ON THE VECTOR SUM OF CONTINUA

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In this note we investigate certain properties of a set formed as the vector sum of continua. Our interest in this subject arose in connection with the preceding paper *Quasiconvex sets* where we use, but do not prove, item 3 below.

Let X be a normed real vector space of dimension ν with the ν vectors a_λ as a basis (λ representing a variable index ranging over the ν indices $1, \ldots, \nu$). The parallelepipedal lattice consisting of all integral linear combinations $a = \sum \alpha_\lambda a_\lambda$ of the basis vectors a_λ, the coefficients α_λ being integers, will be denoted by A.

Consider ν continua Q_λ in X such that the λth continuum Q_λ contains the origin 0 and the basis vector a_λ. Let $Q = \sum Q_\lambda$ be the vector sum of these ν continua Q_λ, that is, the set of all vector sums $q = \sum q_\lambda$ with $q_\lambda \subset Q_\lambda$. A simple example of such a set is the solid parallepiped $P = \sum P_\lambda$ where P_λ is the line segment joining 0 and a_λ.

We shall prove that any vector sum Q, formed as above described, possesses the following properties:

(1) Q is a continuum, (2) $X = A + Q$,

(3) Q has interior points, (4) $\mu(P) \leqslant \mu(Q)$,

where μ is a measure on the space X invariant under translation.

1. We are to show that Q is a continuum: compact and connected.

We first demonstrate that Q is compact. To this end let $q^\gamma = \sum q_\lambda{}^\gamma$ with $q_\lambda{}^\gamma \subset Q_\lambda$ be a sequence of points in Q, γ running through the sequence Γ of positive integers. It is required to find a point $q \subset Q$ with $q^\gamma \rightarrow q$ as γ runs through some subsequence of Γ. Consider the sequence of points $q_1{}^\gamma$ in Q_1 as γ runs through Γ. Since Q_1 is compact a point $q_1 \subset Q_1$ and a subsequence Γ_1 of Γ exists with $q_1{}^\gamma \rightarrow q_1$ as γ runs through Γ_1. Since Q_2 is compact a point $q_2 \subset Q_2$ and a subsequence Γ_2 of Γ_1 exists with $q_2{}^\gamma \rightarrow q_2$ as γ runs through Γ_2. Continuing this process recursively to the νth stage we obtain ν points $q_\lambda \subset Q_\lambda$ and ν sequences $\Gamma_1, \ldots, \Gamma_\nu$ each a subsequence of the preceding one such that $q_\lambda{}^\gamma \rightarrow q_\lambda$ as γ runs through Γ_λ and hence also as γ runs through Γ_ν. Putting $q = \sum q_\lambda$ we have $q \subset Q$, since $q_\lambda \subset Q_\lambda$, and

$$q^\gamma = \sum q_\lambda{}^\gamma \rightarrow \sum q_\lambda = q$$

as γ runs through Γ_ν. This proves Q compact.

We next demonstrate that Q is connected by constructing for each point

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$q \subset Q$ a connected set C containing the origin 0 and the point q. Let $q = \sum q_\lambda$ with $q_\lambda \subset Q_\lambda$; and define the points c_λ and sets C_λ as follows:

$$c_0 = 0, \qquad c_\lambda = c_{\lambda-1} + q_\lambda, \qquad C_\lambda = c_{\lambda-1} + Q_\lambda, \qquad C = \cup C_\lambda.$$

We note that C_λ, being a translation of Q_λ, is connected and, since Q_λ contains 0 and q_λ, contains $c_{\lambda-1}$ and c_λ. Consequently the set C is connected and contains $c_0 = 0$ and $c_\nu = q$. This proves Q connected.

2. Since the vectors a_λ constitute a basis for the ν dimensional vector space X, every vector x in the space may be expressed uniquely in the form

$$x = \sum \xi_\lambda(x) a_\lambda,$$

the coordinate functionals $\xi_\lambda(x)$ being linear. Thus the function

$$\omega(x) = \max |\xi_\lambda(x)|$$

has the properties of a norm. Now any two norm topologies on X are topologically equivalent, so we shall for convenience assume that ω is the norm on X and write $\omega(x) = |x|$. Observe that with this norm we have $|p| \leqslant 1$ for all $p \subset P$.

Every real number ξ can be uniquely partitioned into an integer α and a remainder θ with $0 \leqslant \theta < 1$ so that $\xi = \alpha + \theta$. Let this partition for the coordinate functional $\xi_\lambda(x)$ be

$$\xi_\lambda(x) = \alpha_\lambda(x) + \theta_\lambda(x)$$

and define

$$a(x) = \sum \alpha_\lambda(x) a_\lambda, \qquad p(x) = \sum \theta_\lambda(x) a_\lambda .$$

Then $a(x) \subset A$, $p(x) \subset P$, and

$$x = a(x) + p(x),$$

(which shows $X = A + P$).

Fix the positive number $\epsilon > 0$. Since Q_λ is connected, any two of its points may be connected in it by an ϵ chain. Thus there exists an ϵ chain C_λ^ϵ of points of Q_λ running from 0 to a_λ. Consider the path Q_λ^ϵ obtained by drawing the line segments joining consecutive points along the chain C_λ^ϵ. This path begins at 0 and ends at a_λ, and is at all of its points within ϵ distance of some point of Q_λ, namely a point of C_λ^ϵ. Clearly a continuous mapping q_λ^ϵ can be constructed which maps the closed real unit interval $I: 0 \leqslant \theta \leqslant 1$ onto the path Q_λ^ϵ so that $q_\lambda^\epsilon(0) = 0$ and $q_\lambda^\epsilon(1) = a_\lambda$. Let $\xi = \alpha + \theta$ be the partition of the real number ξ into its integral part α and remainder θ; and define

$$f_\lambda^\epsilon(\xi) = f_\lambda^\epsilon(\alpha + \theta) = \alpha a_\lambda + q_\lambda^\epsilon(\theta).$$

We note that the mapping f_λ^ϵ is continuous for all real ξ. This is obvious for non-integral ξ and also for integral ξ approached from above, since q_λ^ϵ is continuous on I and the integral part of ξ becomes constant. Furthermore for integral $\xi = \alpha + 1$ (not a partition) f_λ^ϵ is also continuous from below; for, as $\theta \to 1$ from below, we have

$$f_\lambda^\epsilon(\alpha + \theta) = \alpha a_\lambda + q_\lambda^\epsilon(\theta) \to \alpha a_\lambda + q_\lambda^\epsilon(1) = (\alpha + 1)a_\lambda = f_\lambda^\epsilon(\alpha + 1).$$

Thus f_λ^ϵ is continuous.

Now define the mapping f^ϵ of the space X into itself as follows:

$$f^\epsilon(x) = \sum f_\lambda^\epsilon(\xi_\lambda(x)) = a(x) + q^\epsilon(x)$$

where

$$q^\epsilon(x) = \sum q_\lambda^\epsilon(\theta_\lambda(x)).$$

Since f_λ^ϵ and ξ_λ are continuous mappings, f^ϵ is also continuous. Every point of Q_λ^ϵ is within ϵ distance of some point of Q_λ, so we see that $q^\epsilon(x)$ is within $\nu\epsilon$ distance of some point of Q. Now

$$x - f^\epsilon(x) = p(x) - q^\epsilon(x),$$

so

$$|x - f^\epsilon(x)| \leqslant 1 + \nu\epsilon + \delta$$

where $\delta = \max |q|$ $(q \subset Q)$. Thus $x - f^\epsilon(x)$ is uniformly bounded for all x in X.

Let S be the open unit sphere: $|s| < 1$. The mapping h defined by the formula

$$s = h(x) = \frac{x}{1 + |x|}$$

is a homeomorphism contracting X onto S whose inverse mapping is

$$x = h^{-1}(s) = \frac{s}{1 - |s|}, \quad |s| < 1.$$

With several applications of the triangle inequality it may be shown that the homeomorphism h satisfies the following norm inequality which we shall call the h-inequality:

$$|h(x) - h(y)| \leqslant |x - y| \,.\, (1 - |h(x)| \,.\, |h(y)|).$$

Consider now the mapping g^ϵ of the closed sphere $\bar{S} : |s| \leqslant 1$ into itself defined by

$$g^\epsilon(s) = \begin{cases} hf^\epsilon h^{-1}(s), & |s| < 1, \\ s, & |s| = 1. \end{cases}$$

This mapping g^ϵ is, as we shall demonstrate, continuous on $\bar{S}$. It is clearly continuous when confined either to S or to its boundary. Thus it remains to show that $g^\epsilon(s^\gamma) \to g^\epsilon(s)$ as $s^\gamma \to s$, the points s^γ being in S and the point s on the boundary of S. Let

$$x^\gamma = h^{-1}(s^\gamma),$$

$$y^\gamma = f^\epsilon(x^\gamma),$$

$$d^\gamma = s^\gamma - g^\epsilon(s^\gamma) = h(x^\gamma) - h(y^\gamma).$$

Since, as we have already noted, the vectors $x^\gamma - f^\epsilon(x^\gamma) = x^\gamma - y^\gamma$ are uniformly bounded independently of γ, and $s^\gamma \to s$ (so that $|s^\gamma| \to |s| = 1$), we have $|x^\gamma| \to \infty$ and $|y^\gamma| \to \infty$. Therefore $|h(x^\gamma)| \to 1$ and $|h(y^\gamma)| \to 1$, so it follows from the h-inequality that $|d^\gamma| \to 0$ and hence that $d^\gamma \to 0$. Consequently

$$g^\epsilon(s^\gamma) - g^\epsilon(s) = s^\gamma - s - d^\gamma \to 0,$$

which proves g^ϵ to be a continuous mapping of the closed sphere $\bar{S}$ into itself leaving the boundary fixed. According to a variant form of the Brouwer fixed point theorem for the closed sphere $\bar{S}$ the mapping g^ϵ is then an onto mapping: $g^\epsilon(\bar{S}) = \bar{S}$, whence $g^\epsilon(S) = S$. Therefore f^ϵ is also an onto mapping:

$$f^\epsilon(X) = f^\epsilon h^{-1}(S) = h^{-1}g^\epsilon(S) = h^{-1}(S) = X.$$

Our present task is to show that $X = A + Q$; that is, for any point $x \subset X$ the equation $x = a + q$ is solvable with $a \subset A$ and $q \subset Q$. Choose a sequence of positive numbers ϵ^γ such that $\epsilon^\gamma \to 0$ as γ runs through the sequence Γ of positive integers. Since $f^{\epsilon^\gamma}(X) = X$ there exists a point x^γ such that

$$x = f^{\epsilon^\gamma}(x^\gamma) = a(x^\gamma) + q^{\epsilon^\gamma}(x^\gamma).$$

Put $a(x^\gamma) = a^\gamma$; and, since $q^{\epsilon^\gamma}(x^\gamma)$ is within $\nu\epsilon^\gamma$ distance of Q, replace it by $q^\gamma + e^\gamma$, where $q^\gamma \subset Q$ and $|e^\gamma| < \nu\epsilon^\gamma$. Thus

$$x = a^\gamma + q^\gamma + e^\gamma.$$

Since the q^γ and e^γ are bounded so also are the a^γ. But any bounded subset of A is finite so a^γ is constant, say a, for infinitely many integers γ, say for the sequence Γ_a. Q being compact, a point $q \subset Q$ and a subsequence Γ_{aq} of Γ_a exist with $q^\gamma \to q$ as γ runs through Γ_{aq}. Therefore

$$x = a^\gamma + q^\gamma + e^\gamma \to a + q$$

as γ runs through Γ_{aq}, since $a^\gamma = a$, $q^\gamma \to q$, and $e^\gamma \to 0$. This proves $x = a + q$.

3. We have shown that $X = A + Q$. Thus the space X is the union as a ranges over the countable set A of the translates $a + Q$ of Q. Since the entire space X is of second category, at least one of these translates of Q, and hence Q itself, is somewhere dense. Therefore the set Q, being closed, contains interior points.

4. The set Q is closed and hence measurable, so every translate $a + Q$ of Q is measurable, and has the measure $\mu(Q)$; similarly every translate $a + P$ of P is measurable and has measure $\mu(P)$.

Define A^β for each integer $\beta \geqslant 0$ to be that subset of A consisting of the $(2\beta + 1)^\nu$ integral linear combinations $a = \sum \alpha_\lambda a_\lambda$ of the basis vectors a_λ, the coefficients α_λ being integers such that $|\alpha_\lambda| \leqslant \beta$. Observe that by our selection of norm we have $|a| \leqslant \beta$ for every $a \subset A^\beta$. Let $Q^\beta = A^\beta + Q$ and $P^\beta = A^\beta + P$.

The set Q^β is the union as a ranges over the set A^β of the $(2\beta + 1)^\nu$ measurable sets $a + Q$ each having measure $\mu(Q)$. Therefore Q^β is measurable and

$$\mu(Q^\beta) \leqslant (2\beta + 1)^\nu \mu(Q).$$

Every plane, being closed, is measurable. Suppose that every plane has measure zero. The intersection of any two translates of P by distinct vectors of A being a planar set (possibly null) then has measure zero. Two such sets may be called μ-disjoint. Consequently P^β is the union as a ranges over A^β of the $(2\beta + 1)^\nu$ measurable pairwise μ-disjoint sets $a + P$ each having measure $\mu(P)$; so P is measurable and has measure

$$\mu(P^\beta) = (2\beta + 1)^\nu \mu(P).$$

We now show that $P^\beta \subset Q^{\beta+\gamma}$, where γ is any fixed integer $\geqslant \delta + 1$ and $\delta = \max |q|$ $(q \subset Q)$. Suppose, to the contrary, that some point x exists with $x \subset P^\beta$ and $x \not\subset Q^{\beta+\gamma}$. Since $x \subset P^\beta$ we have $x = a_p + p$ where $a_p \subset A^\beta$ and $p \subset P$, whence

$$|x| = |a_p + p| \leqslant |a_p| + |p| \leqslant \beta + 1.$$

Now $X = A + Q$ so $x = a_q + q$ where $a_q \subset A$ and $q \subset Q$. However $a_q \not\subset A^{\beta+\gamma}$ since $x \not\subset Q^{\beta+\gamma}$, so

$$|x| = |a_q + q| \geqslant |a_q| - |q| > \beta + \gamma - \delta \geqslant \beta + 1.$$

in contradiction to the preceding inequality. This contradiction proves $P^\beta \subset Q^{\beta+\gamma}$. Therefore

$$(2\beta + 1)^\nu \mu(P) = \mu(P^\beta) \leqslant \mu(Q^{\beta+\gamma}) \leqslant (2\beta + 2\gamma + 1)^\nu \mu(Q).$$

Dividing this inequality through by $(2\beta + 1)^\nu$ and letting $\beta \to \infty$ we obtain the desired inequality $\mu(P) \leqslant \mu(Q)$.

If, finally, some plane has positive measure, then it is possible by suitable translation to insert into any sphere infinitely many disjoint parallel planar portions all having the same positive measure, so that every set with interior points, in particular P and Q, has infinite measure.

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