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A PROOF OF THE EQUIVALENCE OF HELLY'S AND KRASNOSELSKI'S THEOREMS

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ABSTRACT. We show that Krasnoselski's Theorem, which is usually derived from Helly's Theorem, is in fact equivalent to it.

Recall that a set $S \subset R^n$ is said to be *starshaped* at $\bar{s} \in S$ if, for each $s \in S$, the line segment $[s, \bar{s}]$ is contained in S. For any $x \in S$, $S_x = \{s : [x, s] \subset S\}$ is called the *star* of x in S and x is said to *see* points in S_x. Thus S is starshaped if $\bigcap \{S_x : x \in S\} \neq \emptyset$. This intersection called the *kernel* of S, denoted ker (S), is convex [4]. Toranzos [3] has alternatively characterized ker (S) as the intersection of the maximal convex subsets of S. Finally a point $\bar{s}$ in S is *regular* [4] if $S_{\bar{s}}$ lies on one side of a hyperplane containing $\bar{s}$. A classical result of Krasnoselski's is the following:

(K) THEOREM ([2], [4]). *Suppose $S \subset R^n$ is compact and is such that for every set $F \subset S$ consisting of $n+1$ regular points there is some $\bar{x} \in S$ with $\bar{x}$ seeing F. Then S is starshaped.*

This theorem is proved by applying Helly's Theorem [1] to the convex hulls of stars of regular points in S and arguing that any point in their intersection sees S. In this note we will show that in fact Helly's Theorem can in turn be deduced from (K). As is well known [1], Helly's Theorem and Caratheodory's Theorem are equivalent so that both are equivalent to (K). We first recall Helly's Theorem:

(H) THEOREM ([1], [4]). *A finite class C of N convex sets in R^n is such that $N \geq n+1$, and to every subclass which contains $n+1$ members there corresponds a point of R^n which belongs to every member of the subclass. Under these conditions there is a point which belongs to every member of the given class C.*

We will need the following proposition for our construction.

PROPOSITION. *Suppose that C is a compact convex set and L is a line through the origin. If $x \notin C+L$ and $C \subset B(0, M) = \{x : \|x\| \leq M\}$ there exists a point y in*

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$C+L$ of norm M such that

$$[x, y] \cap (C+L)=\{y\}.$$

Proof. $C+L$ is a closed convex set so we can separate x and $C+L$ strictly. That is: there is a linear functional f with

$$f(x)<\inf \{f(y): y \in C+L\}=m.$$

Then f is identically zero on L. Let $f(c_0)=\inf \{f(c): c \in C\}$. Choose $y_t=c_0+td$, where d is a non-zero point in L. Then since $\|y_0\| \leq M$ and $\|y_t\|$ tends to infinity there is a point y_{t_0} with $\|y_{t_0}\|=M$. Also, if $z_r=rx+(1-r)y_{t_0}$ with $0<r<1$, we have $f(z_r)<m$ and $z_r \notin C+L$. □

THEOREM. (K) *implies* (H).

Proof. We will construct a starshaped set S with $\ker(S)=\cap\{C_i: 1 \leq i \leq N\}$. We may suppose as in [4] that C consists of compact polyhedra given for $1 \leq i \leq N$ by

$$C_i=\{x: a_j \cdot x \leq r_j, \qquad n_i<j \leq n_{i+1}\}. \tag{1}$$

We may also assume (by an approximation argument) that all the vectors a_j are pairwise independent. Now for each a_j pick a non-zero vector d_j with $a_j \cdot d_j=0$, subject to the added proviso that the d_j are pairwise independent.

Let L_j be the line through zero and d_j and set

$$E_j=C_i+L_j \qquad n_i<j \leq n_{i+1}. \tag{2}$$

Then each E_j is convex and closed and there is a constant $M>0$, with each $C_j \subset B(0, M)$, and such that

$$j_1 \neq j_2 \quad \text{and} \quad x \in E_{j_1} \cap E_{j_2} \quad \text{implies} \quad \|x\|<M-1. \tag{3}$$

If not, we must find two sets E_{j_1} and E_{j_2} whose intersection is (linearly) unbounded. This implies that E_{j_1} and E_{j_2} share a half line and thus d_{j_1} and d_{j_2} are multiples so that $j_1=j_2$.

Now let

$$S=\bigcup\{E_j \cap B(0, M): n_1<j \leq n_{N+1}\}. \tag{4}$$

Suppose $\{s_1, \ldots, s_{n+1}\} \subset S$. It is easily seen that if $s_k \in E_{j_k}=C_{i_k}+L_{j_k}(n_{i_k}<j_k \leq n i_{k+1})$ that any point $\bar{x}$ in the (non-empty) intersection of $\{C_{i_k}: 1 \leq k \leq n+1\}$ sees each s_k because $\bar{x} \in C_{i_k} \subset E_{j_k}$ is convex. Thus, by (K), S being compact is starshaped and $\ker(S) \neq \varnothing$.

Now each set $E_j \cap B(0, M)$ is a maximal convex subset. This follows from (3) and the proposition. By Taranzos' observation

$$\ker(S) \subset C \cap\{E_j: n_1 \leq j \leq n_{N+1}\}. \tag{5}$$

Suppose in fact that $y \in E_j$ for all j. Then if i is fixed and $n_i < j \leq n_{i+1}$, we have $y \in C_i + L_j$ and

$$a_j \cdot y \leq \sup \{a_j \cdot c : c \in C_i\} + \sup \{a_j \cdot l : l \in L_j\} \leq r_j + 0 = r_j. \tag{6}$$

Thus $y \in C_i$ for each i and we have finally

$$\emptyset \neq \ker(S) = \cap \{C_i : 1 \leq i \leq N\} \tag{7}$$

which establishes (H). $\square$

REMARK (i) We did not in fact use the full strength of (K) as we did not need to consider regular points.

(ii) In light of the equivalence of (H) and (K) it would be interesting to see a "direct" proof of (K).

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