

EXISTENCE AND STABILITY OF PERIODIC SOLUTIONS OF A RAYLEIGH TYPE EQUATION

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Abstract

In this work, we shall be concerned with the following forced Rayleigh type equation:

$$x''(t) + f(x'(t)) + g(t, x(t)) = e(t).$$

Under certain assumptions, some criteria for guaranteeing the existence, uniqueness and asymptotic stability (in the Lyapunov sense) of periodic solutions of this equation are presented by applying the Manásevich–Mawhin continuation theorem, Floquet theory, Lyapunov stability theory and some analysis techniques. Moreover, an example is provided to demonstrate the applications of our results.

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1. Introduction

In this note we consider the existence, uniqueness and asymptotic stability of periodic solutions of the following forced Rayleigh type equation:

$$x''(t) + f(x'(t)) + g(t, x(t)) = e(t), \quad (1.1)$$

where $f \in C^2(\mathbb{R}, \mathbb{R})$, $e \in C(\mathbb{R}, \mathbb{R})$, $g \in C^1(\mathbb{R}^2, \mathbb{R})$, $e(t)$ and $g(t, x)$ are T -periodic functions in t with $T > 0$, and $f(0) = 0$.

It is well known that the British mathematical physicist Lord Rayleigh (John William Strutt, 1842–1919), a Nobel Prize Laureate in Physics in 1904, introduced an equation of the form

$$x''(t) + f(x'(t)) + ax(t) = 0 \quad (1.2)$$

to model the oscillations of a clarinet reed; for details, see [16]. One of the most difficult problems connected with the study of the Rayleigh equation (1.2) is whether

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it can support limit cycles. Limit cycles are very important in science. They model systems that exhibit self-sustained oscillations. In other words, these systems oscillate even in the absence of external periodic forcing.

During the past several years, many authors have studied the self-sustained oscillations in acoustics. Fletcher [3] showed that there exist particular ranges of acoustic impedance for the inlet and outlet ducts to the valve within which self-sustained valve oscillation is possible. Keith and Rand [6], Shimizu and Morioka [15] and Shimizu [14] investigated the existence conditions for limit cycles of the Rayleigh equation, that is, conditions under which auto-oscillations occur. The question of periodic solutions was also studied for different generalizations of the Rayleigh equation, for example (1.1). In [4], Gaines and Mawhin introduced some continuation theorems and applied them in discussing the existence of solutions of certain ordinary differential equations. In particular, a specific example was provided in [4, p. 99] on how T -periodic solutions could be obtained by means of these theorems for (1.1) with $e(t) = 0$. In the course of derivations, it was realized that once a priori bounds for the T -periodic solutions of the homotopic equations is known, then standard procedures will allow these theorems to imply the existence of T -periodic solutions to the original equation. Applying these approaches, the authors of [8–10, 17, 18] continued to discuss the Rayleigh equation and obtained many new results on the existence of T -periodic solutions of (1.1), and generalized some results in [4]. Other authors researched the stability of periodic solutions of the Duffing type equations

$$x''(t) + bx'(t) + g(t, x(t)) = 0 \quad (1.3)$$

and

$$x''(t) + bx'(t) + g(t, x(t)) = e(t). \quad (1.4)$$

In [13], Ortega studied the stability of periodic solutions of (1.3) and obtained some stability results by topological index. Then Lazer and McKenna [7] obtained stability results by converting (1.3) to a fixed-point problem. Recently, Chen and Li [1] studied the rate of decay of stable periodic solutions of (1.4) and determined a sharp rate of exponential decay for a solution that is near to the unique periodic solution. However, to our knowledge, there are much fewer results on the stability of periodic solutions of (1.1). Hence it is essential to continue to study periodic solutions of (1.1) in this case.

The main purpose of this paper is to establish some sufficient conditions for guaranteeing the existence, uniqueness and asymptotic stability of T -periodic solutions of (1.1), by applying the Manásevich–Mawhin continuation theorem, Floquet theory, Lyapunov stability theory and some analysis techniques.

The following notation will be used throughout the rest of this paper.

$$|x|_\infty = \max_{t \in [0, T]} |x(t)|, \quad |x'|_\infty = \max_{t \in [0, T]} |x'(t)|, \quad |x|_k = \left(\int_0^T |x(t)|^k dt \right)^{1/k}.$$

Set

$$C_T^1 := \{x \in C^1(\mathbb{R}, \mathbb{R}) \mid x(t + T) = x(t)\}$$

and

$$C_T := \{x \in C(\mathbb{R}, \mathbb{R}) : x(t + T) = x(t)\},$$

which are two Banach spaces with the norms

$$\|x\|_{C_T^1} = \max\{|x|_\infty, |x'|_\infty\}, \quad \|x\|_{C_T} = |x|_\infty.$$

Our main theorem is as follows.

THEOREM 1.1. *Assume that there exist constants $C_1 \geq 0$, $C_2 \geq 0$ and $D_1 \geq 0$ such that:*

(H₁) $(g(t, u) - g(t, v))(u - v) > 0$, for all $t, u, v \in \mathbb{R}$ with $u \neq v$;

(H₂) $|f(t, u) - f(t, v)| \leq C_1|u - v|$, for all $t, u, v \in \mathbb{R}$;

(H₃) $|g(t, u) - g(t, v)| \leq C_2|u - v|$, for all $t, u, v \in \mathbb{R}$;

(H₄) $(C_1/2\pi)T + (C_2/4\pi)T^2 < 1$;

(H₅) $x(g(t, x) - e(t)) > 0$, for all $|x| \geq D_1$ and $t \in \mathbb{R}$;

(H₆) $\int_0^T [g'_x(t, x) - \frac{1}{4}(f'(y))^2] dt > 0$ and $f'(y) > 0$, for $x \in I_1$ and $y \in I_2$;

(H₇)

$$\left\{ g'_x(t, x) - \frac{1}{2}f''(y)[e(t) - f(y) - g(t, x)] - \frac{1}{4}(f'(y))^2 \right\} < \frac{1}{T^2},$$

for $t \in \mathbb{R}$, $x \in I_1$ and $y \in I_2$; where

$$G = \max_{t \in [0, T]} |g(t, 0)|, \quad M_0 = \frac{(G + |e|_\infty + C_2 D_1)\sqrt{T}}{1 - (C_1/2\pi)T - (C_2/4\pi)T^2},$$

$$M_1 = D_1 + \frac{\sqrt{T^3}}{4\pi} M_0, \quad M_2 = \frac{1}{2}\sqrt{T} M_0,$$

$$I_1 = [-M_1, M_1] \quad \text{and} \quad I_2 = [-M_2, M_2].$$

Then (1.1) has a unique T -periodic solution which is asymptotically stable.

In Section 2 we introduce some lemmas which will help us to obtain the main theorem, the proof of which is given in Section 3. The paper concludes with an illustrative example is provided to demonstrate the application of our results.

2. Lemmas

In this section, we shall introduce some lemmas which will help us to obtain our main theorem. First, let us recall the Manásevich–Mawhin continuation theorem, which is useful in obtaining the existence of T -periodic solutions of (1.1).

For the periodic boundary value problem

$$(\varphi_p(x'(t)))' = h(t, x, x'), \quad x(0) = x(T), \quad x'(0) = x'(T), \quad (2.1)$$

where $p > 1$, $\varphi_p : \mathbb{R} \rightarrow \mathbb{R}$, $\varphi_p(s) = |s|^{p-2}s$ is a one-dimensional p -Laplacian, and $h \in C(\mathbb{R}^3, \mathbb{R})$ is T -periodic in the first variable, we have the following result.

LEMMA 2.1. [11] *Let Ω be an open bounded set in C_T^1 . Suppose the following conditions hold:*

(i) *for each $\lambda \in (0,1)$ the problem*

$$(\varphi_p(x'(t)))' = \lambda h(t, x, x'), \quad x(0) = x(T), \quad x'(0) = x'(T)$$

has no solution on $\partial\Omega$;

(ii) *the equation*

$$F(a) := \frac{1}{T} \int_0^T h(t, a, 0) dt = 0$$

has no solution on $\partial\Omega \cap \mathbb{R}$;

(iii) *the Brouwer degree of F is*

$$\text{deg}(F, \Omega \cap \mathbb{R}, 0) \neq 0.$$

Then the periodic boundary value problem (2.1) has at least one T -periodic solution on $\overline{\Omega}$.

REMARK 2.2. In particular, if $p = 2$, (2.1) reduces to the following periodic boundary value problem:

$$x''(t) = h(t, x, x'), \quad x(0) = x(T), \quad x'(0) = x'(T), \tag{2.2}$$

which implies that the above Lemma 2.1 can be used to investigate the existence of T -periodic solutions of (1.1).

Next, two lemmas will be introduced to help us in obtaining the uniqueness of T -periodic solutions of (1.1).

LEMMA 2.3. *If $x \in C^2(\mathbb{R}, \mathbb{R})$ with $x(t + T) = x(t)$, then*

$$|x'|_2^2 \leq \left(\frac{T}{2\pi}\right)^2 |x''|_2^2.$$

PROOF. Lemma 2.3 is a direct consequence of the Wirtinger inequality; see [5, 12] for its proof. □

LEMMA 2.4. *Suppose (H_1) – (H_4) hold. Then (1.1) has at most one T -periodic solution.*

PROOF. Suppose that $x_1(t)$ and $x_2(t)$ are two T -periodic solutions of (1.1). Then

$$[x_1(t) - x_2(t)]'' + [f(x_1'(t)) - f(x_2'(t))] + [g(t, x_1(t) - g(t, x_2(t))] = 0. \tag{2.3}$$

Setting $Z(t) = x_1(t) - x_2(t)$, we obtain, from (2.3), that

$$Z''(t) + [f(x_1'(t)) - f(x_2'(t))] + [g(t, x_1(t) - g(t, x_2(t))] = 0. \tag{2.4}$$

Since $Z(t) = x_1(t) - x_2(t)$ is a continuous T -periodic function in \mathbb{R} , there exist two constants $t_{\max}, t_{\min} \in \mathbb{R}$ such that

$$Z(t_{\max}) = \max_{t \in [0, T]} Z(t) = \max_{t \in \mathbb{R}} Z(t), \quad Z(t_{\min}) = \min_{t \in [0, T]} Z(t) = \min_{t \in \mathbb{R}} Z(t). \quad (2.5)$$

It follows that

$$Z'(t_{\max}) = x_1'(t_{\max}) - x_2'(t_{\max}) = 0, \quad Z''(t_{\max}) \leq 0 \quad (2.6)$$

and

$$Z'(t_{\min}) = x_1'(t_{\min}) - x_2'(t_{\min}) = 0, \quad Z''(t_{\min}) \geq 0. \quad (2.7)$$

Then, by (2.4)–(2.7),

$$\begin{aligned} g(t_{\max}, x_1(t_{\max})) - g(t_{\max}, x_2(t_{\max})) &= -Z''(t_{\max}) - [f(x_1'(t_{\max})) - f(x_2'(t_{\max}))] \\ &= -Z''(t_{\max}) \geq 0 \end{aligned} \quad (2.8)$$

and

$$\begin{aligned} g(t_{\min}, x_1(t_{\min})) - g(t_{\min}, x_2(t_{\min})) &= -Z''(t_{\min}) - [f(x_1'(t_{\min})) - f(x_2'(t_{\min}))] \\ &= -Z''(t_{\min}) \leq 0. \end{aligned} \quad (2.9)$$

From (2.8) and (2.9), we know that there exists a constant $t_0 \in \mathbb{R}$ such that

$$g(t_0, x_1(t_0)) - g(t_0, x_2(t_0)) = 0. \quad (2.10)$$

Now set $t_0 = nT + \tilde{t}_0$, where $\tilde{t}_0 \in [0, T]$ and n is an integer. Note that since $Z(t + T) = Z(t)$, we get, by (H_1) and (2.10), that

$$Z(\tilde{t}_0) = Z(t_0) = x_1(t_0) - x_2(t_0) = 0. \quad (2.11)$$

Hence, for any $t \in [\tilde{t}_0, \tilde{t}_0 + T]$, we obtain

$$|Z(t)| = \left| Z(\tilde{t}_0) + \int_{\tilde{t}_0}^t Z'(s) ds \right| \leq \int_{\tilde{t}_0}^t |Z'(s)| ds$$

and

$$|Z(t)| = \left| Z(\tilde{t}_0 + T) + \int_{\tilde{t}_0 + T}^t Z'(s) ds \right| = \left| - \int_t^{\tilde{t}_0 + T} Z'(s) ds \right| \leq \int_t^{\tilde{t}_0 + T} |Z'(s)| ds.$$

Now combining above two inequalities, we get

$$|Z(t)| \leq \frac{1}{2} \int_0^T |Z'(s)| ds.$$

This, together with the Schwarz inequality, gives

$$|Z|_\infty = \max_{t \in [\tilde{t}_0, \tilde{t}_0+T]} |Z(t)| \leq \frac{1}{2} \int_0^T |Z'(s)| ds \leq \frac{1}{2} |1|_2 |Z'|_2 = \frac{1}{2} \sqrt{T} |Z'|_2. \tag{2.12}$$

Next, multiplying $Z''(t)$ and (2.4) and integrating it from 0 to T , we obtain, by Lemma 2.3, (H₂), (H₃), (2.12) and the Schwarz inequality, that

$$\begin{aligned} |Z''|_2^2 &= - \int_0^T [f(x'_1(t)) - f(x'_2(t))] Z''(t) dt \\ &\quad - \int_0^T [g(t, x_1(t)) - g(t, x_2(t))] Z''(t) dt \\ &\leq \int_0^T |f(x'_1(t)) - f(x'_2(t))| |Z''(t)| dt \\ &\quad + \int_0^T |g(t, x_1(t)) - g(t, x_2(t))| |Z''(t)| dt \\ &\leq \int_0^T C_1 |x'_1(t) - x'_2(t)| |Z''(t)| dt + \int_0^T C_2 |x_1(t) - x_2(t)| |Z''(t)| dt \\ &\leq \int_0^T C_1 |Z'(t)| |Z''(t)| dt + \int_0^T C_2 |Z(t)| |Z''(t)| dt \\ &\leq C_1 |Z'|_2 |Z''|_2 + C_2 \sqrt{T} |Z|_\infty |Z''|_2 \\ &\leq \left(\frac{C_1}{2\pi} T + \frac{C_2}{4\pi} T^2 \right) |Z''|_2^2. \end{aligned} \tag{2.13}$$

Since $Z(t)$, $Z'(t)$ and $Z''(t)$ are continuous T -periodic functions, we get, by Lemma 2.3, (H₄) and (2.12), that

$$Z(t) = Z'(t) = Z''(t) = 0 \quad \text{for all } t \in \mathbb{R},$$

which implies that $x_1(t) \equiv x_2(t)$ for all $t \in \mathbb{R}$. Hence, (1.1) has at most one T -periodic solution. This completes the proof. □

Moreover, consider the homotopic equation of (1.1),

$$x''(t) + \lambda f(x'(t)) + \lambda g(t, x(t)) = \lambda e(t), \quad \lambda \in (0, 1). \tag{2.14}$$

The following lemma will show that the set of all possible T -periodic solutions of (2.14) are bounded in C_T^1 under some restrictive conditions. This result will help us to obtain the existence of T -periodic solutions of (1.1) when applying the above Manásevich–Mawhin continuation theorem.

LEMMA 2.5. *Suppose that (H₂)–(H₅) hold. Then the set of T -periodic solutions of (2.14) are bounded in C_T^1 .*

PROOF. Let $S \subset C_T^1$ be the set of T -periodic solutions of (2.14). If $S = \emptyset$, the proof is done. Suppose that $S \neq \emptyset$, and let $x \in S$. Then there exist two constants $\bar{t}, \underline{t} \in \mathbb{R}$ such that

$$x(\bar{t}) = \max_{t \in \mathbb{R}} x(t) \quad \text{and} \quad x(\underline{t}) = \min_{t \in \mathbb{R}} x(t),$$

implying that

$$x'(\bar{t}) = 0, x''(\bar{t}) \leq 0; \quad x'(\underline{t}) = 0, x''(\underline{t}) \geq 0. \quad (2.15)$$

Note that since $f(0) = 0$, we obtain, by (2.14) and (2.15), that

$$g(\bar{t}, x(\bar{t})) - e(\bar{t}) \geq 0 \quad \text{and} \quad g(\underline{t}, x(\underline{t})) - e(\underline{t}) \leq 0,$$

which implies that there exists a constant $\hat{t} \in \mathbb{R}$ such that

$$g(\hat{t}, x(\hat{t})) - e(\hat{t}) = 0. \quad (2.16)$$

Subsequently, set $\hat{t} = mT + \hat{t}_0$, where $\hat{t}_0 \in [0, T]$ and m is an integer. Then we get, by (H₅) and (2.16), that

$$|x(\hat{t}_0)| = |x(\hat{t})| \leq D_1. \quad (2.17)$$

Hence, for any $t \in [\hat{t}_0, \hat{t}_0 + T]$,

$$|x(t)| = \left| x(\hat{t}_0) + \int_{\hat{t}_0}^t x'(s) ds \right| \leq D_1 + \int_{\hat{t}_0}^t |x'(s)| ds$$

and

$$\begin{aligned} |x(t)| &= \left| x(\hat{t}_0 + T) + \int_{\hat{t}_0 + T}^t x'(s) ds \right| \leq D_1 + \left| - \int_t^{\hat{t}_0 + T} x'(s) ds \right| \\ &\leq D_1 + \int_t^{\hat{t}_0 + T} |x'(s)| ds. \end{aligned}$$

Now combining above two inequalities, we get

$$|x(t)| \leq D_1 + \frac{1}{2} \int_0^T |x'(s)| ds.$$

This together with the Schwarz inequality gives

$$\begin{aligned} |x|_\infty &= \max_{t \in [\hat{t}_0, \hat{t}_0 + T]} |x(t)| \leq D_1 + \frac{1}{2} \int_0^T |x'(s)| ds \\ &\leq D_1 + \frac{1}{2} \|1\|_2 \|x'\|_2 = D_1 + \frac{1}{2} \sqrt{T} \|x'\|_2. \end{aligned} \quad (2.18)$$

Note that, since $f(0) = 0$, we obtain, by Lemma 2.3, (H₂), (H₃), (2.14), (2.18) and the Schwarz inequality, that

$$\begin{aligned}
 |x''|_2^2 &= -\lambda \int_0^T f(x'(t))x''(t) dt - \lambda \int_0^T g(t, x(t))x''(t) dt + \lambda \int_0^T e(t)x''(t) dt \\
 &= -\lambda \int_0^T [f(x'(t)) - f(0)]x''(t) dt \\
 &\quad - \lambda \int_0^T [g(t, x(t)) - g(t, 0) + g(t, 0)]x''(t) dt + \lambda \int_0^T e(t)x''(t) dt \\
 &\leq \int_0^T |f(x'(t)) - f(0)||x''(t)| dt + \int_0^T |g(t, x(t)) - g(t, 0)||x''(t)| dt \\
 &\quad + \int_0^T |g(t, 0)||x''(t)| dt + \int_0^T |e(t)||x''(t)| dt \\
 &\leq \int_0^T C_1|x'(t)||x''(t)| dt + \int_0^T C_2|x(t)||x''(t)| dt \\
 &\quad + G \int_0^T |x''(t)| dt + |e|_\infty \int_0^T |x''(t)| dt \\
 &\leq C_1|x'|_2|x''|_2 + C_2\sqrt{T}|x|_\infty|x''|_2 + (G + |e|_\infty)|1|_2|x''|_2 \\
 &\leq \frac{C_1}{2\pi}T|x''|_2^2 + C_2\sqrt{T}\left(D_1 + \frac{1}{2}\sqrt{T}|x'|_2\right)|x''|_2 + (G + |e|_\infty)\sqrt{T}|x''|_2 \\
 &\leq \left(\frac{C_1}{2\pi}T + \frac{C_2}{4\pi}T^2\right)|x''|_2^2 + (G + |e|_\infty + C_2D_1)\sqrt{T}|x''|_2, \tag{2.19}
 \end{aligned}$$

where $G = \max_{t \in [0, T]} |g(t, 0)|$. Now, by (H₄) and (2.19),

$$|x''|_2 \leq \frac{(G + |e|_\infty + C_2D_1)\sqrt{T}}{1 - (C_1/2\pi)T - (C_2/4\pi)T^2} = M_0. \tag{2.20}$$

Since $x(0) = x(T)$, there exists a constant $\tilde{t} \in [0, T]$ such that $x'(\tilde{t}) = 0$. Then for any $t \in [\tilde{t}, \tilde{t} + T]$,

$$|x'(t)| = \left| x'(\tilde{t}) + \int_{\tilde{t}}^t x''(s) ds \right| \leq \int_{\tilde{t}}^t |x''(s)| ds$$

and

$$|x'(t)| = \left| x'(\tilde{t} + T) + \int_{\tilde{t}+T}^t x''(s) ds \right| \leq \left| - \int_t^{\tilde{t}+T} x''(s) ds \right| \leq \int_t^{\tilde{t}+T} |x''(s)| ds.$$

Now combining above two inequalities, we get, by the Schwarz inequality, that

$$|x'(t)| \leq \frac{1}{2} \int_0^T |x''(s)| ds \leq \frac{1}{2} |1|_2|x''|_2 = \frac{1}{2}\sqrt{T}|x''|_2,$$

which, together with (2.20), implies that

$$|x'|_{\infty} = \max_{t \in [\bar{t}, \bar{t}+T]} |x'(t)| \leq \frac{1}{2} \sqrt{T} |x''|_2 = \frac{1}{2} \sqrt{T} M_0 = M_2. \quad (2.21)$$

Moreover, according to Lemma 2.3, (2.18) and (2.20),

$$|x|_{\infty} \leq D_1 + \frac{1}{2} \sqrt{T} |x'|_2 \leq D_1 + \frac{\sqrt{T^3}}{4\pi} |x''|_2 \leq D_1 + \frac{\sqrt{T^3}}{4\pi} M_0 = M_1. \quad (2.22)$$

Let $M = \max\{M_1, M_2\}$; then we know that

$$|x|_{\infty} \leq M \quad \text{and} \quad |x'|_{\infty} \leq M.$$

This completes the proof. \square

REMARK 2.6. According to the proof of Lemma 2.5, we can easily conclude that, if $x(t)$ is a T -periodic solution of (1.1), then (2.21) and (2.22) also hold under the conditions of Lemma 2.5.

Finally, for convenience of use, we recall a well-known principle of linearized stability for periodic systems; for details, see for example [2, pp. 321–322]. Consider the periodic boundary value problem

$$\begin{cases} x' = F(t, x), \\ x(0) = x(T), \end{cases} \quad (2.23)$$

where $F : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous function that is T -periodic in t and has continuous first-order partial derivative with respect to x . Let p_0 be a T -periodic solution of (2.23); then we associate the T -periodic solution p_0 with the linearized equation

$$y' = F'_x(t, p_0)y. \quad (2.24)$$

The following theorem, which is due to Lyapunov (see [3, Theorem 2.1, p. 322]), shows the connections between the asymptotic stability of the T -periodic solution p_0 of (2.23) and the characteristic exponents of (2.24).

LEMMA 2.7. *If the characteristic exponents associated with (2.24) all have negative real parts, then the T -periodic solution p_0 of (2.23) is asymptotically stable.*

3. Proof of Theorem 1.1

Now we are in the position to present the proof of Theorem 1.1. The proof falls naturally in two parts. We begin by considering a existence and uniqueness result, before turning to asymptotic stability.

According to Remark 2.2, Lemma 2.1 can be used to obtain the existence of T -periodic solutions of (1.1). Set

$$h(t, x, x') = e(t) - f(x') - g(t, x). \quad (3.1)$$

Then (2.14) is equivalent to

$$x''(t) = \lambda h(t, x(t), x'(t)), \quad \lambda \in (0, 1). \tag{3.2}$$

By Lemma 2.5, there exists $M_3 > M$ such that, for any T -periodic solution $x(t)$ of (2.14) or (3.2),

$$|x|_\infty < M_3 \quad \text{and} \quad |x'|_\infty < M_3. \tag{3.3}$$

Let

$$\Omega = \{x \in C_T^1 : |x|_\infty < M_3, |x'|_\infty < M_3\}.$$

Then we know that (3.2) has no solution on $\partial\Omega$ as $\lambda \in (0, 1)$, so condition (i) of Lemma 2.1 is satisfied.

On the other hand, since

$$F(a) := \frac{1}{T} \int_0^T h(t, a, 0) dt,$$

for any $x \in \partial\Omega \cap \mathbb{R}$, $x = M_3$ or $x = -M_3$, and $f(0) = 0$, we obtain, by (H₅), that

$$\begin{aligned} F(M_3) &= \frac{1}{T} \int_0^T [e(t) - g(t, M_3)] dt < 0, \\ F(-M_3) &= \frac{1}{T} \int_0^T [e(t) - g(t, -M_3)] dt > 0, \end{aligned} \tag{3.4}$$

which imply that condition (ii) of Lemma 2.1 is also satisfied. Moreover, define

$$H(x, \mu) = -\mu x + (1 - \mu) \frac{1}{T} \int_0^T [e(t) - g(t, x)] dt;$$

in view of (3.4), we obtain that

$$xH(x, \mu) < 0 \quad \text{for all } x \in \partial\Omega \cap \mathbb{R} \text{ and } \mu \in [0, 1],$$

thus $H(x, \mu)$ is a homotopic transformation. From (3.4), together with the homotopic invariance theorem,

$$\begin{aligned} \deg(F, \Omega \cap \mathbb{R}, 0) &= \deg\left(\frac{1}{T} \int_0^T [e(t) - g(t, x)] dt, \Omega \cap \mathbb{R}, 0\right) \\ &= \deg(-x, \Omega \cap \mathbb{R}, 0) \neq 0. \end{aligned}$$

This implies that condition (iii) of Lemma 2.1 is satisfied. Therefore, it follows from Lemma 2.1 and Remark 2.2 that there exists a T -periodic solution of (1.1). The uniqueness of T -periodic solutions of (1.1) is guaranteed by Lemma 2.4.

We now turn to asymptotic stability. Let $x_0(t)$ be the unique T -periodic solution of (1.1). Then we know from Lemma 2.5 and Remark 2.6 that

$$|x_0|_\infty \leq M_1 \quad \text{and} \quad |x'_0|_\infty \leq M_2. \tag{3.5}$$

Moreover, according to (2.23) and (2.24), it is easy to see that the linearized equation of (1.1) is

$$y''(t) + f'(x'_0(t))y'(t) + g'_x(t, x_0(t))y(t) = 0. \tag{3.6}$$

In order to show that $x_0(t)$ is asymptotically stable, Lemma 2.7 will be applied.

First, we show that (3.6) does not have real Floquet (or characteristic) multipliers. If not, then there is a real Floquet multiplier α and a nontrivial solution $y(t)$ of (3.6) such that $y(t + T) = \alpha y(t)$. Set $y(t) = \exp(-\frac{1}{2} \int_0^t f'(x'_0(s)) ds)u(t)$; then $u(t)$ is a non-trivial solution of the equation

$$u''(t) + [g'_x(t, x_0(t)) - \frac{1}{2} f''(x'_0(t))x''_0(t) - \frac{1}{4} (f'(x'_0(t)))^2]u(t) = 0 \tag{3.7}$$

with the Floquet multiplier $\beta = \alpha \exp(\frac{1}{2} \int_0^T f'(x'_0(s)) ds)$ (that is, $u(t + T) = \beta u(t)$).

Now we show that the following claim is true.

CLAIM. There exists a $t_0 \in [0, T]$ such that

$$u(t_0) = 0. \tag{3.8}$$

Assume, by way of contradiction, that (3.8) does not hold. Then $u(t) \neq 0$ for all $t \in [0, T]$. Dividing (3.7) by $u(t)$ and integrating it from 0 to T , we obtain, noticing that $\int_0^T f''(x'_0(t))x''_0(t) dt = 0$ and $u'(0)/u(0) = u'(T)/u(T)$, that

$$\int_0^T \left(\frac{u'(t)}{u(t)} \right)^2 dt + \int_0^T \left[g'_x(t, x_0(t)) - \frac{1}{4} (f'(x'_0(t)))^2 \right] dt = 0, \tag{3.9}$$

which together with (3.5) contradicts the condition (H₆) that $\int_0^T [g'_x(t, x) - \frac{1}{4} (f'(y))^2] dt > 0$ for $x \in I_1$ and $y \in I_2$. This implies that (3.8) holds. Therefore we know that this $u(t)$ is a non-trivial solution of the Dirichlet boundary value problem

$$\begin{aligned} u''(t) + [g'_x(t, x_0(t)) - \frac{1}{2} f''(x'_0(t))x''_0(t) - \frac{1}{4} (f'(x'_0(t)))^2]u(t) &= 0, \\ u(t_0 + T) = u(t_0) &= 0. \end{aligned} \tag{3.10}$$

Since $x''_0(t) = e(t) - f(x'_0(t)) - g(t, x_0(t))$, multiplying (3.10) by $u(t)$ and integrating it from t_0 to $t_0 + T$, then, by (3.5) and (H₇),

$$\begin{aligned} &\int_{t_0}^{t_0+T} u'(t)^2 dt \\ &= \int_{t_0}^{t_0+T} \left[g'_x(t, x_0(t)) - \frac{1}{2} f''(x'_0(t))x''_0(t) - \frac{1}{4} (f'(x'_0(t)))^2 \right] u(t)^2 dt \\ &< \frac{1}{T^2} \int_{t_0}^{t_0+T} u(t)^2 dt. \end{aligned} \tag{3.11}$$

For convenience of use, we recall the Dirichlet–Poincaré inequality

$$\int_a^b |f(t)|^2 dt \leq (b - a)^2 \int_a^b |f'(t)|^2 dt \quad \text{where } f \in C^1 \text{ and } f(a) = f(b) = 0.$$

We immediately see that (3.11) contradicts the Dirichlet–Poincaré inequality. Therefore, (3.6) does not have real Floquet multipliers.

Next, we show that the characteristic exponents associated with (3.6) all have negative real parts. In order to do this, let us consider a system equivalent to (3.6):

$$X'(t) = A(t)X(t), \tag{3.12}$$

where the vector function $X(t) = (x(t), x'(t))^T$ and $A(t)$ is the matrix function

$$A(t) = \begin{bmatrix} 0 & 1 \\ -g'_x(t, x_0(t)) & -f'(x'_0(t)) \end{bmatrix}.$$

Let $\alpha_1 = e^{T\mu_1}$ and $\alpha_2 = e^{T\mu_2}$ be the Floquet multipliers of (3.12) and μ_1 and μ_2 be the characteristic exponents associated with α_1 and α_2 . Then it follows from the above discussion that α_1 and α_2 are a pair of complex conjugates. Applying Liouville’s theorem (see [1, Problem 1, p. 285]),

$$\alpha_1\alpha_2 = e^{\int_0^T \text{trace}A(t) dt} = e^{\int_0^T -f'(x'_0(t)) dt} \tag{3.13}$$

and

$$\text{Re}(\mu_1) = \text{Re}(\mu_2) = \frac{1}{2}\text{Re}(\mu_1 + \mu_2) = \frac{1}{2T} \ln(\alpha_1\alpha_2) = \frac{1}{2T} \int_0^T -f'(x'_0(t)) dt. \tag{3.14}$$

By assumption (H₆), which says that $f'(y) > 0$ for $y \in I_2$, (3.14) leads to $\text{Re}(\mu_1) = \text{Re}(\mu_2) < 0$.

Finally, applying Lemma 2.7, we obtain that $x_0(t)$ is asymptotically stable. This completes the proof of Theorem 1.1. □

REMARK 3.1. If $f(0) \neq 0$, the problem of the existence, uniqueness and asymptotic stability of T -periodic solutions to (1.1) can be converted into the equation

$$x''(t) + f_1(x'(t)) + g(t, x(t)) = e_1(t), \tag{3.15}$$

where $f_1(x'(t)) = f(x'(t)) - f(0)$, $e_1(t) = e(t) - f(0)$. As $f_1(0) = 0$, (3.15) can be studied by means of Theorem 1.1 in this paper.

4. An illustrative example

In this section, we apply the main results obtained in previous sections to an example.

EXAMPLE 4.1. Consider the existence, uniqueness and asymptotic stability of 2π -periodic solutions of the Rayleigh type equation

$$x''(t) + \frac{1}{\pi^4} \arctan x'(t) + \frac{1}{\pi^4}(\cos^2 t + 1)x(t) = \frac{1}{\pi^3} \sin t, \tag{4.1}$$

where $T = 2\pi$. Here $f(x') = (1/\pi^4) \arctan x'$, $g(t, x) = (1/\pi^4)(\cos^2 t + 1)x$ and $e(t) = (1/\pi^3) \sin t$.

PROOF. For convenience, let $\pi = 3.1416$. Set $C_1 = (1/\pi^4)$, $C_2 = (2/\pi^4)$ and $D_1 = 2\pi$. Then it is easy to check that conditions (H₁)–(H₅) of Theorem 1.1 are satisfied. Now we check that (H₆) and (H₇) also hold. According to Theorem 1.1, we can compute that $G = 0$, $M_0 = 0.4376$, $M_1 = 6.8317$, $M_2 = 0.5485$, $I_1 = [-6.8317, 6.8317]$ and $I_2 = [-0.5485, 0.5485]$.

For $x \in I_1$ and $y \in I_2$,

$$\begin{aligned} \int_0^{2\pi} \left[g'_x(t, x) - \frac{1}{4}(f'(y))^2 \right] dt &= \int_0^{2\pi} \left[\frac{1}{\pi^4}(\cos^2 t + 1) - \frac{1}{4\pi^8(1+y^2)^2} \right] dt \\ &\geq \int_0^{2\pi} \left[\frac{1}{\pi^4}(\cos^2 t + 1) - \frac{1}{4\pi^8} \right] dt \\ &= \frac{3}{\pi^3} - \frac{1}{2\pi^7} > 0 \end{aligned}$$

and

$$f'(y) = \frac{1}{\pi^4(1+y^2)} > 0,$$

which imply that (H₆) holds.

On the other hand,

$$\begin{aligned} g'_x(t, x) - \frac{1}{2}f''(y)[e(t) - f(y) - g(t, x)] - \frac{1}{4}(f'(y))^2 \\ = \frac{1}{\pi^4}(\cos^2 t + 1) + \frac{y}{\pi^4(1+y^2)^2} \left[\frac{1}{\pi^3} \sin t - \frac{1}{\pi^4} \arctan y - \frac{1}{\pi^4}(\cos^2 t + 1)x \right] \\ - \frac{1}{4\pi^8(1+y^2)^2} \\ < \frac{2}{\pi^4} + \frac{1}{2\pi^4} \times \left[\frac{1}{\pi^3} + \frac{1}{\pi^4} \times \frac{\pi}{2} + \frac{2}{\pi^4} \times 6.8317 \right] \\ \approx 0.0216 < \frac{1}{T^2} = \frac{1}{4\pi^2} \approx 0.0253, \end{aligned}$$

which implies that (H₇) holds. Hence (4.1) has a unique 2π -periodic solution which is asymptotically stable. \square

REMARK 4.2. It is easy to see that the results in [1, 4, 7–11, 13–18] and the references therein cannot be applied to (4.1) in order to guarantee the existence, uniqueness and asymptotic stability of 2π -periodic solutions, so our results are new and complement the existing results.

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