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# BRANCHING SYSTEMS FOR HIGHER-RANK GRAPH C*-ALGEBRAS

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**Abstract.** We define branching systems for finitely aligned higher-rank graphs. From these, we construct concrete representations of higher-rank graph C*-algebras on Hilbert spaces. We prove a generalized Cuntz–Krieger uniqueness theorem for periodic single-vertex 2-graphs. We use this result to give a sufficient condition under which representations of periodic single-vertex 2-graph C*-algebras arising from branching systems are faithful.

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**1. Introduction.** Higher-rank graphs, or $k$-graphs, are combinatorial objects that generalize directed graphs. In [**23**], Kumijian and Pask introduced higher-rank graph C*-algebras for row-finite higher-rank graphs without sources, as generalizations of graph algebras and the higher-rank Cuntz–Krieger algebras constructed by Robertson and Steger [**28**]. Since then, driven by the fact that higher-rank graph C*-algebras include a larger class than graph C*-algebras, while still can be studied via combinatorial methods, intense research has been done in the subject, see [**4, 5, 7, 8, 21, 24, 26, 27, 32**], for example.

Branching systems arise in disciplines such as random walk, symbolic dynamics and scientific computing (see, for example, [**9, 20, 30**]). More recently, stimulated by Bratteli–Jorgensen's work connecting representations of the Cuntz algebra arising from iterated function systems with wavelets (see [**2,3**]), a large number of papers have studied representations of graph algebras from branching systems (see [**6, 12–19**], etc). Farsi et al. have studied connections of representations of finite higher-rank graphs C*-algebras arising from semibranching function systems with wavelets, Kubo–Martin–Schwinger (KMS) states (see [**10, 11**]).

It is our intention to connect the theory of higher-rank graph C*-algebras with the branching system theory. Notice that when developing the theory of higher-rank graphs, some additional hypotheses are usually assumed, such as finiteness, row finiteness, local convexity or finite alignment. Of these, finite alignment is the most general one, and we try to study the branching system theory of higher-rank graphs in this generality as much as we can. As the paper goes on, to obtain interesting results, we reduce the generality to row-finite higher-rank graphs without sources. Eventually, we restrict to single-vertex 2-graphs, which have been studied in depth by Davidson and Yang (see [**7**]).

The structure of the paper is as follows. Section 2 is devoted to recalling the material on higher-rank graph C*-algebras. In Section 3, we define branching systems for finitely aligned higher-rank graphs. Using the space of boundary paths of a higher-rank graph, we build a branching system associated to any finitely aligned higher-rank graph. We then show how branching systems induce representations of higher-rank graph C*-algebras, which generalizes results in [**15**]. In Section 4, we look into some examples of higher-rank graphs and build branching systems on $\mathbb{R}$ for these graphs, which include higher-rank graphs that are not row-finite. It is usually not easy to decide if a representation of a higher-rank graph C*-algebra is faithful. Therefore, in Section 5, we focus on studying periodic single-vertex 2-graphs, and we aim to provide a sufficient condition for representations induced from branching systems to be faithful. To do so, we first extend the general Cuntz–Krieger uniqueness theorem proved by Brown et al.in [**4**, Theorem 7.10], in the same spirit of Szymański's result for graph algebras (see [**31**, Theorem 1.2]) and the author's result for ultragraph algebras (see [**12**, Theorem 7.4]). We finish the section by building branching systems for periodic single-vertex 2-graphs such that the induced representations are faithful.

**2. Preliminaries.** Throughout this paper, the notation $\mathbb{N}$ stands for the set of all nonnegative integers; the notation $\mathbb{N}_+$ stands for the set of all positive integers; and all measure spaces are assumed to be $\sigma$-finite.

In this section, we recall the definition of $k$-graph C*-algebras from [**23, 26, 27**].

DEFINITION 2.1 ([**23**, Definition 1.1]). Let $k \in \mathbb{N}_+$. A small category $\Lambda$ is called a *$k$-graph* if there exists a functor $d : \Lambda \to \mathbb{N}^k$ satisfying the *factorization property*, that is, for $\mu \in \Lambda, n, m \in \mathbb{N}^k$ with $d(\mu) = n + m$, there exists unique $\nu, \alpha \in \Lambda$ such that $d(\nu) = n, d(\alpha) = m, s(\nu) = r(\alpha), \mu = \nu\alpha$. The functor $d$ is called the degree map of $\Lambda$.

Let $(\Lambda_1, d_1), (\Lambda_2, d_2)$ be two $k$-graphs. A functor $f : \Lambda_1 \to \Lambda_2$ is called a *morphism* if $d_2 \circ f = d_1$.

Throughout this paper, all $k$-graphs are assumed to be countable.

EXAMPLE 2.2 ([**27**, p. 211]). Let $k \in \mathbb{N}_+$ and let $n \in (\mathbb{N} \cup \{\infty\})^k$. Define $\Omega_{k,n} := \{(p, q) \in \mathbb{N}^k \times \mathbb{N}^k : p \leq q \leq n\}$. For $(p, q), (q, m) \in \Omega_{k,n}$, define $(p, q) \cdot (q, m) := (p, m); r(p, q) := (p, p); s(p, q) := (q, q)$; and $d(p, q) := q - p$. Then, $(\Omega_{k,n}, d)$ is a $k$-graph.

NOTATION 2.3 ([**27**, p. 211]). Let $k \in \mathbb{N}_+$, let $\Lambda$ be a $k$-graph. Denote by

$$X_\Lambda := \bigcup_{n\in(\mathbb{N}\cup\{\infty\})^k} \{x : \Omega_{k,n} \to \Lambda : x \text{ is a graph morphism}\}.$$

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Fix a graph morphism $x : \Omega_{k,n} \to \Lambda$ for some $n \in (\mathbb{N} \cup \{\infty\})^k$. For $\mu \in \Lambda$ with $s(\mu) = x(0, 0)$, denote by $\mu x : \Omega_{k,d(\mu)+n} \to \Lambda$ the unique graph morphism such that $(\mu x)(0, d(\mu)) = \mu, (\mu x)(d(\mu), m) = x(0, m - d(\mu))$ for all $d(\mu) \leq m \leq n$. For $\mathbb{N}^k \ni m \leq n$, denote by $\sigma^m(x) : \Omega_{k,n-m} \to \Lambda$ the unique graph morphism such that $\sigma^m(x)(0, l) = x(m, m + l)$ for all $\mathbb{N}^k \ni l \leq n - m$. Moreover, for $A \subset \Lambda, B \subset X_\Lambda$, denote by $AB := \{\mu x : \mu \in A, x \in B, s(\mu) = x(0, 0)\}$.

The following lemma might be well-known, however we could not find any reference to it.

Lemma 2.4. *Let $k \in \mathbb{N}_+$, let $\Lambda$ be a $k$-graph, let $\mu \in \Lambda$, and let $B \subset X_\Lambda$. Then, $\sigma^{d(\mu)} : \mu B \to s(\mu)B$ is a bijection.*

*Proof.* It is straightforward to see. Indeed $\sigma^{d(\mu)}(\mu x) = x$ for all $\mu x \in \mu B$ and the inverse map of $\sigma^{d(\mu)}$ is to attach $\mu$ to the elements of $B$.

Definition 2.5 ([**27**, Definition 2.8]). Let $k \in \mathbb{N}_+$ and let $\Lambda$ be a $k$-graph. Define

$$\Lambda^{\leq\infty} := \bigcup_{n\in(\mathbb{N}\cup\{\infty\})^k} \{x : \Omega_{k,n} \to \Lambda \text{ is a graph morphism } : \exists \mathbb{N}^k \ni n_x \leq n, \text{ s.t.}$$
$$\forall m \in \mathbb{N}^k \text{ with } n_x \leq m \leq n, \text{ we have } m_i = n_i \implies x(0, m)\Lambda^{e_i} = \emptyset\}.$$

$\Lambda^{\leq\infty}$ is called the *boundary path space* of $\Lambda$. The range and degree maps may be extended to boundary paths $x : \Omega_{k,n} \to \Lambda$ by setting $r(x) := x(0, 0)$ and $d(x) = n$.

Notation 2.6. *Let $k \in \mathbb{N}_+$. Denote by $e_1, \ldots, e_k$ the standard basis of $\mathbb{N}^k$. For $i \geq 1$, denote by*

$$e_i := \begin{cases} e_1 & \text{if } i = 1, k+1, 2k+1, 3k+1 \ldots; \\ \cdots & \\ e_k & \text{if } i = k, k+k, 2k+k, 3k+k, \ldots. \end{cases}$$

*For $n, m \in \mathbb{N}^k$, denote by $|n| := n_1 + \cdots + n_k; n \vee m := (\max\{n_i, m_i\})_{i=1}^k$; and $n \wedge m := (\min\{n_i, m_i\})_{i=1}^k$. Furthermore, for $z \in \mathbb{T}^k$, denote by $z^n := z_1^{n_1} \ldots z_k^{n_k}$.*

Notation 2.7 ([**27**, Definitions 2.2, 2.4]). Let $k \in \mathbb{N}_+$ and let $\Lambda$ be a $k$-graph. For $n \in \mathbb{N}^k$, denote by $\Lambda^n := d^{-1}(n)$. For $A, B \subset \Lambda$, define $AB := \{\mu\nu : \mu \in A, \nu \in B, s(\mu) = r(\nu)\}$. For $\mu, \nu \in \Lambda$, define $\Lambda^{\min}(\mu, \nu) := \{(\alpha, \beta) \in \Lambda \times \Lambda : \mu\alpha = \nu\beta, d(\mu\alpha) = d(\mu) \vee d(\nu)\}$. For $v \in \Lambda^0$, a subset $E$ of $v\Lambda$ is said to be *exhaustive* for $v$ if, for any $\mu \in v\Lambda$, there exists $\nu \in E$ such that $\Lambda^{\min}(\mu, \nu) \neq \emptyset$.

Definition 2.8 ([**27**, Definition 2.2]). Let $k \in \mathbb{N}_+$. A $k$-graph $\Lambda$ is said to be *finitely aligned* if, for any $\mu, \nu \in \Lambda$, we have that $\Lambda^{\min}(\mu, \nu)$ is a finite set.

Definition 2.9 ([**27**, Definition 2.5]). Let $k \in \mathbb{N}_+$ and let $\Lambda$ be a finitely aligned $k$-graph. A *Cuntz–Krieger $\Lambda$-family* in a C*-algebra $B$ is a family of partial isometries $\{S_\mu\}_{\mu\in\Lambda}$ satisfying

(1) $\{S_v\}_{v\in\Lambda^0}$ is a family of mutually orthogonal projections;
(2) $S_{\mu\nu} = S_\mu S_\nu$ if $s(\mu) = r(\nu)$;
(3) $S_\mu^* S_\nu = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\mu,\nu)} S_\alpha S_\beta^*$ for all $\mu, \nu \in \Lambda$; and
(4) $\prod_{\mu\in E}(S_v - S_\mu S_\mu^*) = 0$ for all $v \in \Lambda^0$, for all finite exhaustive set $E \subset v\Lambda$.

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The C*-algebra generated by a universal Cuntz–Krieger $\Lambda$-family, denoted by $\{s_\mu\}_{\mu\in\Lambda}$, is called the *k-graph C*-algebra* of $\Lambda$ and is denoted by $C^*(\Lambda)$.

REMARK 2.10. By [**27**, Proposition 2.12], each $s_\mu$ is nonzero.

THEOREM 2.11 ([**27**, Theorem C.1]). *Let $k \in \mathbb{N}_+$, let $\Lambda$ be a finitely aligned $k$-graph, and let $\{S_\mu : \mu \in \left(\bigcup_{i=1}^k \Lambda^{e_i}\right) \cup \Lambda^0\}$ be a family of partial isometries in a C*-algebra $B$ satisfying*

(1) $\{S_v\}_{v\in\Lambda^0}$ *is a family of mutually orthogonal projections;*
(2) $S_\mu S_\nu = S_\alpha S_\beta$ *if* $\mu, \nu, \alpha, \beta \in \left(\bigcup_{i=1}^k \Lambda^{e_i}\right) \cup \Lambda^0$, $\mu\nu = \alpha\beta$;
(3) $S_\mu^* S_\nu = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\mu,\nu)} S_\alpha S_\beta^*$ *for all* $\mu, \nu \in \left(\bigcup_{i=1}^k \Lambda^{e_i}\right) \cup \Lambda^0$*; and*
(4) $\prod_{\mu\in E}(S_v - S_\mu S_\mu^*) = 0$ *for all* $v \in \Lambda^0$, *for all finite exhaustive set* $E \subset v\left(\bigcup_{i=1}^k \Lambda^{e_i}\right)$.
*Then, there exists a unique Cuntz–Krieger $\Lambda$-family $\{T_\mu\}_{\mu\in\Lambda}$ in $B$ such that $T_\mu = S_\mu$ for all $\mu \in \left(\bigcup_{i=1}^k \Lambda^{e_i}\right) \cup \Lambda^0$.*

DEFINITION 2.12 ([**23**, Definition 1.4]). Let $k \in \mathbb{N}_+$, let $\Lambda$ be a $k$-graph. Then, $\Lambda$ is said to be *row-finite* if $|v\Lambda^n| < \infty$ for all $v \in \Lambda^0$, $n \in \mathbb{N}^k$. $\Lambda$ is said to *have no sources* if $v\Lambda^n \neq \emptyset$ for all $v \in \Lambda^0$, $n \in \mathbb{N}^k$.

PROPOSITION 2.13 ([**27**, Proposition B.1]). *Let $k \in \mathbb{N}_+$ and let $\Lambda$ be a row-finite $k$-graph without sources. Then, a family of partial isometries $\{S_\mu\}_{\mu\in\Lambda}$ in a C*-algebra $B$ is a Cuntz–Krieger $\Lambda$-family if and only if*

(1) $\{S_v\}_{v\in\Lambda^0}$ *is a family of mutually orthogonal projections;*
(2) $S_{\mu\nu} = S_\mu S_\nu$ *if* $s(\mu) = r(\nu)$;
(3) $S_\mu^* S_\mu = S_{s(\mu)}$ *for all* $\mu \in \Lambda$*; and*
(4) $S_v = \sum_{\mu\in v\Lambda^n} S_\mu S_\mu^*$ *for all* $v \in \Lambda^0$, $n \in \mathbb{N}^k$.

REMARK 2.14. Conditions (1)–(4) of Proposition 2.13 are exactly the definition of a Cuntz–Krieger family for row-finite without sources $k$-graphs, as given originally by Kumjian and Pask in [**23**].

The following proposition is a special case of [**27**, Theorem C.1].

PROPOSITION 2.15. *Let $k \in \mathbb{N}_+$, let $\Lambda$ be a row-finite $k$-graph without sources, and let $\{S_\mu : \mu \in \left(\bigcup_{i=1}^k \Lambda^{e_i}\right) \cup \Lambda^0\}$ be a family of partial isometries in a C*-algebra $B$ satisfying*

(1) $\{S_v\}_{v\in\Lambda^0}$ *is a family of mutually orthogonal projections;*
(2) $S_\mu S_\nu = S_\alpha S_\beta$ *if* $\mu, \nu, \alpha, \beta \in \left(\bigcup_{i=1}^k \Lambda^{e_i}\right) \cup \Lambda^0$, $\mu\nu = \alpha\beta$;
(3) $S_\mu^* S_\mu = S_{s(\mu)}$ *for all* $\mu \in \left(\bigcup_{i=1}^k \Lambda^{e_i}\right) \cup \Lambda^0$*; and*
(4) $S_v = \sum_{\mu\in v\Lambda^{e_i}} S_\mu S_\mu^*$ *for all* $v \in \Lambda^0$, $i = 1, \ldots, k$.
*Then, there exists a unique Cuntz–Krieger $\Lambda$-family $\{T_\mu\}_{\mu\in\Lambda}$ in $B$ such that $T_\mu = S_\mu$ for all $\mu \in \left(\bigcup_{i=1}^k \Lambda^{e_i}\right) \cup \Lambda^0$.*

*Proof.* First of all, we prove the uniqueness. Let $\{T_\mu\}_{\mu\in\Lambda}, \{T'_\mu\}_{\mu\in\Lambda}$ be Cuntz–Krieger $\Lambda$-families in $B$ such that $T_\mu = T'_\mu = S_\mu$ for all $\mu \in \left(\bigcup_{i=1}^k \Lambda^{e_i}\right) \cup \Lambda^0$. For $\mu \in \Lambda \setminus \Lambda^0$, by the factorization property of $\Lambda$, we can write $\mu$ in the form of $\mu^1 \cdots \mu^n$, where $\mu^1, \ldots, \mu^n \in \bigcup_{i=1}^k \Lambda^{e_i}$. So, by the assumption and by Condition (2) of Definition 2.9, $T_\mu = T_{\mu^1} \cdots T_{\mu^n} = T'_{\mu^1} \cdots T'_{\mu^n} = T'_\mu$.

Next, we prove the existence. For $\mu \in \left(\bigcup_{i=1}^k \Lambda^{e_i}\right) \cup \Lambda^0$, define $T_\mu := S_\mu$. For $\mu \in \Lambda \setminus \left(\left(\bigcup_{i=1}^k \Lambda^{e_i}\right) \cup \Lambda^0\right)$, by the factorization property of $\Lambda$, we can write $\mu$ in the form

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of $\mu^1 \cdots \mu^n$, where $\mu^1, \ldots, \mu^n \in \bigcup_{i=1}^k \Lambda^{e_i}$. Define $T_\mu := S_{\mu^1} \cdots S_{\mu^n}$ (the factorization property of $\Lambda$ and Condition (2) imply this is well-defined).

For $\mu \in \bigcup_{i=1}^k \Lambda^{e_i}$, by Condition 3, $T_\mu T_{s(\mu)} = T_\mu$. By Condition (4), $T_{r(\mu)} T_\mu = T_\mu$. So, for $\mu \in \Lambda$, we have $T_\mu T_{s(\mu)} = T_\mu$, $T_{r(\mu)} T_\mu = T_\mu$, $T_\mu^* T_\mu = T_{s(\mu)}$, hence $T_\mu$ is a partial isometry. For $v \in \Lambda^0, n \in \mathbb{N}^k$, we show that $T_v = \sum_{\mu \in v\Lambda^n} T_\mu T_\mu^*$. We prove by the induction on $|n|$. The equality holds for $|n| = 0, 1$ by Condition (4). Suppose the equality holds for $|n| = N \geq 1$. When $|n| = N + 1$, write $n = p + q$ such that $|p| = N$, $|q| = 1$. Then, by the induction assumption and by Condition (4), we have

$$T_v = \sum_{\alpha \in v\Lambda^p} T_\alpha T_\alpha^* = \sum_{\alpha \in v\Lambda^p} \sum_{\beta \in s(\alpha)\Lambda^q} T_{\alpha\beta} T_{\alpha\beta}^* = \sum_{\mu \in v\Lambda^n} T_\mu T_\mu^*.$$

So, $\{T_\mu\}_{\mu \in \Lambda}$ satisfies Conditions (1)–(4) of Proposition 2.13. Hence, by Proposition 2.13 $\{T_\mu\}_{\mu \in \Lambda}$ is a Cuntz–Krieger $\Lambda$-family.

NOTATION 2.16 ([**23**, Section 3]). Let $k \in \mathbb{N}_+$, let $\Lambda$ be a row-finite $k$-graph without sources. Then, there exists a *gauge action,* which is a strongly continuous group homomorphism $\gamma : \mathbb{T}^k \to \mathrm{Aut}(C^*(\Lambda))$ such that $\alpha_z(s_\mu) = z^{d(\mu)} s_\mu$ for all $z \in \mathbb{T}^k$, $\mu \in \Lambda$. The *fixed point algebra* is the algebra $C^*(\Lambda)^\gamma = \overline{span}\{s_\mu s_\nu^* : d(\mu) = d(\nu)\}$. The gauge action yields a faithful expectation $\Phi$ from $C^*(\Lambda)$ onto $C^*(\Lambda)^\gamma$ such that $\Phi(s_\mu s_\nu^*) = \delta_{0,d(\mu)-d(\nu)} s_\mu s_\nu^*$ for all $\mu, \nu \in \Lambda$.

DEFINITION 2.17 ([**23**, Definition 4.3]). Let $k \in \mathbb{N}_+$ and let $\Lambda$ be a row-finite $k$-graph without sources. Denote by $\Lambda^\infty$ the set of *infinite paths*, which consists of all graph morphisms from $\Omega_{k,(\infty,\ldots,\infty)}$ to $\Lambda$. Then, $\Lambda$ is said to be *aperiodic* if, for any $v \in \Lambda^0$, there exists $x \in v\Lambda^\infty$ such that $\sigma^n(x) \neq \sigma^m(x)$ for all $n \neq m \in \mathbb{N}^k$.

The following theorem is the Cuntz–Krieger uniqueness theorem for row-finite higher-rank graphs without sources.

THEOREM 2.18 ([**23**, Theorem 4.6]). Let $k \in \mathbb{N}_+$, let $\Lambda$ be a row-finite $k$-graph without sources, and let $\pi : C^*(\Lambda) \to B$ be a homomorphism. Suppose that $\Lambda$ is aperiodic. Then, $\pi$ is injective if and only if $\pi(s_v) \neq 0$ for all $v \in \Lambda^0$.

DEFINITION 2.19 ([**4**]). Let $k \in \mathbb{N}_+$ and let $\Lambda$ be a row-finite $k$-graph without sources. A pair $(\mu, \nu) \in \Lambda \times \Lambda$ is called a *cycline pair* if $s(\mu) = s(\nu)$ and $\mu x = \nu x$ for all $x \in s(\mu)\Lambda^\infty$. The C*-subalgebra $\mathcal{M} := C^*(\{s_\mu s_\nu^* : (\mu, \nu)$ is a cycline pair $\})$ is called the *cycline subalgebra* of $C^*(\Lambda)$. Moreover, the C*-subalgebra $\mathcal{D} := C^*(\{s_\mu s_\mu^* : \mu \in \Lambda\})$ is called the *diagonal* of $C^*(\Lambda)$.

NOTATION 2.20 ([**5**, p. 2581]). *Let $k \in \mathbb{N}_+$, let $\Lambda$ be a row-finite $k$-graph without sources. Define the set of* periodicity *of $\Lambda$ by* $\mathrm{Per}(\Lambda) := \{d(\mu) - d(\nu) : (\mu, \nu)$ *is a cycline pair* $\}$. *By [**32**, Theorem 4.6] $\Lambda$ is aperiodic if and only if* $\mathrm{Per}(\Lambda) = \{0\}$.

The following theorem is the general Cuntz–Krieger uniqueness theorem for row-finite higher-rank graphs without sources.

THEOREM 2.21 ([**4**, Theorem 7.10]). *Let $k \in \mathbb{N}_+$, let $\Lambda$ be a row-finite $k$-graph without sources, and let $\pi : C^*(\Lambda) \to B$ be a homomorphism. Then, $\pi$ is injective if and only if $\pi$ is injective on $\mathcal{M}$.*

**3. Branching systems of higher-rank graphs.** In this section, we introduce the notion of branching systems of higher-rank graphs. The branching system definition

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will invoke the Radon–Nikodym derivative, and we refer the reader to [**29**] for background on this material.

Notice that when studying the branching systems of higher-rank graphs, we always consider those graphs satisfying certain hypotheses, like finiteness, row finiteness, local convexity or finite alignment. Of these assumptions finite alignment is the most general one, and we develop the theory of branching systems for higher-rank graphs in this generality as much as we can.

### **3.1. Finitely aligned case.**

DEFINITION 3.1. Let $k \in \mathbb{N}_+$, let $\Lambda$ be a finitely aligned $k$-graph, let $(X, \eta)$ be a measure space and let $\{R_\mu, D_v\}_{\mu \in \bigcup_{i=1}^k \Lambda^{e_i}, v \in \Lambda^0}$ be a family of measurable subsets of $X$. Suppose that

(1) $R_\mu \cap R_\nu \overset{\eta-a.e.}{=} \emptyset$ if $\mu \neq \nu \in \Lambda^{e_i}$ for some $1 \leq i \leq k$;
(2) $D_v \cap D_w \overset{\eta-a.e.}{=} \emptyset$ if $v \neq w \in \Lambda^0$;
(3) $R_\mu \overset{\eta-a.e.}{\subseteq} D_{r(\mu)}$ for all $\mu \in \bigcup_{i=1}^k \Lambda^{e_i}$;
(4) for each $\mu \in \bigcup_{i=1}^k \Lambda^{e_i}$, there exist two measurable maps $f_\mu : D_{s(\mu)} \to R_\mu$ and $f_\mu^{-1}$ : $R_\mu \to D_{s(\mu)}$ such that $f_\mu \circ f_\mu^{-1} \overset{\eta-a.e.}{=} \mathrm{id}_{R_\mu}, f_\mu^{-1} \circ f_\mu \overset{\eta-a.e.}{=} \mathrm{id}_{D_{s(\mu)}}$, the pushforward measure $\eta \circ f_\mu$, of $f_\mu^{-1}$ in $D_{s(\mu)}$, is absolutely continuous with respect to $\eta$ in $D_{s(\mu)}$, and the pushforward measure $\eta \circ f_\mu^{-1}$, of $f_\mu$ in $R_\mu$, is absolutely continuous with respect to $\eta$ in $R_\mu$. Denote the Radon–Nikodym derivative $d(\eta \circ f_\mu)/d\eta$ by $\Phi_{f_\mu}$ and the Radon–Nikodym derivative $d(\eta \circ f_\mu^{-1})/d\eta$ by $\Phi_{f_\mu^{-1}}$;
(5) for $\mu, \nu, \alpha, \beta \in \bigcup_{i=1}^k \Lambda^{e_i}$ with $\mu\nu = \alpha\beta$, we have $f_\mu \circ f_\nu \overset{\eta-a.e.}{=} f_\alpha \circ f_\beta$;
(6) for $\mu, \nu \in \bigcup_{i=1}^k \Lambda^{e_i}$ with $r(\mu) = r(\nu)$ and $d(\mu) \neq d(\nu)$, we have $f_\mu(D_{s(\mu)} \setminus \bigcup_{(\alpha,\beta)\in\Lambda^{\min}(\mu,\nu)} R_\alpha) \cap f_\nu(D_{s(\nu)} \setminus \bigcup_{(\alpha,\beta)\in\Lambda^{\min}(\mu,\nu)} R_\beta)) \overset{\eta-a.e.}{=} \emptyset$; and
(7) for any $v \in \Lambda^0$, and for any finite exhaustive set $E \subset \bigcup_{i=1}^k v\Lambda^{e_i}$ for $v$, we have $\bigcup_{\mu \in E} R_\mu \overset{\eta-a.e.}{=} D_v$.

We call $\{D_v, R_\mu, f_\mu\}_{\mu \in \bigcup_{i=1}^k \Lambda^{e_i}, v \in \Lambda^0}$ a $\Lambda$-*branching system* on $(X, \eta)$.

REMARK 3.2. Informally, we can think of the maps $f_\mu$ as 'representing' the partial isometries $S_\mu$, so that the subsets $D_{s(\mu)}$ 'represent' the initial projection of $S_\mu$ and the subsets $R_\mu$ 'represent' the final projection of $S_\mu$. With this in mind, the conditions we impose on the definition of a branching system become intuitive, except Condition 6 that we feel deserves further explaining. We will keep a rather informal tone in this remark in order to explain the intuition behind this condition. Notice that we need to rephrase Condition 3 of Theorem 2.11 as one of our conditions on a branching system. Reading it directly, we would like that $f_\mu^{-1} f_\nu|_{R_\beta} = f_\alpha f_\beta^{-1}$, for all $\beta$ such that $(\alpha, \beta) \in \Lambda^{\min}(\mu, \nu)$ (notice that since for fixed $\mu, \nu$ there exists $j \in \mathbb{N}^k$ such that if $(\alpha, \beta) \in \Lambda^{\min}(\mu, \nu)$, then $\beta \in \Lambda^j$, we have that the $\{R_\beta : (\alpha, \beta) \in \Lambda^{\min}(\mu, \nu)\}$ forms a collection of a.e. disjoint sets). But if $(\alpha, \beta) \in \Lambda^{\min}(\mu, \nu)$, then $\mu\alpha = \nu\beta$ and hence $f_\mu f_\alpha = f_\nu f_\beta$ (by Condition 5), so that $f_\alpha = f_\mu^{-1} f_\nu f_\beta$ and $f_\mu^{-1} f_\nu|_{R_\beta} = f_\alpha f_\beta^{-1}$ is satisfied. Notice that we also desire that on $D_{s(\nu)} \setminus \bigcup_{(\alpha,\beta)\in\Lambda^{\min}(\mu,\nu)} R_\beta$ the equality $f_\mu^{-1} f_\nu|_{R_\beta} = f_\alpha f_\beta^{-1}$ not necessarily hold. This is the reason we require Condition 6.

In the following theorem, we build a branching system associated to any finitely aligned higher-rank graph using the space of boundary paths of a higher-rank graph.

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THEOREM 3.3. *Let $k \in \mathbb{N}_+$ and let $\Lambda$ be a finitely aligned k-graph. Then, there exists a $\Lambda$-branching system.*

*Proof.* Let $X := \Lambda^{\leq\infty}$, and let $\eta$ be the counting measure on $X$. For $v \in \Lambda^0$, define $D_v := v\Lambda^{\leq\infty}$. For $\mu \in \bigcup_{i=1}^k \Lambda^{e_i}$, define $R_\mu := \mu\Lambda^{\leq\infty}$. By [27, Lemma 2.11], $D_v, R_\mu$ are nonempty. It is straightforward to see that $\{R_\mu, D_v\}_{\mu\in\bigcup_{i=1}^k \Lambda^{e_i}, v\in\Lambda^0}$ satisfies Conditions (1)–(3) of Definition 3.1.

For $\mu \in \bigcup_{i=1}^k \Lambda^{e_i}$, Lemma 2.4 yields a bijection $\sigma^{d(\mu)} : R_\mu \to D_{s(\mu)}$. Denote by $f_\mu := (\sigma^{d(\mu)})^{-1}$. Since $\eta$ is the counting measure on $X$, it is straightforward to see that the pushforward measure $\eta \circ f_\mu$, of $f_\mu^{-1}$ in $D_{s(\mu)}$, is absolutely continuous with respect to $\eta$ in $D_{s(\mu)}$, and the pushforward measure $\eta \circ f_\mu^{-1}$, of $f_\mu$ in $R_\mu$, is absolutely continuous with respect to $\eta$ in $R_\mu$. So, Condition (4) of Definition 3.1 holds.

Fix $\mu, \nu, \alpha, \beta \in \bigcup_{i=1}^k \Lambda^{e_i}$ with $\mu\nu = \alpha\beta$. Then, for $x \in D_{s(\nu)} = D_{s(\beta)}$, we have

$$f_\mu \circ f_\nu(x) = f_\mu(\nu x) = \mu(\nu x) = \alpha(\beta x) = f_\alpha(\beta x) = f_\alpha \circ f_\beta(x).$$

So, Condition (5) of Definition 3.1 holds.

Fix $\mu, \nu \in \bigcup_{i=1}^k \Lambda^{e_i}$ with $r(\mu) = r(\nu)$ and $d(\mu) \neq d(\nu)$. Suppose that there exist $x \in D_{s(\mu)} \setminus \bigcup_{(\alpha,\beta)\in\Lambda^{\min}(\mu,\nu)} R_\alpha$ and $y \in D_{s(\nu)} \setminus \bigcup_{(\alpha,\beta)\in\Lambda^{\min}(\mu,\nu)} R_\beta$ such that $f_\mu(x) = f_\nu(y)$. Let $z := \mu x = \nu y$. Then, there exists $n \geq d(\mu) \vee d(\nu)$ such that $z : \Omega_{k,n} \to \Lambda$ is a graph morphism. So, $z(0, d(\mu) \vee d(\nu)) = z(0, d(\mu))z(d(\mu), d(\mu) \vee d(\nu)) = \mu\alpha_0$ for some $\alpha_0 \in \Lambda^{d(\nu)}$; and $z(0, d(\mu) \vee d(\nu)) = z(0, d(\nu))z(d(\nu), d(\mu) \vee d(\nu)) = \nu\beta_0$ for some $\beta_0 \in \Lambda^{d(\mu)}$. Hence, $(\alpha_0, \beta_0) \in \Lambda^{\min}(\mu, \nu)$ and $z = \mu \cdot \alpha_0 \cdot \sigma^{d(\mu)\vee d(\nu)}(z) = \nu \cdot \beta_0 \cdot \sigma^{d(\mu)\vee d(\nu)}(z)$. By [27, Lemma 2.10], $x \in R_{\alpha_0}, y \in R_{\beta_0}$, which is a contradiction. Therefore, Condition (6) of Definition 3.1 holds.

Fix $v \in \Lambda^0$, and fix a finite exhaustive set $E \subset \bigcup_{i=1}^k v\Lambda^{e_i}$ for $v$. It is straightforward to see that $\bigcup_{\mu\in E} R_\mu \subset D_v$. We prove the reverse inclusion. Fix a graph morphism $x : \Omega_{k,n} \to \Lambda$ in $D_v$ (notice that $n \neq 0$). Suppose that $x \notin \bigcup_{\mu\in E} R_\mu$, for a contradiction. By the definition of $D_v$, there exists $\mathbb{N}^k \ni n_x \leq n$ such that whenever $n_x \leq m \leq n, m_i = n_i$, we have $x(0, m)\Lambda^{e_i} = \emptyset$. Since $E$ is exhaustive, there exists $\mu^1 \in E$ such that $\Lambda^{\min}(\mu^1, x(0, n_x)) \neq \emptyset$. Take an arbitrary $(\alpha, \beta) \in \Lambda^{\min}(\mu^1, x(0, n_x))$. Then, $d(\beta) = d(\mu^1)$. So, $n_x + d(\mu^1) \leq n$. Since $x \notin \bigcup_{\mu\in E} R_\mu$ and $E$ is exhaustive, there exists $\mu^2 \in E \setminus \{\mu\}$ such that $\Lambda^{\min}(\mu^2, x(0, n_x + d(\mu^1))) \neq \emptyset$. Then, $n_x + d(\mu^1) + d(\mu^2) \leq n$. Inductively, we deduce that $n_x + \sum_{\mu\in E} d(\mu) \leq n$. Then, we are not able to find any path in $\mu \in E$ such that $\Lambda^{\min}(\mu, x(0, n_x + \sum_{\mu\in E} d(\mu)))$ because $x \notin \bigcup_{\mu\in E} R_\mu$. Hence, we get a contradiction and therefore Condition (7) of Definition 3.1 holds. □

Before we show that a branching system induces a representation of $C^*(\Lambda)$ on $L^2(X)$, we need the following lemma.

LEMMA 3.4. *Let $k \in \mathbb{N}_+$, let $\Lambda$ be a finitely aligned k-graph, and let $\{R_\mu, D_v, f_\mu : \mu \in \bigcup_{i=1}^k \Lambda^{e_i}, v \in \Lambda^0\}$ be a $\Lambda$-branching system on a measure space $(X, \eta)$. Fix $\mu, \nu \in \bigcup_{i=1}^k \Lambda^{e_i}$ with $s(\mu) = r(\nu)$. Then, $\eta \circ f_\mu \circ f_\nu, \eta \circ f_\nu, \eta$ are measures on $D_{s(\nu)}$. Furthermore, we have that $\eta \circ f_\mu \circ f_\nu$ is absolutely continuous with respect to $\eta \circ f_\nu$, and $\eta \circ f_\nu$ is absolutely continuous with respect to $\eta$. Hence,*

$$d(\eta \circ f_\mu \circ f_\nu)/d(\eta) = (\Phi_{f_\mu} \circ f_\nu) \cdot \Phi_{f_\nu}.$$

*Proof.* It is straightforward to see that $\eta \circ f_\mu \circ f_\nu$ is absolutely continuous with respect to $\eta \circ f_\nu$, and $\eta \circ f_\nu$ is absolutely continuous with respect to $\eta$ by Condition 4

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of Definition 3.1. By the chain rule, we have

$$d(\eta \circ f_\mu \circ f_\nu)/d(\eta) = d(\eta \circ f_\mu \circ f_\nu)/d(\eta \circ f_\nu) \cdot \Phi_{f_\nu}.$$

We show that $d(\eta \circ f_\mu \circ f_\nu)/d(\eta \circ f_\nu) = \Phi_{f_\mu} \circ f_\nu$. For any measurable set $E \subset D_{s(\nu)}$, we have

$$\eta \circ f_\mu \circ f_\nu(E) = \int (d(\eta \circ f_\mu \circ f_\nu)/d(\eta \circ f_\nu)) \cdot \chi_E \, \mathrm{d}(\eta \circ f_\nu).$$

Let $F := f_\nu(E)$. Then,

$$\begin{aligned} \eta \circ f_\mu \circ f_\nu(E) &= \eta \circ f_\mu(F) \\ &= \int \Phi_{f_\mu} \chi_F \, \mathrm{d}\eta \\ &= \int (\Phi_{f_\mu} \cdot \chi_F) \circ f_\nu \, \mathrm{d}(\eta \circ f_\nu) \\ &= \int (\Phi_{f_\mu} \circ f_\nu) \cdot \chi_E \, \mathrm{d}(\eta \circ f_\nu). \end{aligned}$$

So, $d(\eta \circ f_\mu \circ f_\nu)/d(\eta \circ f_\nu) = \Phi_{f_\mu} \circ f_\nu$ and we are done. □

Next, we show that branching systems induce representations of higher-rank graph C*-algebras, which is a generalization of [**15**, Theorem 2.2] (see also [**10, 25**]).

THEOREM 3.5. *Let $k \in \mathbb{N}_+$, let $\Lambda$ be a finitely aligned $k$-graph, and let $\{D_v, R_\mu, f_\mu\}_{\mu \in \bigcup_{i=1}^k \Lambda^{e_i}, v \in \Lambda^0}$ be a $\Lambda$-branching system on a measure space $(X, \eta)$. Then, there exists a unique representation $\pi : C^*(\Lambda) \to B(L^2(X, \eta))$ such that $\pi(s_\mu)(\phi) = \chi_{R_\mu} \Phi_{f_\mu^{-1}}^{1/2} (\phi \circ f_\mu^{-1})$ and $\pi(s_v)(\phi) = \chi_{D_v} \phi$, for all $\mu \in \bigcup_{i=1}^k \Lambda^{e_i}$, $v \in \Lambda^0$ and $\phi \in \mathcal{L}^2(X, \eta)$.*

*Proof.* For $\mu \in \bigcup_{i=1}^k \Lambda^{e_i}$, and for $\phi \in L^2(X, \eta)$, we have

$$\int_{R_\mu} |\Phi_{f_\mu^{-1}}^{1/2} (\phi \circ f_\mu^{-1})|^2 \, \mathrm{d}\eta = \int_{R_\mu} |\phi \circ f_\mu^{-1}|^2 \, \mathrm{d}(\eta \circ f_\mu^{-1}) = \int_{D_{s(\mu)}} |\phi|^2 \, \mathrm{d}\eta < \infty.$$

Define $S_\mu : L^2(X, \eta) \to L^2(X, \eta)$ by $S_\mu(\phi) := \Phi_{f_\mu^{-1}}^{1/2}(\phi \circ f_\mu^{-1})$. It is straightforward to see that $S_\mu \in B(L^2(X, \eta))$. For $\phi_1, \phi_2 \in L^2(X, \eta)$, we have

$$\begin{aligned} \langle \phi_1, S_\mu(\phi_2) \rangle &= \int_{R_\mu} \phi_1 \cdot \Phi_{f_\mu^{-1}}^{1/2} \cdot \overline{\phi_2 \circ f_\mu^{-1}} \, \mathrm{d}\eta \\ &= \int_{R_\mu} (\Phi_{f_\mu}^{-1/2} \circ f_\mu^{-1}) \cdot \phi_1 \cdot \overline{\phi_2 \circ f_\mu^{-1}} \, \mathrm{d}\eta \\ &= \int_{D_{s(\mu)}} \Phi_{f_\mu}^{-1/2} \cdot (\phi_1 \circ f_\mu) \cdot \overline{\phi_2} \, \mathrm{d}(\eta \circ f_\mu) \\ &= \int_{D_{s(\mu)}} \Phi_{f_\mu} \cdot \Phi_{f_\mu}^{-1/2} \cdot (\phi_1 \circ f_\mu) \cdot \overline{\phi_2}) \, \mathrm{d}\eta \\ &= \int_{D_{s(\mu)}} \Phi_{f_\mu}^{1/2} \cdot (\phi_1 \circ f_\mu) \cdot \overline{\phi_2} \, \mathrm{d}\eta. \end{aligned}$$

So, $S_\mu^*(\phi) = \chi_{D_{s(\mu)}} \cdot \Phi_{f_\mu}^{1/2} \cdot (\phi \circ f_\mu)$ for all $\phi \in L^2(X, \eta)$.

Notice that, for $\mu \in \bigcup_{i=1}^k \Lambda^{e_i}$ and $\phi \in L^2(X, \eta)$, we have $S_\mu S_\mu^*(\phi) \overset{\eta-a.e.}{=} \chi_{R_\mu}\phi$. So, $S_\mu$ is a partial isometry.

For $v \in \Lambda^0$, define $S_v : L^2(X, \eta) \to L^2(X, \eta)$ by $S_v(\phi) := \chi_{D_v}\phi$.

We will show that the family $\{S_\mu, S_v\}_{\mu \in \bigcup_{i=1}^k \Lambda^{e_i}, v \in \Lambda^0}$ satisfy the conditions of Theorem 2.11.

Condition (1) of Theorem 2.11 follows from Condition (2) of Definition 3.1. Condition (2) of Theorem 2.11 follows from Condition (5) of Definition 3.1 and Lemma 3.4.

Next, we check Condition (3) of Theorem 2.11.

Fix $\mu, \nu \in \bigcup_{i=1}^k \Lambda^{e_i}$.

**Case 1**. $\mu = \nu$. Then, $\Lambda^{\min}(\mu, \nu) = \{(s(\mu), s(\mu))\}$. Since $S_\mu^* S_\nu = S_\mu^* S_\mu = S_{s(\mu)}$, Condition (3) of Theorem 2.11 holds.

**Case 2**. $\mu \neq \nu$, $d(\mu) = d(\nu) = e_i$ for some $1 \leq i \leq k$. Then, $\Lambda^{\min}(\mu, \nu) = \emptyset$. By Condition (1) of Definition 3.1, we have $S_\mu^* S_\nu = 0$. So, Condition (3) of Theorem 2.11 holds.

**Case 3**. $d(\mu) \neq d(\nu)$. Then, $d(\mu) = e_i, d(\nu) = e_j$, for some $1 \leq i \neq j \leq k$. For $(\alpha, \beta) \in \Lambda^{\min}(\mu, \nu)$, we have $S_\mu S_\alpha = S_\nu S_\beta$ because we just verified Condition (2) of Theorem 2.11. Then, $S_\alpha S_\beta^* = S_\mu^* S_\mu S_\alpha S_\beta^* = S_\mu^* S_\nu S_\beta S_\beta^*$. So,

$$\sum_{(\alpha,\beta)\in\Lambda^{\min}(\mu,\nu)} S_\alpha S_\beta^* = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\mu,\nu)} S_\mu^* S_\nu S_\beta S_\beta^*.$$

We claim that $\sum_{(\alpha,\beta)\in\Lambda^{\min}(\mu,\nu)} S_\mu^* S_\nu S_\beta S_\beta^* = S_\mu^* S_\nu$. Fix $\phi \in L^2(X, \eta)$. Then,

$$\begin{aligned}
\sum_{(\alpha,\beta)\in\Lambda^{\min}(\mu,\nu)} S_\mu^* S_\nu S_\beta S_\beta^* \phi &= \sum_{(\alpha,\beta)\in\Lambda^{\min}(\mu,\nu)} S_\mu^* S_\nu (\chi_{R_\beta}\phi) \\
&= S_\mu^* S_\nu (\chi_{\bigcup_{(\alpha,\beta)\in\Lambda^{\min}(\mu,\nu)} R_\beta} \cdot \phi) \\
&\quad \text{(By Condition (1) of Definition 3.1} \\
&\quad \text{and by the finite alignment of } \Lambda) \\
&= \Phi_{f_\mu}^{1/2} \cdot (\Phi_{f_\nu^{-1}}^{1/2} \circ f_\mu) \cdot (\phi \circ f_\nu^{-1} \circ f_\mu) \cdot \\
&\quad (\chi_{\bigcup_{(\alpha,\beta)\in\Lambda^{\min}(\mu,\nu)} R_\beta} \circ f_\nu^{-1} \circ f_\mu) \\
&= \Phi_{f_\mu}^{1/2} \cdot (\Phi_{f_\nu^{-1}}^{1/2} \circ f_\mu) \cdot (\phi \circ f_\nu^{-1} \circ f_\mu) \\
&= S_\mu^* S_\nu \phi \text{ (By Condition 6 of Definition 3.1).}
\end{aligned}$$

So, we finish proving the claim, and hence Condition (3) of Theorem 2.11 holds.

Finally, we check Condition (4) of Theorem 2.11. Fix $v \in \Lambda^0$, fix a finite exhaustive set $E \subset \bigcup_{i=1}^k v\Lambda^{e_i}$, and fix $\phi \in L^2(X, \eta)$. It is straightforward to see that $\prod_{\mu\in E}(S_v - S_\mu S_\mu^*)(\phi) = \prod_{\mu\in E}(\chi_{D_v} - \chi_{R_\mu})\phi$. So, by Condition (7) of Definition 3.1, we have $\prod_{\mu\in E}(\chi_{D_v} - \chi_{R_\mu})\phi = 0$. Hence, Condition (4) of Theorem 2.11 holds. Therefore, by Theorem 2.11 there exists a unique Cuntz–Krieger $\Lambda$-family $\{T_\mu\}_{\mu\in\Lambda}$ in $B(L^2(X, \eta))$ such that $T_\mu = S_\mu$ for all $\mu \in \left(\bigcup_{i=1}^k \Lambda^{e_i}\right) \cup \Lambda^0$. By the universal property of $C^*(\Lambda)$, there exists a unique representation $\pi : C^*(\Lambda) \to B(L^2(X, \eta))$ such that $\pi(s_\mu) = T_\mu$ for all $\mu \in \Lambda$. $\square$

**3.2. Row-finite without sources case.** In this subsection, we simplify the definition of branching systems for row-finite $k$-graphs without sources.

DEFINITION 3.6. Let $k \in \mathbb{N}_+$, let $\Lambda$ be a row-finite $k$-graph without sources, let $(X, \eta)$ be a measure space and let $\{R_\mu, D_v\}_{\mu\in\bigcup_{i=1}^k \Lambda^{e_i}, v\in\Lambda^0}$ be a family of measurable subsets of $X$. Suppose that

(1) $R_\mu \cap R_\nu \overset{\eta-a.e.}{=} \emptyset$ if $\mu \neq \nu \in \Lambda^{e_i}$ for some $1 \leq i \leq k$;
(2) $D_v \cap D_w \overset{\eta-a.e.}{=} \emptyset$ if $v \neq w \in \Lambda^0$;
(3) for each $\mu \in \bigcup_{i=1}^k \Lambda^{e_i}$, there exist two measurable maps $f_\mu : D_{s(\mu)} \to R_\mu$ and $f_\mu^{-1} : R_\mu \to D_{s(\mu)}$ such that $f_\mu \circ f_\mu^{-1} \overset{\eta-a.e.}{=} \mathrm{id}_{R_\mu}, f_\mu^{-1} \circ f_\mu \overset{\eta-a.e.}{=} \mathrm{id}_{D_{s(\mu)}}$, the pushforward measure $\eta \circ f_\mu$, of $f_\mu^{-1}$ in $D_{s(\mu)}$, is absolutely continuous with respect to $\eta$ in $D_{s(\mu)}$, and the pushforward measure $\eta \circ f_\mu^{-1}$, of $f_\mu$ in $R_\mu$, is absolutely continuous with respect to $\eta$ in $R_\mu$. Denote the Radon–Nikodym derivative $d(\eta \circ f_\mu)/d\eta$ by $\Phi_{f_\mu}$, and the Radon–Nikodym derivative $d(\eta \circ f_\mu^{-1})/d\eta$ by $\Phi_{f_\mu^{-1}}$;
(4) for $\mu, \nu, \alpha, \beta \in \bigcup_{i=1}^k \Lambda^{e_i}$ with $\mu\nu = \alpha\beta$, we have $f_\mu \circ f_\nu \overset{\eta-a.e.}{=} f_\alpha \circ f_\beta$;
(5) for $v \in \Lambda^0$, and for $1 \leq i \leq k$, we have $\bigcup_{\mu\in v\Lambda^{e_i}} R_\mu \overset{\eta-a.e.}{=} D_v$.

We call $\{D_v, R_\mu, f_\mu\}_{\mu\in\bigcup_{i=1}^k \Lambda^{e_i}, v\in\Lambda^0}$ a $\Lambda$-*branching system* on $(X, \eta)$.

Next, we show that, for row-finite without sources $k$-graphs, the above definition coincides with Definition 3.1.

PROPOSITION 3.7. *Let $k \in \mathbb{N}_+$, let $\Lambda$ be a row-finite $k$-graph without sources, and let $(X, \eta)$ be a measure space. Suppose that $\{D_v, R_\mu, f_\mu\}$ is a $\Lambda$-branching system in the sense of Definition 3.1. Then, $\{D_v, R_\mu, f_\mu\}$ is a $\Lambda$-branching system in the sense of Definition 3.6. Conversely suppose that $\{D_v, R_\mu, f_\mu\}$ is a $\Lambda$-branching system in the sense of Definition 3.6. Then, $\{D_v, R_\mu, f_\mu\}$ is a $\Lambda$-branching system in the sense of Definition 3.1.*

*Proof.* Firstly suppose that $\{D_v, R_\mu, f_\mu\}$ is a $\Lambda$-branching system in the sense of Definition 3.1. Then, it is straightforward to see that Conditions (1)–(4) of Definition 3.6 hold. For $v \in \Lambda^0$, and for $1 \leq i \leq k$, $v\Lambda^{e_i}$ is a finite exhaustive set for $v$ (see [**27**, Lemma B.2]). Then, Condition (7) of Definition 3.1 implies Condition (5) of Definition 3.6. So, $\{D_v, R_\mu, f_\mu\}$ is a $\Lambda$-branching system in the sense of Definition 3.6.

Conversely suppose that $\{D_v, R_\mu, f_\mu\}$ is a $\Lambda$-branching system in the sense of Definition 3.6. Then, it is straightforward to see that Conditions (1)–(5) of Definition 3.1 hold. For $\mu, \nu \in \bigcup_{i=1}^k \Lambda^{e_i}$ with $r(\mu) = r(\nu), d(\mu) \neq d(\nu)$, for $\alpha \in s(\mu)\Lambda^{d(\nu)}$ with $\mu\alpha = \nu'\beta, \nu \neq \nu', d(\nu) = d(\nu')$, Condition (4) of Definition 3.6 implies that $f_\mu(R_\alpha) = f_\mu \circ f_\alpha(D_{s(\alpha)}) = f_{\nu'} \circ f_\beta(D_{s(\beta)}) \subset R_{\nu'}$. So, $f_\mu(R_\alpha) \cap f_\nu(D_{s(\nu)}) = \emptyset$. So, Condition 6 of Definition 3.1 holds. For $v \in \Lambda^0$, for a finite exhaustive set $E \subset \bigcup_{i=1}^k v\Lambda^{e_i}$ for $v$, suppose that $\eta(D_v \setminus \bigcup_{\mu\in E} R_\mu) \neq 0$, for a contradiction. Since $\Lambda$ is row-finite without sources, there exists $\mu \in v\Lambda^{e_1+\cdots+e_k}$ such that $\eta((D_v \setminus \bigcup_{\mu\in E} R_\mu) \cap f_\mu(D_{s(\mu)})) \neq 0$. Since $E$ is exhaustive and $E \subset \bigcup_{i=1}^k v\Lambda^{e_i}$, there exist $\alpha \in E$ with $d(\alpha) < d(\mu)$ and $\beta \in \Lambda$ such that $\mu = \alpha\beta$. By Condition (4) of Definition 3.6, we have $f_\mu(D_{s(\mu)}) = f_\alpha \circ f_\beta(D_{s(\beta)}) \subset f_\alpha(D_{s(\alpha)}) = R_\alpha$, which implies that $(D_v \setminus \bigcup_{\mu\in E} R_\mu) \cap f_\mu(D_{s(\mu)}) = \emptyset$. So, $\eta((D_v \setminus \bigcup_{\mu\in E} R_\mu) \cap f_\mu(D_{s(\mu)})) = 0$. However, this contradicts with $\eta((D_v \setminus \bigcup_{\mu\in E} R_\mu) \cap f_\mu(D_{s(\mu)})) \neq 0$. Hence, Condition (7) of Definition 3.1 holds. Therefore, $\{D_v, R_\mu, f_\mu\}$ is a $\Lambda$-branching system in the sense of Definition 3.1. ☐

REMARK 3.8. Proposition 3.7 yields that for branching systems of row-finite $k$-graphs without sources, Definition 3.1 is equivalent to Definition 3.6, and Definition 3.6 has an easier formulation than Definition 3.1. Therefore, from now on, whenever we consider branching systems of row-finite $k$-graphs without sources, we will not distinguish which definition we refer to.

NOTATION 3.9. *Let $k \in \mathbb{N}_+$, let $\Lambda$ be a row-finite $k$-graph without sources, let $\{D_v, R_\mu, f_\mu\}_{\mu\in\bigcup_{i=1}^k \Lambda^{e_i}, v\in\Lambda^0}$ be a $\Lambda$-branching system on $(X,\eta)$, and let $\pi : C^*(\Lambda) \to B(L^2(X,\eta))$ be the representation obtained from Theorem 3.5. For $n \geq 1$, $\mu = \mu_1 \cdots \mu_n$, where $\mu_1, \ldots, \mu_n \in \bigcup_{i=1}^k \Lambda^{e_i}$, denote by $f_\mu := f_{\mu_1} \circ \cdots \circ f_{\mu_n}$ ($f_\mu$ is well-defined due to Condition (4) of Definition 3.6); denote by $\Phi_{f_\mu}$ the Radon–Nikodym derivative $d(\eta \circ f_\mu)/d\eta$; and denote by $\Phi_{f_\mu^{-1}}$ the Radon–Nikodym derivative $d(\eta \circ f_\mu^{-1})/d\eta$. It is straightforward to verify that $\pi(s_\mu)(\phi) = \chi_{R_\mu}\Phi_{f_\mu^{-1}}^{1/2}(\phi \circ f_\mu^{-1})$, $\pi(s_\mu)^*(\phi) = \chi_{D_{s(\mu)}}\Phi_{f_\mu}^{1/2}(\phi \circ f_\mu)$, $\pi(s_\mu^* s_\mu)(\phi) = \chi_{D_{s(\mu)}}\phi$ and $\pi(s_\mu s_\mu^*)(\phi) = \chi_{f_\mu(D_{s(\mu)})}\phi$, for all $\phi \in \mathcal{L}^2(X,\eta)$.*

### 3.3. Semibranching function systems.

Farsi et al. in [**10**] defined $\Lambda$-semibranching function systems for a finite $k$-graph without sources $\Lambda$ (being finite means that $|\Lambda^n| < \infty$ for all $n \in \mathbb{N}^k$). In this subsection, we find connections between $\Lambda$-semibranching function systems and $\Lambda$-branching systems.

The following definition is inspired by [**10**, Definitions 3.1, 3.2].

DEFINITION 3.10. Let $\Lambda$ be a finite $k$-graph without sources, let $(X,\eta)$ be a measure space, let $\{\mathcal{D}_\mu, \mathcal{R}_\mu\}_{\mu\in\left(\bigcup_{i=1}^k \Lambda^{e_i}\right)\cup\Lambda^0}$ be a family of measurable subsets of $X$. Suppose that

(1) for each $\mu \in \left(\bigcup_{i=1}^k \Lambda^{e_i}\right) \cup \Lambda^0$, there exist two measurable maps $\tau_\mu : \mathcal{D}_\mu \to \mathcal{R}_\mu$ and $\tau_\mu^{-1} : \mathcal{R}_\mu \to \mathcal{D}_\mu$ such that $\tau_\mu \circ \tau_\mu^{-1} \overset{\eta-a.e.}{=} \mathrm{id}_{\mathcal{R}_\mu}$, $\tau_\mu^{-1} \circ \tau_\mu \overset{\eta-a.e.}{=} \mathrm{id}_{\mathcal{D}_\mu}$, the pushforward measure $\eta \circ \tau_\mu$, of $\tau_\mu^{-1}$ in $\mathcal{D}_\mu$, is absolutely continuous with respect to $\eta$ in $\mathcal{D}_\mu$, and the pushforward measure $\eta \circ \tau_\mu^{-1}$, of $\tau_\mu$ in $\mathcal{R}_\mu$, is absolutely continuous with respect to $\eta$ in $\mathcal{R}_\mu$;

(2) for $n \in \{0, e_1, \ldots, e_k\}$, $X \overset{\eta-a.e.}{=} \bigcup_{\mu\in\Lambda^n} \mathcal{R}_\mu$;

(3) for $n \in \{0, e_1, \ldots, e_k\}$, for $\mu \neq \nu \in \Lambda^n$, $\mathcal{R}_\mu \cap \mathcal{R}_\nu \overset{\eta-a.e.}{=} \emptyset$;

(4) for $v \in \Lambda^0$, $\tau_v \overset{\eta-a.e.}{=} \mathrm{id}$, $\eta(\mathcal{D}_v) > 0$;

(5) for $\mu \in \bigcup_{i=1}^k \Lambda^{e_i}$, we have $\mathcal{R}_\mu \overset{\eta-a.e.}{\subset} \mathcal{D}_{r(\mu)}$, $\mathcal{D}_\mu = \mathcal{D}_{s(\mu)}$;

(6) for $n \in \{0, e_1, \ldots, e_k\}$, define a measurable map $\tau^n : X \to X$ by $\tau^n|_{\mathcal{R}_\mu} := \tau_\mu^{-1}$ for all $\mu \in \Lambda^n$. Then, $\tau^n \circ \tau^m = \tau^m \circ \tau^n$ for all $n, m \in \{0, e_1, \ldots, e_k\}$.

We call $\{\mathcal{R}_\mu, \mathcal{D}_\mu, \tau_\mu, \tau^n : \mu \in \left(\bigcup_{i=1}^k \Lambda^{e_i}\right) \cup \Lambda^0, n \in \{0, e_1, \ldots, e_k\}\}$ a *partial $\Lambda$-semibranching function system* on $(X,\eta)$.

REMARK 3.11. Let $\Lambda$ be a finite $k$-graph without sources. For $\mu = \mu_1 \cdots \mu_n \in \Lambda \setminus \Lambda^0$ where $\mu_1, \ldots, \mu_n \in \bigcup_{i=1}^k \Lambda^{e_i}$, define $\mathcal{D}_\mu := \mathcal{D}_{s(\mu)}$, define $\tau_\mu := \tau_{\mu_1} \circ \cdots \circ \tau_{\mu_n}$, and define $\mathcal{R}_\mu := \tau_\mu(\mathcal{D}_\mu)$. For $n = (n_1, \ldots, n_k) \in \mathbb{N}^k \setminus \{0\}$, define $\tau^n := n_1\tau^{e_1} \circ \cdots \circ n_k\tau^{e_k}$. Then, $\{\mathcal{R}_\mu, \mathcal{D}_\mu, \tau_\mu, \tau^n : \mu \in \Lambda, n \in \mathbb{N}^k\}$ is in fact a $\Lambda$-*semibranching function system* on $(X,\eta)$ as introduced by Farsi et al. in [**10**].

PROPOSITION 3.12. *Let $\Lambda$ be a finite $k$-graph without sources, let $(X,\eta)$ be a measure space, and let $\{D_v, R_\mu, f_\mu\}_{\mu\in\bigcup_{i=1}^k \Lambda^{e_i}, v\in\Lambda^0}$ be a $\Lambda$-branching system on $(X,\eta)$. Suppose that $\eta(D_v) > 0$ for all $v \in \Lambda^0$, and that $X = \bigcup_{v\in\Lambda^0} D_v$. For $v \in \Lambda^0$, define $\mathcal{D}_v = \mathcal{R}_v :=$*

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*$D_v$, define $\tau_v : \mathcal{D}_v \to \mathcal{R}_v$ to be the identity map. For $\mu \in \bigcup_{i=1}^k \Lambda^{e_i}$, define $\mathcal{D}_\mu := D_{s(\mu)}$, define $\mathcal{R}_\mu := R_\mu$, and define $\tau_\mu := f_\mu$. For $n \in \{0, e_1, \ldots, e_k\}$, define a measurable map $\tau^n : X \to X$ by $\tau^n|_{\mathcal{R}_\mu} := \tau_\mu^{-1}$ for all $\mu \in \Lambda^n$. Then, $\{\mathcal{R}_\mu, \mathcal{D}_\mu, \tau_\mu, \tau^n : \mu \in \left(\bigcup_{i=1}^k \Lambda^{e_i}\right) \cup \Lambda^0, n \in \{0, e_1, \ldots, e_k\}\}$ is a partial $\Lambda$-semibranching function system on $(X, \eta)$.*

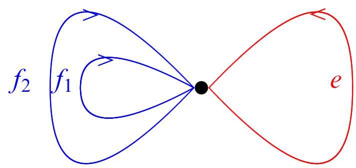
*Proof.* It is straightforward to see. □

Proposition 3.13. *Let $\Lambda$ be a finite k-graph without sources, let $(X, \eta)$ be a measure space, and let $\{\mathcal{R}_\mu, \mathcal{D}_\mu, \tau_\mu, \tau^n : \mu \in \left(\bigcup_{i=1}^k \Lambda^{e_i}\right) \cup \Lambda^0, n \in \{0, e_1, \ldots, e_k\}\}$ be a partial $\Lambda$-semibranching function system on $(X, \eta)$. For $v \in \Lambda^0$, define $D_v := \mathcal{D}_v$. For $\mu \in \bigcup_{i=1}^k \Lambda^{e_i}$, define $R_\mu := \mathcal{R}_\mu$, and define $f_\mu := \tau_\mu$. Then, $\{D_v, R_\mu, f_\mu\}_{\mu \in \bigcup_{i=1}^k \Lambda^{e_i}, v \in \Lambda^0}$ is a $\Lambda$-branching system on $(X, \eta)$.*

*Proof.* It is straightforward to see. □

**4. Examples of $\Lambda$-branching systems on $\mathbb{R}$ with the Lebesgue measure.** In this section, we will present many examples of branching systems on $\mathbb{R}$. As we mentioned before, due to the large combinatorial possibilities permitted by the factorization property on a coloured graph, we are not able to provide a general construction of branching systems on $\mathbb{R}$. Instead, in the examples, we provide an algorithmic way to build branching systems on $\mathbb{R}$, covering many examples of $k$-graphs in the literature.

Example 4.1. Let $\Gamma$ be the following 2-coloured graph, where $\Gamma^0 = \{v\}$, $\Gamma^{e_1} = \{f_1, f_2 : n \in \mathbb{N}\}$ and $\Gamma^{e_2} = \{e\}$. This is an example in [**11**, Section 4].




There are two 2-graphs $\Lambda_2$ and $\Lambda_3$ associated to the 2-coloured graph $\Gamma$. The factorization rules for $\Lambda_2$ are given by

$$f_1 e = e f_1 \text{ and } f_2 e = e f_2,$$

and the factorization rules for $\Lambda_3$ are given by

$$f_1 e = e f_2 \text{ and } f_2 e = e f_1.$$

We will define a $\Lambda_2$-branching system and a $\Lambda_3$-branching system in the sense of Definition 3.6 on [0, 1] with the Lebesgue measure.

Define $D_v = R_e := [0, 1]$, define $R_{f_1} := [0, \frac{1}{2}]$ and define $R_{f_2} = [\frac{1}{2}, 1]$.

For $\Lambda_2$, let $f_e$ be the identity map, let $f_{f_1}, f_{f_2}$ be any bijective differentiable maps from $D_v$ onto $R_{f_1}, R_{f_2}$, respectively. It is straightforward to check that $\{D_v, R_e, R_{f_1}, R_{f_2}, f_e, f_{f_1}, f_{f_2}\}$ is a $\Lambda_2$-branching system in the sense of Definition 3.6.

For $\Lambda_3$, define $f_e(x) := x + \frac{1}{2}$ if $x \in [0, \frac{1}{2}]$; define $f_e(x) := x - \frac{1}{2}$ if $x \in [\frac{1}{2}, 1]$; define $f_{f_1}(x) := \frac{1}{2}x$; define $f_{f_2}(x) := \frac{1}{2}x + \frac{3}{4}$ if $x \in [0, \frac{1}{2}]$; and define $f_{f_2}(x) := \frac{1}{2}x + \frac{1}{4}$ if

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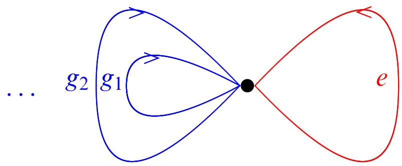
$x \in [\frac{1}{2}, 1]$. It is straightforward to check that $\{D_v, R_e, R_{f_1}, R_{f_2}, f_e, f_{f_1}, f_{f_2}\}$ is a $\Lambda_3$-branching system in the sense of Definition 3.6. $\square$

EXAMPLE 4.2. Consider $\Lambda$ as the following 2-coloured graph, where $\Lambda^0 = \{v\}$, $\Lambda^{e_1} = \{g_n : n \in \mathbb{N}_+\}$ and $\Lambda^{e_2} = \{e\}$.




There are uncountably many possible factorizations in the above graph, each giving a different 2-graph. For each of these 2-graphs, we build a branching system below.

Fix a factorization and let $d : \Lambda \to \mathbb{N}^2$ be the degree map. Notice that $d(g_i e) = e_1 + e_2 = e_2 + e_1$, and by the factorization property, there is a unique $g_j$ such that $g_i e = e g_j$. So, we get a map $h : \mathbb{N}_+ \to \mathbb{N}_+$ such that $g_i e = e g_{h(i)}$. Moreover, note that $h$ is injective, because if we suppose that $h(i) = h(j)$, then $g_i e = e g_{h(i)} = e g_{h(j)} = g_j e$ and then, again by the factorization property, we get $g_i = g_j$. The map $h$ is also surjective, since if $j \in \mathbb{N}_+$ then, by the factorization property, there exist a unique $i$ such that $g_i e = e g_j$.

Our goal is to define a $\Lambda$-branching system on the interval $(0, 1]$ with the Lebesgue measure. Define $D_v = (0, 1] = R_e$ and $R_{g_i} = (\frac{1}{i+1}, \frac{1}{i}]$, for each $i \in \mathbb{N}_+$. Now we need to define the bijective maps $\{f_{g_i}\}_{i \in \mathbb{N}_+}$ and $f_e$.

First of all, for each $i \in \mathbb{N}_+$, define the set $B_i := \{h^n(i) : n \in \mathbb{Z}\}$. There are two possible configurations for the $B_i$: if $h^n(i) = h^m(i)$, for some $n, m \in \mathbb{Z}$, then $B_i = \{i, h(i), h^2(i), \ldots, h^k(i)\}$, where the elements $h^s(i)$ are pairwise distinct and $h^{k+1}(i) = i$; if $h^n(i) \neq h^m(i)$ for each $m, n$, then $B_i = \{\ldots, h^{-2}(i), h^{-1}(i), i, h(i), h^2(i), \ldots\}$.

It is not hard to see that for $i \neq j$, $B_i = B_j$ or $B_i \cap B_j = \emptyset$. So, by choosing an appropriate set $F \subseteq \mathbb{N}_+$, we get that $\mathbb{N}_+$ is the disjoint union $\mathbb{N}_+ = \sqcup_{i \in F} B_i$.

Now we define the bijective map $f_e : D_v \to R_e$. First, we define this map on each set $R_{g_i}$. Fix an $i \in F$. Suppose first that $B_i = \{i, h(i), \ldots, h^k(i)\}$, and $h^{k+1}(i) = i$. Define, for each $n \in \{1, \ldots, k+1\}$, the increasing linear maps $f_e : R_{g_{h^n(i)}} \to R_{g_{h^{n-1}(i)}}$, and piece them together to obtain the map $f_e : \bigcup_{n=0}^k R_{g_{h^n(i)}} \to \bigcup_{n=0}^k R_{g_{h^n(i)}}$. It follows by definition that $f_e^{k+1}$ is the identity map. For $B_i = \{\ldots, h^{-2}(i), h^{-1}(i), i, h(i), h^2(i), \ldots\}$ define $f_e : R_{g_{h^n(i)}} \to R_{g_{h^{n-1}(i)}}$ as being the increasing linear diffeomorphism, for each $n \in \mathbb{Z}$. So, we get a bijective measurable map $f_e : D_v \to R_e$, with the property that, for each $i \in \mathbb{N}_+$, $f_e(R_{g_{h(i)}}) = R_{g_i}$ and, moreover, if $h^{k+1}(i) = i$, for some $k \in \mathbb{N}$, then $f_e : \bigcup\limits_{n=0}^{k} R_{g_{h^n(i)}} \to \bigcup\limits_{n=0}^{k} R_{g_{h^n(i)}}$ is such that $f_e^{k+1}$ is the identity map (restricted to this set).

It remains to define the maps $f_{g_i} : D_v \to R_{g_i}$ for each $i \in \mathbb{N}_+$.

Let $i \in F$. If $B_i = \{i, h(i), h^2(i), \ldots, h^k(i)\}$ with $h^{k+1}(i) = i$, define $f_{g_i} : D_v \to R_{g_i}$ as being the increasing linear diffeomorphism, and define inductively $f_{g_{h^n(i)}} = f_e^{-1} \circ f_{g_{h^{n-1}(i)}} \circ f_e$ for $n \in \{1, \ldots, k\}$. Notice that, then the equality $f_e f_{g_{h^n(i)}} = f_{g_{h^{n-1}(i)}} f_e$ holds for each $n \in \{1, \ldots, k\}$. To see that the equality $f_e f_{g_{h^{k+1}(i)}} = f_{g_{h^k(i)}} f_e$ also holds, note that

$$f_{g_{h^k(i)}} f_e = f_e^{-1} f_{g_{h^{k-1}(i)}} f_e^2 = f_e^{-2} f_{g_{h^{k-2}(i)}} f_e^3 = \cdots = f_e^{-k} f_{g_i} f_e^{k+1}$$

$$= f_e^{-k-1} f_e f_{g_i} f_e^{k+1} = f_e f_{g_i} = f_e f_{g_{h^{k+1}(i)}},$$

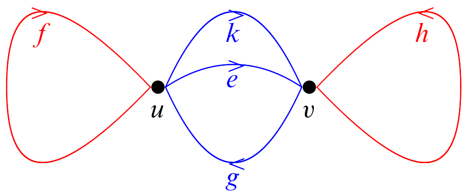
since $f_e^{k+1}$ is the identity map and $h^{k+1}(i) = i$.

If $B_i = \{h^n(i) : n \in \mathbb{Z}\}$, with $h^n(i) \neq h^m(i)$ for each $n, m$, let $f_{g_i} : D_v \to R_{g_i}$ be the increasing linear diffeomorphism and define inductively $f_{g_{h^n(i)}} = f_e^{-1} \circ f_{g_{h^{n-1}(i)}} \circ f_e$ and $f_{g_{h^{-n}(i)}} = f_e \circ f_{g_{h^{-n+1}(i)}} \circ f_e^{-1}$ for $n \geq 1$.

It is easy to see that Conditions (1)–(6) of Definition 3.1 are satisfied. Condition (7) also holds, because each exhaustive finite set $E \subseteq v\Lambda^{e_1} \cup v\Lambda^{e_2}$ must contain $e$, $R_e = D_v$ and any other $R_{g_i} \subset D_v$. $\square$

REMARK 4.3. To simplify notation, and when no confusion arises, from now on we will denote the map $f_e$ associated to an edge $e$ just by $e$.

EXAMPLE 4.4. We next turn our attention to the 2-graphs given in [**26**, p. 102]. Recall the 2-coloured graph, where $f$ and $h$ have degree $(0, 1) \in \mathbb{N}^2$ and $k$, $e$ and $g$ have degree $(1, 0)$:




There are two possible factorizations. One is $kf = hk$, $ef = he$ and $gh = fg$. For this 2-graph a branching system is obtained similarly to what we did in Example 4.1, defining the maps associated to the loops in the graph as the identity. We will focus in the second possible factorization, that is

$$he = kf, \; hk = ef \text{ and } gh = fg.$$

Let $D_u = [0, 1]$ and $D_v = [1, 2]$. Notice that the sets $\{g\}, \{f\}, \{h\}, \{e, k\}$ are exhaustive, hence $R_g = R_f = [0, 1]$ and $R_h = [1, 2] = R_e \cup R_k$. Let $R_e = [1, \frac{3}{2}]$ and $R_k = [\frac{3}{2}, 2]$. From the factorization, we obtain the information on how to break up the definition of the function $h$. Let $h|_{[1,\frac{3}{2}]} \to [\frac{3}{2}, 2]$ be defined by $h(x) = x + \frac{1}{2}$ and $h|_{[\frac{3}{2},2]} \to [1, \frac{3}{2}]$ be defined by $h(x) = x - \frac{1}{2}$ (notice that $h^2 = id$). Let $g : [1, 2] \to [0, 1]$ be defined by $g(x) = x - 1$, $e : [0, 1] \to [1, \frac{3}{2}]$ be defined by $e(x) = \frac{1}{2}x + 1$ and define the remaining functions via the factorization, that is, $f := ghg^{-1}$ and $k := h^{-1}ef = hef = hef^{-1}$ (notice that $f^2 = id$). $\square$

EXAMPLE 4.5. Let $\Lambda$ be the 2-graph whose 1-skeleton is given below, and where the edges $g_i$ have degree $(0, 1)$ and the edges $e_i$ have degree $(1, 0)$.

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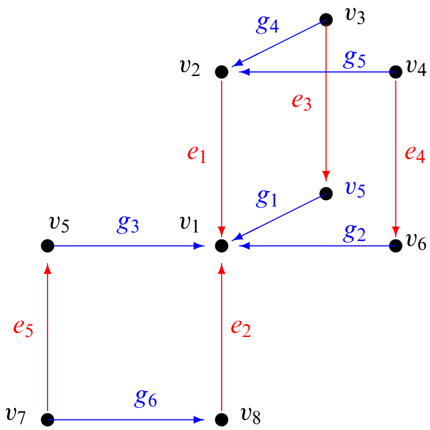
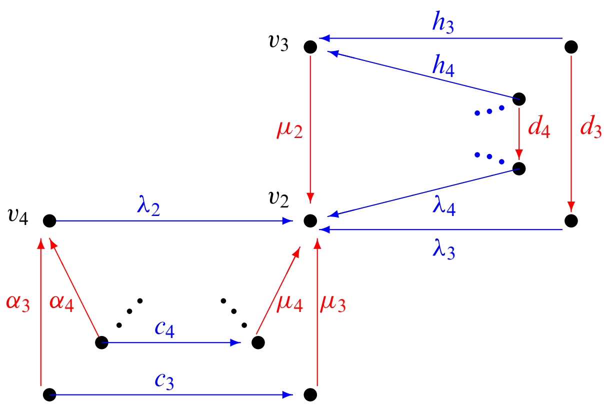



Notice that the sets $\{e_1, e_2\}$, $\{g_1, g_2, g_3\}$, $\{e_1, g_3\}$ and $\{e_2, g_1, g_2\}$ are exhaustive for $v_1$ and $\{g_4, g_5\}$ is exhaustive for $v_2$. So, take $D_{v_i} = [i-1, i]$, $R_{e_1} = [0, \frac{1}{2}]$, $R_{e_2} = [\frac{1}{2}, 1]$, $R_{g_3} = [\frac{1}{2}, 1]$, $R_{g_2} = [\frac{1}{4}, \frac{1}{2}]$, $R_{g_1} = [0, \frac{1}{4}]$, $R_{g_4} = [1, \frac{3}{2}]$ and $R_{g_5} = [\frac{3}{2}, 2]$. Let $f_{e_1}|_{R_{g_4}}$ be the affine map onto $R_{g_1}$ and $f_{e_1}|_{R_{g_5}}$ be the affine map onto $R_{g_2}$. Define $f_{g_4}$ and $f_{e_3}$ as affine maps and let $f_{g_1} := f_{e_1} f_{g_4} f_{e_3}^{-1}$. Analogously one define the reminder maps and sets. $\square$

EXAMPLE 4.6. Next, we construct a branching system for the 2-graph $\Lambda_2$ given in [**27**, Example A.2]. We reproduce a picture of the 1-skeleton below.




In this example, the edges $h_i$, $c_i$ and $\lambda_i$ have degree $(1, 0)$ and edges $\alpha_i$, $\mu_i$ and $d_i$ have degree $(0, 1)$.

To construct a branching system, first we need to enumerate the edges and vertices. Respecting the labels already given in the example, we enumerate the red edges in $v_2\Lambda_2$ by $\mu_2, \mu_3, \ldots$, the blue edges in $v_2\Lambda_2$ by $\lambda_2, \lambda_3, \ldots$, the blue edges in $s(\mu_2)$ by $h_3, h_4, \ldots$, the red edges in $s(\lambda_2)$ by $\alpha_3, \alpha_4, \ldots$, for $i \neq 2$, we denote the red edge whose range is $s(\lambda_i)$ by $d_i$ and the blue edge whose range is $s(\mu_i)$ by $c_i$.

With the above labels, the factorization reads

$$\mu_2 h_n = \lambda_n d_n \text{ and } \lambda_2 \alpha_i = \mu_i c_i.$$

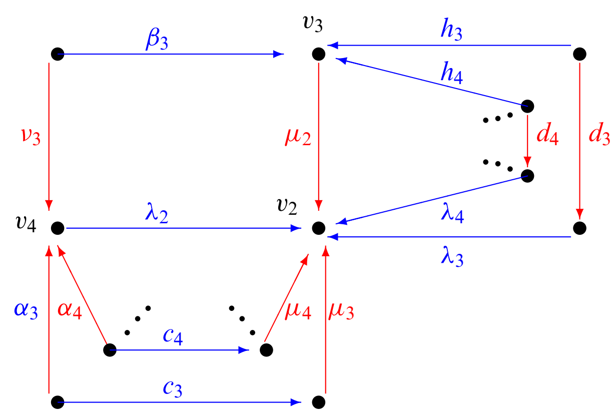
As usual, to obtain a branching system, we define the sets associated to vertices as intervals. In particular, let $D_{v_2} = [0, 1]$ and $D_{v_3} = [1, 2]$, where $v_3 = s(\mu_2)$. We will focus on defining the branching system for these vertices (once this is done it is clear how to extend it to the remaining vertices).

Notice that $\{\lambda_2, \mu_2\}$ is exhaustive. So, let $R_{\mu_2} = [0, \frac{1}{2}]$ and $R_{\lambda_2} = [\frac{1}{2}, 1]$. Also, let $R_{h_{i+2}} = [1 + \frac{1}{2^i}, 1 + \frac{1}{2^{i-1}}]$, $i = 1, 2, \ldots$ and let $R_{\lambda_3} = [0, \frac{1}{4}]$, $R_{\lambda_4} = [\frac{1}{4}, \frac{1}{4} + \frac{1}{8}]$ and so on. Define $\mu_2|_{R_{h_n}}$ as the affine map onto $R_{\lambda_n}$. Also, let $h_n$ and $d_n$ be affine bijective maps. Following the factorization define, for $n \neq 2$, $\lambda_n := \mu_2 h_n d_n^{-1}$.

Proceeding analogously one defines $\mu_i (i \neq 2)$, $c_i$, $\alpha_i$, $\lambda_2$ and so a branching system is obtained. $\square$

EXAMPLE 4.7. Next, we construct a branching system for the 2-graph $\Lambda_3$ given in [**27**, Example A.3]. Differently from [**27**], we will keep all the notations of our previous example (A.2). This example is the same as the example before, with the addition of two edges, called $\beta_3$ (of degree $(1, 0)$) and $\alpha_3$ (of degree $(0, 1)$) in [**27**] (we will call $\alpha_3$ of $v_3$, since in our setting $\alpha_3$ is already taken). Notice that this is a particularly interesting example, since there is no finite exhaustive subset of $v_2\Lambda$ whose range projections are orthogonal as mentioned in [**27**, Example A.3].

We reproduce a picture of the 1-skeleton below.




The factorization is the same as before, with one more factorization property:

$$\mu_2 h_n = \lambda_n d_n \ , \ \lambda_2 \alpha_i = \mu_i c_i \text{ and } \mu_2 \beta_3 = \lambda_2 \nu_3.$$

As before, we will describe the branching system mainly at $v_2$. Notice that $\{\lambda_2, \mu_2\}$ is an exhaustive set in $v_2$. Furthermore, the new factorization property implies that $R_{\mu_2}$ and $R_{\lambda_2}$ can not be disjoint. Let $D_{v_2} = [0, 1]$, $D_{v_3} = [1, 2]$, where $v_3 = s(\mu_2)$, and $D_{v_4} = [2, 3]$, where $v_4 = s(\lambda_2)$. Define

$$R_{\mu_2} = \left[0, \frac{3}{4}\right] \text{ and } R_{\lambda_2} = \left[\frac{1}{2}, 1\right].$$

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Break $D_{v_3}$ in infinitely many intervals of positive length, namely, $\{R_{\beta_3}, R_{h_{i+2}} : i = 1, 2, \ldots\}$ and break $D_{v_4}$ in infinitely many intervals of positive length, namely, $\{R_{\nu_3}, R_{\alpha_{i+2}} : i = 1, 2, \ldots\}$. Now, break $[\frac{3}{4}, 1]$ in infinitely many intervals of positive length, say $R_{\mu_3}, R_{\mu_4}, \ldots$ and break $[0, \frac{1}{2}]$ in infinitely many intervals of positive length, say $R_{\lambda_3}, R_{\lambda_4}, \ldots$.

Proceeding similarly to the previous example, we define $\mu_2|_{R_{h_n}}$ as the affine map onto $R_{\lambda_n}, n = 3, 4, \ldots$ and $\mu_2|_{R_{\beta_3}}$ as the affine map onto $R_{\mu_2} \cap R_{\lambda_2}$. We also let $h_n$ and $d_n$ be affine bijective maps and following the factorization define, for $n \neq 2, \lambda_n := \mu_2 h_n d_n^{-1}$, $\lambda_2|_{R_{\nu_3}} := \mu_2 \beta_3 \nu_3^{-1}$, where $\beta_3$ and $\nu_3$ are affine bijective. The remainder of the definition of a branching systems follows analogously to above and the previous example. □

**5. Faithful representations of periodic single-vertex 2-graphs C*-algebras via branching systems.** In this section, we exclusively study the branching systems of periodic single-vertex 2-graphs and we intend to find a sufficient condition for representations of periodic single-vertex 2-graph C*-algebras, induced from branching systems, to be faithful.

First of all, we recall the work of Davidson and Yang on the periodicity of single-vertex 2-graphs in [**7**] (they also studied the structure of single-vertex $k$-graph C*-algebras in [**8**]).

THEOREM 5.1 ([**7**, Theorems 3.1, 3.4]). *Let $\Lambda$ be a single-vertex 2-graph. Suppose that $|\Lambda^{e_1}|, |\Lambda^{e_2}| \geq 2$. Then, the following are equivalent.*

(1) *$\Lambda$ is periodic;*
(2) *$\mathrm{Per}(\Lambda) = \mathbb{Z}(a, -b)$ for some positive integers $a, b$;*
(3) *there exist positive integers $p, q$ with $|\Lambda^{e_1}|^p = |\Lambda^{e_2}|^q$ and a bijection $h : \prod_{i=1}^{p} \Lambda^{e_1} \to \prod_{i=1}^{q} \Lambda^{e_2}$ such that for $\mu \in \prod_{i=1}^{p} \Lambda^{e_1}$, $\nu \in \prod_{i=1}^{q} \Lambda^{e_2}$, we have $\mu\nu = h(\mu)h^{-1}(\nu)$ (we can identify $\prod_{i=1}^{p} \Lambda^{e_1}, \prod_{i=1}^{q} \Lambda^{e_2}$ with elements in $\Lambda$).*

NOTATION 5.2. *Let $\Lambda$ be a periodic single-vertex 2-graph with $|\Lambda^{e_1}|, |\Lambda^{e_2}| \geq 2$. Let $(a, -b)$ be the generator of $\mathrm{Per}(\Lambda)$, and let $h : \prod_{i=1}^{a} \Lambda^{e_1} \to \prod_{i=1}^{b} \Lambda^{e_2}$ obtained from the above theorem. Then, for each $\mu \in \prod_{i=1}^{a} \Lambda^{e_1}, (\mu, h(\mu))$ is a cycline pair. By [7, Lemma 5.3] there is a distinguished unitary $W := \sum_{\mu \in \prod_{i=1}^{a} \Lambda^{e_1}} s_{h(\mu)} s_\mu^*$ in $C^*(\Lambda)$.*

LEMMA 5.3. *Let $\Lambda$ be a periodic single-vertex 2-graph with $|\Lambda^{e_1}|, |\Lambda^{e_2}| \geq 2$. We inherit the notation from Notation 5.2. Then, the spectrum of $W$ contains the unit circle.*

*Proof.* It is sufficient to show that $C^*(W) \cong C(\mathbb{T})$, via a unital isomorphism that identifies $W$ with the identity function on $\mathbb{T}$. By [**22**, Proposition 3.11] it is sufficient to show that there exists an expectation $\Phi : C^*(W) \to \mathbb{C} \cdot 1_{C^*(\Lambda)}$ such that $\Phi(W^n) = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$, and that $\Phi(1_{C^*(\Lambda)}) = 1_{C^*(\Lambda)}$. Let $\gamma$ be the gauge action on $C^*(\Lambda)$. Then, $\gamma$ induces a strongly continuous homomorphism from $\mathbb{T}^2$ to $\mathrm{Aut}(C^*(W))$. Denote by $\iota : \mathbb{T} \to \mathbb{T}^2$, the embedding $z \mapsto (1, z)$. So, we obtain a strongly continuous homomorphism $\gamma \circ \iota : \mathbb{T} \to \mathrm{Aut}(C^*(W))$. Therefore, $\gamma \circ \iota$ yields the desired expectation $\Phi : C^*(W) \to \mathbb{C} \cdot 1_{C^*(\Lambda)}$ and hence we are done. □

The following theorem is an extension of the general Cuntz–Krieger uniqueness theorem of Brown–Nagy–Reznikoff (see Theorem 2.21).

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THEOREM 5.4. *Let $\Lambda$ be a periodic single-vertex 2-graph with $|\Lambda^{e_1}|, |\Lambda^{e_2}| \geq 2$, let $\mathcal{A}$ be a $C^*$-algebra and let $\varphi : C^*(\Lambda) \to \mathcal{A}$ be a homomorphism. We inherit the notation from Notation 5.2. Then, $\varphi$ is injective if and only if*

*(1) $\varphi(1_{C^*(\Lambda)}) \neq 0$;*
*(2) the spectrum of $\varphi(W)$ contains the unit circle.*

*Proof.* First of all, suppose that $\varphi$ is injective. It is straightforward to see that Condition (1) holds. By Lemma 5.3, Condition (2) holds.

Conversely suppose that Conditions (1) and (2) hold. By Theorem 2.21, it is sufficient to prove that $\varphi$ is injective on $\mathcal{M}$.

The faithful expectation $\Phi$ from Notation 2.16 restricts to a faithful expectation from $\mathcal{M}$ onto $\mathcal{D}$ satisfying that for $d \in \mathcal{D}, n \in \mathbb{Z}$, if $n = 0$, then $\Phi(dW^n) = d$; and if $n \neq 0$, then $\Phi(dW^n) = 0$.

Since $\varphi(1_{C^*(\Lambda)}) \neq 0$, [**27**, Theorem 3.1] gives that $\varphi$ is injective on $C^*(\Lambda)^\gamma$. Since $\mathcal{D} \subset C^*(\Lambda)^\gamma$, $\varphi$ is injective on $\mathcal{D}$.

By Condition 2, there exists an expectation $\Psi : \varphi(C^*(W)) \to \varphi(\mathbb{C} \cdot 1_{C^*(\Lambda)})$ such that for $n \in \mathbb{Z}$, if $n = 0$, then $\Psi(\varphi(W^n)) = \varphi(1_{C^*(\Lambda)})$; and if $n \neq 0$, then $\Psi(\varphi(W^n)) = 0$. As shown in the proof of [**32**, Theorem 6.2], $\mathcal{M} = \overline{\operatorname{span}}\{s_\mu s_\mu^* W^n : \mu \in \Lambda, n \in \mathbb{Z}\}$ and $\mathcal{M}$ is unital abelian, where $\mathcal{M}$ is the cycline subalgebra of $C^*(\Lambda)$ given in Definition 2.19. We aim to construct a linear map $\Gamma : \operatorname{span}\{\varphi(s_\mu s_\mu^* W^n) : \mu \in \Lambda, n \in \mathbb{Z}\} \to \varphi(\mathcal{D})$ such that if $n = 0$, then $\Gamma(\varphi(s_\mu s_\mu^* W^n)) = \varphi(s_\mu s_\mu^*)$; and if $n \neq 0$, then $\Gamma(\varphi(s_\mu s_\mu^* W^n)) = 0$, where $\mathcal{D}$ is the diagonal of $C^*(\Lambda)$ given in Definition 2.19. In order to prove that $\Gamma$ is well-defined, we show that it is contractive. Fix distinct $\mu_1, \ldots, \mu_L \in \Lambda$ with $d(\mu_1) = \cdots = d(\mu_L)$, for $1 \leq i \leq L$, fix $\{z_{ij}\}_{j \in \mathbb{Z}} \subset \mathbb{C}$ with at most finitely many nonzero. Then, we compute that

$$\begin{aligned}
\Big\| \sum_{i=1}^{L} z_{i0} \varphi(s_{\mu_i} s_{\mu_i}^*) \Big\| &= \max_{1 \leq i \leq L} \big\| z_{i0} \varphi(s_{\mu_i} s_{\mu_i}^*) \big\| \\
&\leq \max_{1 \leq i \leq L} \Big\| \varphi(s_{\mu_i} s_{\mu_i}^*) \sum_{j \in \mathbb{Z}} z_{ij} \varphi(W^j) \Big\| \\
&\quad \text{(since } \Psi \text{ is an expectation on } \varphi(C^*(W))) \\
&= \Big\| \sum_{i=1}^{L} \varphi(s_{\mu_i} s_{\mu_i}^*) \sum_{j \in \mathbb{Z}} z_{ij} \varphi(W^j) \Big\|.
\end{aligned}$$

By Condition (4) of Proposition 2.13, every element in $\operatorname{span}\{\varphi(s_\mu s_\mu^* W^n) : \mu \in \Lambda, n \in \mathbb{Z}\}$ has the form $\sum_{i=1}^{L} \varphi(s_{\mu_i} s_{\mu_i}^*) \sum_{j \in \mathbb{Z}} z_{ij} \varphi(W^j)$. Hence, we obtain a linear idempotent map $\Gamma$ of norm 1 from $\varphi(\mathcal{M})$ onto $\varphi(\mathcal{D})$. By [**1**, II.6.10.2], $\Gamma$ is an expectation. Finally, by [**22**, Proposition 3.11], $\varphi$ is injective on $\mathcal{M}$. So, we are done. □

Now we present a sufficient condition for representations of periodic single-vertex 2-graphs induced from branching systems to be faithful.

THEOREM 5.5. *Let $\Lambda$ be a periodic single-vertex 2-graph with the vertex $v$ and $|\Lambda^{e_1}|, |\Lambda^{e_2}| \geq 2$. We inherit the notation from Notation 5.2. Let $\{D_v, R_\mu, f_\mu : \mu \in \bigcup_{i=1}^{k} \Lambda^{e_i}\}$ be a $\Lambda$-branching system on a measure space $(X, \eta)$ such that $\eta(D_v) \neq 0$, and let $\pi : C^*(\Lambda) \to B(L^2(X, \eta))$ be the representation induced from the $\Lambda$-branching system. Suppose that for any finite subset $\mathcal{F}$ of $\mathbb{Z} \setminus \{0\}$, there exist $\mu \in \prod_{i=1}^{a} \Lambda^{e_1}$ and a*

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*measurable subset $E$ of $f_\mu(D_v)$ such that $\eta(E) \neq 0$ and $(f_\mu \circ f_{h(\mu)}^{-1})^n(E) \cap E \overset{\eta-a.e.}{=} \emptyset$ for all $n \in \mathcal{F}$. Then, $\pi$ is faithful.*

*Proof.* Since $\eta(D_v) \neq 0$, we have that $\pi(1_{C^*(\Lambda)}) \neq 0$. By [**4**, Proposition 4.1], we have $\pi(s_\mu s_\mu^*) = \pi(s_{h(\mu)} s_{h(\mu)}^*)$ for all $\mu \in \prod_{i=1}^a \Lambda^{e_1}$. So, $f_\mu(D_v) \overset{\eta-a.e.}{=} f_{h(\mu)}(D_v)$ for all $\mu \in \prod_{i=1}^a \Lambda^{e_1}$; $f_\mu(D_v) \cap f_\nu(D_v) \overset{\eta-a.e.}{=} \emptyset$ for distinct $\mu, \nu \in \prod_{i=1}^a \Lambda^{e_1}$; and $D_v \overset{\eta-a.e.}{=} \bigcup_{\mu \in \prod_{i=1}^a \Lambda^{e_1}} f_\mu(D_v)$. By Theorem 5.4, in order to prove that $\pi$ is injective, we only need to show that the spectrum of $\pi(W)$ in $C^*(\pi(W))$ contains the unit circle. By [**22**, Proposition 3.11], it suffices to construct an expectation $\Phi : C^*(\pi(W)) \to \mathbb{C} \cdot \pi(1_{C^*(\Lambda)})$ such that $\Phi(\pi(1_{C^*(\Lambda)})) = \pi(1_{C^*(\Lambda)})$, $\Phi(\pi(W^n)) = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$. Fix $\{z_n\}_{n \in \mathbb{Z}} \subset \mathbb{C}$ with at most finitely many nonzero. Let $\mathcal{F} := \{0 \neq n \in \mathbb{Z} : z_n \neq 0\}$. By the assumption of the theorem, there exist $\mu_0 \in \prod_{i=1}^a \Lambda^{e_1}$ and a measurable subset $E$ of $f_{\mu_0}(D_v)$ such that $\eta(E) \neq 0$ and $(f_{\mu_0} \circ f_{h(\mu_0)}^{-1})^n(E) \cap E \overset{\eta-a.e.}{=} \emptyset$ for all $n \in \mathcal{F}$. Take an arbitrary function $\phi \in L^2(X, \eta)$ with $\|\phi\| = 1$ and $\mathrm{supp}(\phi) \overset{\eta-a.e.}{\subset} E$. Then,

$$
\begin{aligned}
\Big\| \sum_{n \in \mathbb{Z}} z_n \pi(W^n) \Big\|^2 &= \Big\| z_0 \pi(1_{C^*(\Lambda)}) + \sum_{n \in \mathcal{F}} z_n \pi(W^n) \Big\| \\
&\geq \Big\| z_0 \phi \chi_{D_v} + \sum_{n \in \mathcal{F}} z_n \pi(W^n)(\phi) \Big\|^2 \\
&= \int_X \Big| z_0 \phi \chi_{D_v} + \sum_{n \in \mathcal{F}} z_n \pi(W^n)(\phi) \Big|^2 \mathrm{d}\eta \\
&\geq \int_E \Big| z_0 \phi + \sum_{n \in \mathcal{F}} z_n \pi(W^n)(\phi) \Big|^2 \mathrm{d}\eta \\
&= \int_E \Big| z_0 \phi + \sum_{n \in \mathcal{F}} z_n \pi(s_{h(\mu_0)} s_{\mu_0}^*)(\phi) \Big|^2 \mathrm{d}\eta \\
&= \int_E \big| z_0 \phi \big|^2 \mathrm{d}\mu \\
&= |z_0|^2.
\end{aligned}
$$

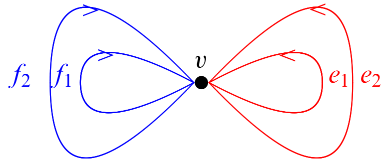
So, we get the required expectation $\Phi$ and hence $\pi$ is injective. □

In the following, we modify the construction of the branching systems in Theorem 3.3 and we obtain a branching system for each periodic single-vertex 2-graph so that the associated representation is faithful.

EXAMPLE 5.6. Let $\Lambda$ be a periodic single-vertex 2-graph with $|\Lambda^{e_1}|, |\Lambda^{e_2}| \geq 2$. Let $X := [0, 1] \times \Lambda^\infty$. Define $D_v := X$. For $e \in \Lambda^{e_1}$, define $R_e := [0, 1] \times e\Lambda^\infty$, define $F_e : D_v \to R_e$ by $F_e(t, x) := (t^2, ex)$ (see Lemma 2.4). For $f \in \Lambda^{e_2}$, define $R_f := [0, 1] \times f\Lambda^\infty$, define $F_f : D_v \to R_f$ by $F_f(t, x) := (\sqrt{t}, fx)$. Then, $\{D_v, R_\mu, F_\mu\}_{\mu \in \bigcup_{i=1}^2 \Lambda^{e_i}}$ is a $\Lambda$-branching system. By Theorem 5.5, the induced representation is faithful.

We finish this section by building a branching system on $\mathbb{R}^2$ for a periodic single-vertex 2-graph such that the associated representation is faithful.

EXAMPLE 5.7. Consider the flip C*-algebra from the 2-coloured graph of [**7**, Example 4.3],




with the factorization rule $e_if_j = f_ie_j$, for $i, j \in \{1, 2\}$. For this 2-graph, here called $\Lambda$, we obtain a $\Lambda$-branching system on the measure space $([0, 1] \times [-1, 1], \eta)$ in the sense of Theorem 5.5, where $\eta$ is the Lebesgue measure in $\mathbb{R}^2$.

PROPOSITION 5.8. *Let $\Lambda$ be the 2-graph as above. Let $D_v = [0, 1] \times [-1, 1]$, $R_{e_1} = R_{f_1} = [0, 1] \times [0, 1]$, and $R_{e_2} = R_{f_2} = [0, 1] \times [-1, 0]$. Define the maps $f_{e_1}, f_{f_1} : D_v \to R_{e_1}$ by $f_{e_1}((x, y)) = (x^2, \frac{y}{2} + \frac{1}{2})$ and $f_{f_1}((x, y)) = (\sqrt{x}, \frac{y}{2} + \frac{1}{2})$, and the maps $f_{e_2}, f_{f_2} : D_v \to R_{f_2}$ by $f_{e_2}((x, y)) = (x^2, \frac{y}{2} - \frac{1}{2})$ and $f_{f_2}((x, y)) = (\sqrt{x}, \frac{y}{2} - \frac{1}{2})$. Then, the representation of $C^*(\Lambda)$ arising from this $\Lambda$-branching system is faithful.*

*Proof.* It is easy to see that all the conditions of Definition 3.1 are satisfied. Let $\pi : C^*(\Lambda) \to B(L^2([0, 1] \times [-1, 1], \eta))$be the *-homomorphism induced by this $\Lambda$-branching system. Note that each cycline pair is of the form $(e_i, f_i)$. So, to apply Theorem 5.5, it is enough to show that, for each finite set $\mathcal{F} \subseteq \mathbb{Z} \setminus \{0\}$, there exists a subset $E \subseteq f_{e_1}(D_v)$ with $\eta(E) \neq 0$ and $(f_{e_1} \circ f_{f_1}^{-1})^k(E) \cap E \overset{\eta-a.e.}{=} \emptyset$ for each $k \in \mathcal{F}$. Note that $(f_{e_1} \circ f_{f_1}^{-1})(x, y) = (x^4, y)$.

Let $E = [\frac{1}{4}, \frac{1}{2}] \times [0, 1]$. Then, $(f_{e_1} \circ f_{f_1}^{-1})^k(E) \cap E \overset{\eta-a.e.}{=} \emptyset$ for each $k \in \mathbb{Z} \setminus \{0\}$, and hence by Theorem 5.5, $\pi$ is faithful. $\square$

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