Third Meeting, 12th January 1906.

D. C. M'INTOSH, Esq., M.A., President, in the Chair.

On a Nine-Point Conic, etc.

By P. PINKERTON, M.A.

1. The Pole and Polar Theorem for the Circle may be proved as follows :—

Let P be any point, and C the centre of a circle.

Let AB be the diameter through P, and QQ′ any chord passing through P.

Let AQ, BQ′ meet in M; AQ′, BQ in O; MO, AB in N; and MO, QQ′ in P′.

Then, by the harmonic properties of a complete quadrilateral,

N is the harmonic conjugate of P with respect to A, B; and

P′ is the harmonic conjugate of P with respect to Q, Q′.

N is therefore a fixed point. And NP′ is perpendicular to AB, since O is clearly the orthocentre of triangle MAB. But P′ is *any* point on the polar of P; hence the polar of P is the perpendicular to the diameter of the circle which passes through P, drawn through the harmonic conjugate of P with respect to the extremities of that diameter. *

2. The proof does not hold for any conic, as O is the orthocentre of triangle MAB only when the conic is a circle. The proof, however, could be made general by using the following theorem, of which the theorem regarding the concurrence of the perpendiculars of a triangle is a particular case.

Theorem: If ABC is any triangle and if AD, BE, CF are lines drawn through the vertices such that AD, BC; BE, CA; CF, AB are parallel to pairs of conjugate diameters of a fixed conic, then AD, BE, CF are concurrent.

* This proof was given to me by R. VICKERS, one of my pupils.—P.P.

For, let BE and CF meet at O. Join AO.

Then $$\frac{\sin BAO}{\sin CAO} \cdot \frac{\sin CBO}{\sin ABO} \cdot \frac{\sin ACO}{\sin BCO} = -1;$$

$$\therefore \quad \frac{\sin(\alpha' - \gamma)}{\sin(\alpha' - \beta)} \cdot \frac{\sin(\beta' - \alpha)}{\sin(\beta' - \gamma)} \cdot \frac{\sin(\gamma' - \beta)}{\sin(\gamma' - \alpha)} = 1$$

where the lines BC, CA, AB, OA, OB, OC make, with the x-axis of a rectangular system of reference, angles α, β, γ, α', β', γ'.

Hence $$\frac{\tan\alpha' - \tan\gamma}{\tan\alpha' - \tan\beta} \cdot \frac{\tan\beta' - \tan\alpha}{\tan\beta' - \tan\gamma} \cdot \frac{\tan\gamma' - \tan\beta}{\tan\gamma' - \tan\alpha} = 1;$$

$$\therefore \quad (l' - n)(m' - l)(n' - m) = (l' - m)(m' - n)(n' - l)$$

where $$l = \tan\alpha, \quad l' = \tan\alpha', \text{ etc.}$$

$$\therefore \; ll'(m + m' - n - n') + (l + l')(nn' - mm') + (n + n')mm' - (m + m')nn' = 0.$$

But with the usual notation

$$a + h(m + m') + bmm' = 0$$

$$\text{and} \quad a + h(n + n') + bnn' = 0;$$

$$\therefore \quad m + m' - n - n' : nn' - mm' : (n + n')mm' - (m + m')nn' = b : h : a,$$

whence $$a + h(l + l') + bll' = 0,$$

i.e., AO and BC are parallel to a pair of conjugate diameters of the conic. This proves the theorem.

3. The Pole and Polar Theorem for a conic follows at once by the method of § 1. For if the circle is replaced by a conic, AQ′ and BQ′ are supplementary chords and therefore parallel to a pair of conjugate diameters of the conic; so also are AQ and BQ; therefore so also are NP′ and AB. Hence the theorem.

4. The Theorem of § 2 leads to a proposition in conics, corresponding to a fundamental property of the orthocentre of a triangle, viz.:

If ABC is a triangle whose vertices lie on a conic, whose centre is O, and if the concurrent lines AH, BH, CH be drawn so that AH, BC; BH, CA; CH, AB are parallel to pairs of conjugate diameters of the conic, then AH = 2OL, etc., where L, M, N are the middle points of BC, CA, AB.

Following the proof for the corresponding property of the orthocentre, draw the diameter COZ, and join BZ, AZ.

Then ZB, BC are parallel to conjugate diameters of the conic, since they are supplementary chords.

$$\therefore \text{ ZB is parallel to AH,}$$
$$\text{similarly ZA is parallel to BH;}$$
$$\therefore \text{ ZB} = \text{AH},$$
$$\text{and ZB} = 2\text{OL},$$
$$\therefore \text{ AH} = 2\text{OL}.$$

5. Hence the following, corresponding to the theorem of the nine-point circle.

Let AH, BH, CH meet BC, CA, AB in D, E, F; and let P, Q, R be the middle points of AH, BH, CH; then the nine points L, M, N, P, Q, R, D, E, F lie on a conic whose centre is the middle point of OH.

Take X the middle point of OH.

A conic is determined if we know its centre and three points on it. Consider the conic whose centre is X and which passes through L, M, N.

Now AH = 2OL in magnitude and direction by § 4,

$\therefore$ PH = OL in magnitude and direction,

$\therefore$ X is the middle point of LP,

$\therefore$ P lies on the conic; similarly for Q, R.

Again OL, being the diameter conjugate to BC, is parallel to HD; and the middle point of MN is the middle point of AL, therefore the join of the middle points of LD and MN is parallel to HD or OL and so bisects OH.

Remembering that MN is parallel to LD, we see that D lies on the conic. Similarly E, F lie on the conic. This completes the proof.