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ON MINIMAL ASYMPTOTIC g-ADIC BASES

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Abstract

Let $g \geq 2$ be a fixed integer. Let $\mathbb{N}$ denote the set of all nonnegative integers and let A be a subset of $\mathbb{N}$. Write $r_2(A, n) = \sharp\{(a_1, a_2) \in A^2 : a_1 + a_2 = n\}$. We construct a thin, strongly minimal, asymptotic g-adic basis A of order two such that the set of n with $r_2(A, n) = 2$ has density one.

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1. Introduction

Let $\mathbb{N}$ denote the set of all nonnegative integers and let A be a subset of $\mathbb{N}$. Write $A(x) = \sharp\{n \in A : n \leq x\}$. For $h \geq 2$, let

$$r_h(A, n) = \sharp\{(a_1, a_2, \ldots, a_h) \in A^h : a_1 + a_2 + \cdots + a_h = n\}.$$

Let W be a nonempty subset of $\mathbb{N}$. Denote by $\mathcal{F}^*(W)$ the set of all finite, nonempty subsets of W. For any integer $g \geq 2$, let $A_g(W)$ be the set of all numbers of the form $\sum_{f\in F} a_f g^f$, where $F \in \mathcal{F}^*(W)$ and $1 \leq a_f \leq g - 1$. The set A is called an asymptotic basis of order h if $r_h(A, n) \geq 1$ for all sufficiently large integers n. In particular, A is a basis of order h if $r_h(A, n) \geq 1$ for all $n \geq 0$. An asymptotic basis A of order h is minimal if no proper subset of A is an asymptotic basis of order h. This means that, for any $a \in A$, the set $E_a = hA \backslash h(A \backslash \{a\})$ is infinite. An asymptotic basis A of order h is called *strongly minimal* if, for every $a \in A$, there exists a constant $c = c(a) > 0$ such that $E_a(x) > c(A(x))^{h-1}$ for all x sufficiently large. An asymptotic basis A of order h is called *thin* if there is a constant $c > 0$ such that $A(x) < cx^{1/h}$ for all x sufficiently large.

In 1955, Stöhr [10] introduced the concept of minimal asymptotic bases. In 1956, Härtter [4] proved that minimal asymptotic bases of order h exist for all $h \geq 2$. Nathanson [7] constructed a minimal asymptotic basis of order two and an asymptotic basis of order two no subset of which is minimal. In 2011, Chen and Chen [2] resolved some questions on minimal asymptotic bases posed by Nathanson [8]. For related problems concerning minimal asymptotic bases, see [5, 6, 9, 10]. In 2012, Chen [1]

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proved that there is a basis A of order two such that the set of n with $r_2(A, n) = 2$ has density one. In 2013, Yang [12] extended Chen's theorem to a basis of order h. Recently, the second author of this paper [11] developed Yang's method of proof to establish a more general result.

To our surprise, the structure of the minimal asymptotic basis given by Nathanson [7] is similar to the structure of the basis given by Chen [1]. Motivated by this observation, we obtain the following result.

Theorem 1.1. *For $i = 0, 1$, let $W_i = \{n \in \mathbb{N} \mid n \equiv i \pmod 2\}$. Then $A_g = A_g(W_0) \cup A_g(W_1)$ is a thin, strongly minimal, asymptotic g-adic basis of order two and the set of n with $r_2(A_g, n) = 2$ has density one.*

Remark 1.2. Using [6, Lemma 2] and the same idea as in the proof of [2, Theorem 4], we can extend [2, Theorem 4] to all $g \geq 2$ as follows. Let $h \geq 2$ and let t be the least integer with $t > \max\{1, \log h/ \log g\}$. Let $\mathbb{N} = W_0 \cup \cdots \cup W_{h-1}$ be a partition such that each set W_i is infinite and contains t consecutive integers for $i = 0, 1, \ldots, h-1$. Then $A_g = A_g(W_0) \cup \cdots \cup A_g(W_{h-1})$ is a minimal asymptotic g-adic basis of order h.

2. Proofs

Lemma 2.1 [6, Lemma 1]. *Let $g \geq 2$ be any integer.*

(a) *If W_1 and W_2 are disjoint subsets of $\mathbb{N}$, then $A_g(W_1) \cap A_g(W_2) = \emptyset$.*
(b) *If $W \subseteq \mathbb{N}$ and $W(x) = \theta x + O(1)$ for some $\theta \in (0, 1]$, then there exist positive constants c_1 and c_2 such that $c_1 x^{\theta} < A_g(W)(x) < c_2 x^{\theta}$ for all x sufficiently large.*
(c) *Let $\mathbb{N} = W_0 \cup \cdots \cup W_{h-1}$, where $W_i \neq \emptyset$ for $i = 0, 1, \ldots, h-1$. Then $A_g = A_g(W_0) \cup \cdots \cup A_g(W_{h-1})$ is an asymptotic basis of order h.*

Lemma 2.2 [3, Theorem 143]. *Almost all positive integers, when expressed in any scale, contain a given possible sequence of digits.*

Proof of Theorem 1.1. We shall show that the set A_g satisfies:

(i) A_g is a thin asymptotic basis of order two;
(ii) the set of n with $r_2(A_g, n) = 2$ has density one;
(iii) A_g is a minimal asymptotic basis of order two;
(iv) A_g is strongly minimal.

Proof of (i). By Lemma 2.1(c), for a fixed $g \geq 2$, the set A_g is an asymptotic basis of order two. Since $W_i(x) = \frac{1}{2}x + O(1)$ for $i = 0, 1$, Lemma 2.1(b) implies that there is a constant $c > 0$ such that $A_g(W_i)(x) < cx^{1/2}$ for all i and all x sufficiently large. Thus, $A_g(x) < 2cx^{1/2}$ for all x sufficiently large and A_g is a thin asymptotic basis of order two.

Proof of (ii). Define

$$U = \{n \in \mathbb{N} : n \text{ expressed in the scale } g \text{ contains three consecutive digits } g-1\}.$$

By Lemma 2.2, the set U has density one. We show that $r_2(A_g, n) = 2$ for all $n \in U$.

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For any nonnegative integers m and t, we write $m = \sum_{i\in X} \alpha_i g^i$, where α_i are integers with $0 \le \alpha_i \le g-1$ and X is a set of nonnegative integers, and define

$$T(m,t) = \sum_{i\in X\cap[0,t]} \alpha_i g^i.$$

Let $n = \sum_{i\in I} \beta_i g^i \in U$, where β_i are integers with $0 \le \beta_i \le g-1$, and let $n = a_1 + a_2$, where $a_1, a_2 \in A_g$. Then clearly

$$T(n,t) \le T(a_1,t) + T(a_2,t) \tag{2.1}$$

for all integers $t \ge 0$.

Suppose that $a_s \in A_g(W_1)$, $s = 1,2$. By the definition of U, there exists a positive integer i_0 such that $\beta_{2i_0-1} = \beta_{2i_0} = g-1$. By (2.1),

$$T(a_1, 2i_0) + T(a_2, 2i_0) \ge T(n, 2i_0) \ge (g-1)(g^{2i_0-1} + g^{2i_0}).$$

On the other hand, since $g \ge 2$ and $a_s \in A_g(W_1)$, $s = 1,2$,

$$T(a_1, 2i_0) + T(a_2, 2i_0) \le 2(g-1)\sum_{h=0}^{i_0-1} g^{2h+1} < (g-1)(g^{2i_0-1} + g^{2i_0}),$$

which is a contradiction.

Suppose that $a_s \in A_g(W_0)$, $s = 1,2$. By the definition of U, there exists a positive integer i_0 such that $\beta_{2i_0} = \beta_{2i_0+1} = g-1$. By (2.1),

$$T(a_1, 2i_0+1) + T(a_2, 2i_0+1) \ge T(n, 2i_0+1) \ge (g-1)(g^{2i_0} + g^{2i_0+1}).$$

On the other hand, by $g \ge 2$ and $a_s \in A_g(W_0)$, $s = 1,2$,

$$T(a_1, 2i_0+1) + T(a_2, 2i_0+1) \le 2(g-1)\sum_{h=0}^{i_0} g^{2h} < (g-1)(g^{2i_0} + g^{2i_0+1}),$$

which is a contradiction.

Thus, for any j with $0 \le j \le 1$, there exists an integer s_j with $1 \le s_j \le 2$ such that $a_{s_j} \in A_g(W_j)$. It is clear that s_0, s_1 are distinct. Therefore, by the uniqueness of the representation in the scale g and the definition of A_g, we have $r_2(A_g, n) = 2$.

Proof of (iii). We must show that for each $b \in A_g$, there are infinitely many numbers $m = b + b' = b' + b$ with no other representation as the sum of two elements of A_g.

Fix an integer $i \in \{0,1\}$ and suppose that $b \in A_g(W_i)$. Then

$$b = a_n g^{2n+i} + \sum_{s\in S} a_s g^{2s+i},$$

where S is a finite, possibly empty, set of integers greater than n, $1 \le a_n \le g-1$ and $1 \le a_s \le g-1$ for all $s \in S$.

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For any finite set T of integers greater than n, let

$$m = a_0 g^i + \sum_{s\in S} a_s g^{2s+i} + (g-1)g^{1-i} + \sum_{t\in T} b_t g^{2t+1-i} \quad \text{if } n = 0, \tag{2.2}$$

$$\begin{aligned} m = a_n g^{2n+i} &+ \sum_{s\in S} a_s g^{2s+i} + (g-1)(g^{(1-i)(2n-1-i)} + g^{2n+1-i}) \\ &+ \sum_{t\in T} b_t g^{2t+1-i} \quad \text{if } n > 0, \end{aligned} \tag{2.3}$$

where $1 \le b_t \le g-1$ for all $t \in T$.

By the uniqueness of the g-adic representation of m, no other partition of m as the sum of an element of $A_g(W_0)$ and an element of $A_g(W_1)$ is possible. Now we show that $m \notin 2A_g(W_i)$ and $m \notin 2A_g(W_{1-i})$.

Suppose that $m \in 2A_g(W_i)$. Then there exist $m_1, m_2 \in A_g(W_i)$ such that $m = m_1 + m_2$. Let

$$m_j = \sum_{k\in K} c_k^{(j)} g^{2k+i}, \quad j = 1, 2, \tag{2.4}$$

where K is a set of nonnegative integers, $1 \le c_k^{(j)} \le g-1$ for all $k \in K$ and $c_k^{(j)} = 0$ for all $k \notin K$.

Case 1: $i = 1$. By (2.2) and (2.4), we have $m \equiv g-1 \pmod g$ and $m_1 \equiv m_2 \equiv 0 \pmod g$, which is impossible.

Case 2: $i = 0$. If $n = 0$, then, by (2.2) and (2.4), we have $m \equiv a_0 + (g-1)g \pmod{g^2}$ and $m_1 + m_2 \equiv c_0^{(1)} + c_0^{(2)} \pmod{g^2}$. But

$$0 \le c_0^{(1)} + c_0^{(2)} \le 2(g-1) < a_0 + g(g-1) < g^2,$$

which is a contradiction. If $n > 0$, then, by (2.3) and (2.4),

$$m \equiv a_n g^{2n} + (g-1)g^{2n-1} + (g-1)g^{2n+1} \pmod{g^{2n+2}}$$

and

$$m_1 + m_2 \equiv \sum_{k=0}^{n} (c_k^{(1)} + c_k^{(2)}) g^{2k} \pmod{g^{2n+2}}.$$

Again,

$$0 \le \sum_{k=0}^{n} (c_k^{(1)} + c_k^{(2)}) g^{2k} \le (g-1) \sum_{k=0}^{n} g^{2k+1} < a_n g^{2n} + (g-1)g^{2n-1} + (g-1)g^{2n+1} < g^{2n+2}$$

is a contradiction.

Suppose that $m \in 2A_g(W_{1-i})$. Then there exist $m_1', m_2' \in A_g(W_{1-i})$ such that $m = m_1' + m_2'$. Let

$$m_j' = \sum_{h\in H} d_h^{(j)} g^{2h+1-i}, \quad j = 1, 2, \tag{2.5}$$

where H is a set of nonnegative integers, $1 \le d_h^{(j)} \le g-1$ for all $h \in H$ and $d_h^{(j)} = 0$ for all $h \notin H$.

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Case 1: $i = 1$. If $n = 0$, then, by (2.2) and (2.5), we have $m \equiv a_0 g + g - 1 \pmod{g^2}$ and $m'_1 + m'_2 \equiv d_0^{(1)} + d_0^{(2)} \pmod{g^2}$. Thus,

$$0 \le d_0^{(1)} + d_0^{(2)} < a_0 g + g - 1 < g^2,$$

which is a contradiction. If $n > 0$, then, by (2.3) and (2.5),

$$m \equiv a_n g^{2n+1} + (g-1) + (g-1)g^{2n} \pmod{g^{2n+2}}$$

and

$$m'_1 + m'_2 \equiv \sum_{h=0}^{n} (d_h^{(1)} + d_h^{(2)}) g^{2h} \pmod{g^{2n+2}}$$

and again

$$0 \le \sum_{h=0}^{n} (d_h^{(1)} + d_h^{(2)}) g^{2h} < 2g^{2n+1} - g^{2n} < a_n g^{2n+1} + (g-1) + (g-1)g^{2n} < g^{2n+2}$$

is a contradiction.

Case 2: $i = 0$. If $n = 0$, then, by (2.2) and (2.5), we have $m \equiv a_0 \pmod{g}$ and $m'_1 \equiv m'_2 \equiv 0 \pmod{g}$, which is a contradiction. If $n > 0$, then, by (2.3) and (2.5),

$$m \equiv a_n g^{2n} + (g-1)g^{2n-1} \pmod{g^{2n+1}}$$

and

$$m'_1 + m'_2 \equiv \sum_{h=0}^{n-1} (d_h^{(1)} + d_h^{(2)}) g^{2h+1} \pmod{g^{2n+1}}.$$

But then

$$0 \le \sum_{h=0}^{n-1} (d_h^{(1)} + d_h^{(2)}) g^{2h+1} < 2g^{2n} - g^{2n-1} < a_n g^{2n} + (g-1)g^{2n-1} < g^{2n+1},$$

which is a contradiction.

Proof of (iv). Since A_g is thin, it suffices to prove that there is a constant $c = c(b) > 0$ such that $E_b(x) > cx^{1/2}$ for all x sufficiently large. Choose an integer v such that $v > n$ and $v > s$ for all $s \in S$. Let $x > g^{2(v+1)}$. Define $w \ge v$ by $g^{2(w+1)} \le x < g^{2(w+2)}$. Let T be any subset of $\{n+1, n+2, \dots, w\}$. By (2.2) and (2.3), we know that there are

$$\sum_{i=0}^{w-n} \binom{w-n}{i} (g-1)^i = g^{w-n}$$

choices of m. Moreover,

$$m \le (g-1) \sum_{i=0}^{2w+1} g^i = g^{2w+2} - 1 < x,$$

and so m is counted in $E_b(x)$. Therefore, $E_b(x) \ge g^{w-n} > cx^{1/2}$, where $c = g^{-(n+2)}$.

This completes the proof of Theorem 1.1. □

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