group

$$\left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| a,b \in GF(5), a^2 + b^2 \neq 0 \right\}$$

is generated by $\begin{pmatrix} 22 \\ 32 \end{pmatrix}$ and $\begin{pmatrix} 23 \\ 22 \end{pmatrix}$ and is of order 16. Choosing $a = b = 2$ we get the non-singular linear transformation α. Using the relation (2.1) and the fact

$$\left\{ A^{-1}M_iA \middle| A = \begin{pmatrix} 22 \\ 32 \end{pmatrix}, i = 1,6,9,17 \right\} \subset C$$

it follows that α is a collineation of π. Its action on the ideal points is given by:

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α : (0)(25)(5)(8)(1)(4)(2,7,3,6)(9,19,11,17)(16,22,14,24)
(10,18,12,20)(15,23,13,21).

Choosing $a = 2$, $b = 3$, we get $\hat{\alpha} = 3\alpha^3$ and $\hat{\alpha}$ is therefore a collineation of π. Hence to the lemma.

LEMMA 3.2. *The group* J *of all collineations that fix the ideal points* 0, 25 *and* 5 *is generated by* α, $\hat{\alpha}$ *and* β, *where* $\beta = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$, $B = \begin{pmatrix} 10 \\ 03 \end{pmatrix}$, *and it is of order* 96.

Proof. That $\beta = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$ is a collineation follows as in Lemma 3.1, and its action restricted to the ideal points is given by:

β : (0)(25)(5)(8)(6)(7)(1,3,4,2)(9,18,10,17)(19,12,20,11)
(21,14,22,13)(15,24,16,23).

The group of all collineations that fix the ideal points 0,25 and 5 is induced by inner automorphisms of GL(2,5). The set G is invariant under these inner automorphisms because G consists of all matrices of C which have determinant 1. The actions of α and β reveal that the group of all collineations that fix the ideal points 0,25 and 5 is transitive on the set of ideal points {1,2,3,4,6,7}. The group of all collineations that fix the ideal points 0,25 and 5 is therefore given as a union of cosets of $\langle\alpha,\hat{\alpha}\rangle$ by collineations which fix the ideal points 0,25,5 and send the ideal point 1 onto the ideal points 1,2,3,4,6,7 respectively. The collineations $\alpha, \beta^{-1}, \beta, \beta^2, \beta^{-1}\alpha^{-1}$, $\beta^{-1}\alpha$ fix the ideal points 0,25,5 and send the ideal point 1 onto the ideal points 1,2,3,4,6,7 respectively. Therefore the group of all collineations that fix the ideal points 0,25 and 5 is of order $16 \times 6 = 96$ and is generated by α, $\hat{\alpha}$ and β.

LEMMA 3.3. *The mapping* $\gamma = \begin{pmatrix} I & I \\ I & 4I \end{pmatrix}$ *is a collineation and its action restricted to the ideal points is given by:*

γ : *(0,5)(8,25)(1,4)(2,3)(6,7)(13,24)(14,23)(15,22)(16,21)*
(9)(10)(11)(12)(17)(18)(19)(20).

Proof. Follows easily.

From (2.1) and (2.2) it follows that the mappings $\begin{pmatrix} I & 0 \\ 0 & M \end{pmatrix}$

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and $\begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix}$ are collineations of π for all $M \in \mathcal{G}$. Let

$$H = \left\{ \begin{pmatrix} I & 0 \\ 0 & M \end{pmatrix} \middle| M \in \mathcal{G} \right\} \text{ and } L = \left\{ \begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix} \middle| M \in \mathcal{G} \right\}.$$

The groups H and L are isomorphic to the quaternion group.

LEMMA 3.4. *The group K of all collineations which fix the ideal points 0 and 25 is given by $\langle J,H \rangle$ and is of order 768.*

Proof. Any collineation δ that fixes the ideal points 0 and 25 is of the form $\delta = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ for some $A,B \in GL(2,5)$. Further A and B must satisfy the condition that for each $M \in \mathcal{C}$ there exists $N \in \mathcal{C}$ such that $A^{-1}MB = N$. Taking $M = I$ we get the condition that $A^{-1}B \in \mathcal{C}$. The 1-spread set $\mathcal{C}$ consists of one matrix with determinant zero, 8 matrices with determinant 1 and 16 matrices with determinant 4. This forces $A^{-1}B$ to be of determinant 1 and therefore $A^{-1}B \in \mathcal{G}$. Let $A^{-1}B = M, M \in \mathcal{G}$. Then

$$\delta = \begin{pmatrix} A & 0 \\ 0 & AM \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & M \end{pmatrix}.$$

But $\begin{pmatrix} I & 0 \\ 0 & M \end{pmatrix}$ is a collineation from H. Therefore $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ is a collineation and belongs to J, and $\delta \in \langle J,H \rangle$. Since every element of $\langle J,H \rangle$ fixes the ideal points 0 and 25, $K = \langle J,H \rangle$. Obviously the set of ideal points $\{1,2,3,4,5,6,7,8\}$ is invariant under K, and H is transitive on the set of ideal points $\{1,2,3,4,5,6,7,8\}$. Therefore the order of $K = 8 \times (\text{order of } J) = 768$.

LEMMA 3.5. *The translation complement divides the set of ideal points into two orbits of lengths 10 and 16.*

Proof. Consider the collineation $\tau = \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} \in J$, where $C = M_7$. Its action restricted to the ideal points is

$$\tau : (0)(25)(1,2,4,3)(5,6,8,7)(9,11,16,14)(10,12,15,13)$$
$$(17,20,24,21)(18,19,23,22).$$

The action of $\alpha\tau$ is

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$\alpha\tau$: (0)(25)(1,2,5,6,4,3,8,7)(9,23,10,19,16,18,15,22)
(11,20,12,24,14,21,13,17).

The group $\langle \alpha\tau,\gamma \rangle$ divides the set of ideal points into two orbits, one orbit consisting of the ideal points {0,1,2,3,4,5,6,7,8,25} and the other consisting of the remaining ideal points. Now consider the collineation $\gamma\alpha\tau$, its action restricted to the ideal points is

$\gamma\alpha\tau$: (0,6,1,3,5)(2,8,25,7,4)(9,23,21,18,15)(10,19,16,13,14)
(11,20,12,24,17)(22).

We now claim that no collineation of π maps an ideal point from R = {0,1,2,3,4,5,6,7,8,25} onto any ideal point outside R. If there is such a collineation then the plane is a flag transitive plane of order 25 and is therefore isomorphic to the first or the second of Foulser's flag transitive planes [2,Lemma 6.3,p.203] of order 25. Neither of these planes possesses a collineation fixing only one ideal point and moving all the remaining [6,11]. But our C-plane contains a collineation, viz. $\gamma\alpha\tau$ which fixes only one ideal point. Hence the lemma.

LEMMA 3.6. *Let C' be a $(n-1)$-spread set defined over $GF(q)$, $q = p^r$, p a prime ≥ 3, r an integer > 0 and π' be the translation plane associated with C'. Further suppose that C' satisfies the condition that if $M \in C'$ then $-M \in C'$. Then there exists a collineation which fixes V_∞ and maps V_0 onto V_r if and only if $M + M_r \in C'$ whenever $M \in C'$.*

Proof. See [8].

LEMMA 3.7. *Any collineation of π that fixes the ideal point 25 also fixes the ideal point 0.*

Proof. Suppose μ is a collineation that fixes the ideal point 25 and moves the ideal point 0 onto the ideal point r. In view of Lemma 3.5 there is no loss of generality if we suppose $r = 1$. The 1-spread set C satisfies the condition that $-M \in C$ whenever $M \in C$. Taking $M = M_1$, we find that $M + M_1 \notin C$. Now the lemma follows from Lemma 3.6.

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THEOREM 3.8. *The translation complement is generated by the groups* K *and* $\langle \alpha\tau, \gamma \rangle$ *and is of order* 7680.

Proof. Since the translation complement is transitive on the set of ideal points $\{0,25,1,2,3,4,5,6,7,8\}$ and any collineation that fixes the ideal point 25 also fixes the ideal point 0, the translation complement has the coset decomposition $G = \{Kx_i \mid i = 0,25,1,2,3,4,5,6,7,8\}$, where x_i is a collineation that sends the ideal point 0 onto the ideal point i. Hence the order of G is 7680.

Thus the full collineation group of π is $\langle G, G' \rangle$ where G' is the translation group of π.

THEOREM 3.9. *One of the exceptional planes of order* 25 *of Walker is isomorphic to the C-plane.*

Proof. The collineations $\sigma = 4\gamma\tau^{-1}\alpha =$ and $\eta = \begin{pmatrix} M_3 & 0 \\ 0 & M_1 \end{pmatrix}$ together generate a group isomorphic to $SL(2,5)$ because their

$$\text{conjugates by } \begin{pmatrix} 11 & 33 \\ 32 & 41 \\ 00 & 30 \\ 00 & 03 \end{pmatrix} \text{ are } S_1 = \begin{pmatrix} 10 & 00 \\ 11 & 00 \\ 31 & 10 \\ 13 & 11 \end{pmatrix} \text{ and } T = \begin{pmatrix} 01 & 00 \\ 40 & 00 \\ 00 & 01 \\ 00 & 40 \end{pmatrix}$$

respectively, which generate a group isomorphic to $SL(2,5)$ [12,p.143]. The only translation planes of order 25 which admit $SL(2,5)$ in their translation complements [4,Theorem 49.6,p.254] are

(i) the Desarguesian plane,

(ii) Hall plane,

(iii) Hering's plane, and

(iv) two exceptional Walker planes of order 25.

It has already been noted that π is not Desarguesian [7,Theorem 7.3]. The Hall plane and the Hering's plane of order 25 have the orbit structure 20 and 6 on ℓ_∞ [9,pp.204-209] and therefore they are not isomorphic to π. The first exceptional plane π_1 [4,p.244] of order 25 of Walker has the orbit structure 1 and 25 [9,p,209] and therefore cannot be isomorphic to π. Hence the second exceptional plane $\pi_{2,0}$ [4,p.244] of order 25 of Walker is isomorphic to the C-plane π

discovered earlier.

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