Compositio Math. 142 (2006) 798–810
doi:10.1112/S0010437X06002016

Conformally flat manifolds with nonnegative Ricci curvature

Gilles Carron and Marc Herzlich

Abstract

We show that complete conformally flat manifolds of dimension $n \geqslant 3$ with nonnegative Ricci curvature enjoy nice rigidity properties: they are either flat, or locally isometric to a product of a sphere and a line; or are globally conformally equivalent to $\mathbb{R}^n$ or to a spherical spaceform $\mathbb{S}^n/\Gamma$. This extends previous results due to Cheng, Noronha, Chen, Zhu and Zhu.

Introduction

In this note, we study complete conformally flat manifolds with nonnegative Ricci curvature. It is well known in dimension 2 that the sphere, the plane and their quotients are the only surfaces that can be endowed with a metric of nonnegative curvature. As Riemannian surfaces are always conformally flat, it seems natural to look at higher-dimensional analogues of this fact. Schoen and Yau showed in [SY88] that conformal flatness together with nonnegative scalar curvature (or any variant of it involving the Yamabe constant) still allows much flexibility. In contrast, if one concentrates on stronger curvature conditions such as Ricci curvature bounds, one might expect that they put some quite strong restrictions on the manifold.

For instance, in the case where the manifold is closed, a characterization of the spaceforms and the quotients of $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ has been obtained by Noronha [Nor93]. If one adds extra assumptions such as constant positive scalar curvature, further restrictions can moreover be obtained; see, for instance, Cheng [Che01] for results and references.

More generally, and without any compactness assumption, Zhu proved in [Zhu94] that the *universal covering* of a complete conformally flat manifold of nonnegative Ricci curvature either is conformally equivalent to $\mathbb{S}^n$ or $\mathbb{R}^n$, or is isometric to $\mathbb{R} \times \mathbb{S}^{n-1}$. In the same vein, Chen and Zhu proved in [CZ02] that the only complete non-compact conformally flat manifolds with nonnegative Ricci curvature and fast curvature decay at infinity are the complete flat manifolds. One motivation for this result is that it stands as an analogue on real manifolds of well-known rigidity and gap phenomena on Kähler manifolds with nonnegative holomorphic bisectional curvature [CZ03, MSY81, Ni98, Ni04, NST01, NT03].

The goal of this paper is to complete the above results by exhibiting a full classification of the possible geometries of complete conformally flat manifolds with nonnegative Ricci curvature. As we shall see, quite a bit more can be said in this rather general setting on the topology of those manifolds, and also on their geometry. More precisely, we prove the following.

Received 1 November 2004, accepted in final form 16 October 2005.
2000 Mathematics Subject Classification 53C21 (primary), 53C24, 53C15 (secondary).
Keywords: conformally flat manifold, nonnegative Ricci curvature.
Both authors are supported in part by an ACI program of the French Ministry of Research, and the EDGE Research Training Network HPRN-CT-2000-00101 of the European Union.

Theorem A. *Let (M, g) be a complete conformally flat manifold of dimension $n \geqslant 3$ with nonnegative Ricci curvature. Then, exactly one of the following holds:*

(1) *M is* globally conformally equivalent *to $\mathbb{R}^n$ with a conformal* non-flat *metric with nonnegative Ricci curvature;*

(2) *M is* globally conformally equivalent *to a spaceform of positive curvature, endowed with a conformal metric with nonnegative Ricci curvature;*

(3) *M is* locally isometric *to the cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$;*

(4) *M is* isometric *to a complete flat manifold.*

Obviously, the last three cases do appear. In particular, as the round metric on spherical spaceforms has positive curvature, any small conformal deformation will keep nonnegative Ricci curvature as well, hence the second class is rather large. The first case also occurs: Zhu [Zhu94, p. 196] has found explicit examples of non-flat globally conformally flat metrics with nonnegative Ricci curvature on $\mathbb{R}^n$, which we very quickly review in § 1 below.

The most interesting part of our theorem lies of course in the dichotomy between the first two cases, where deformations are permitted, and the last two cases, where some strong rigidity is obtained. The philosophy of our result is then: a complete conformally flat manifold with nonnegative Ricci curvature is either (globally) *topologically simple* (diffeomorphic to a vector space or a quotient of a sphere), or it is (locally) *metrically rigid*. Hence, only very general assumptions on the conformally flat manifold are enough to yield strong constraints either on its topology or on its geometry. If one compares this with Zhu's classification [Zhu94] of the possible universal coverings, one sees that in the non-compact case (i.e. when quotients of the sphere are excluded), the presence of a nontrivial fundamental group implies metric rigidity.

The vector space $\mathbb{R}^n$ admits non-flat complete conformally flat metrics with nonnegative Ricci curvature. As already alluded to, Chen and Zhu [CZ02] showed that, in the case where the metric satisfies fast curvature decay assumptions at infinity, the metric is flat. It is well known under very general circumstances that fast curvature decay at infinity implies strong constraints on the topology at infinity, but our work implies further that, in the conformally flat case, either the metric is already flat or the topology is precisely that of $\mathbb{R}^n$. The examples we exhibit below of non-flat conformally flat metrics with nonnegative Ricci curvature on $\mathbb{R}^n$ support the idea that the decay rate chosen by Chen and Zhu [CZ02] is indeed optimal. It remains an interesting question to understand whether further constraints can be deduced on non-flat conformally flat metrics with nonnegative Ricci curvature on $\mathbb{R}^n$

1. Non-trivial conformally flat metrics on $\mathbb{R}^n$ with nonnegative Ricci curvature

In this short section, we recall the example of Zhu [Zhu94, p. 196], he exhibited examples of (rotationally symmetric) metrics on $\mathbb{R}^n$ that are *non-flat*, but *complete, conformally flat and with nonnegative Ricci curvature*. Hence this may exist, whereas the existence of any analogous metric on quotients is forbidden by our result. A by-product is the optimality of the curvature decay rate imposed by Chen and Zhu in [CZ02] to get a flat metric: the example lies exactly on the threshold where the result in [CZ02] does not hold anymore.

Zhu's example [Zhu94]

Let f be a real function. From Besse's book [Bes81, Formula 1.159, p. 59] the expression for the Ricci curvature of the metric $\mathrm{e}^{2f} g_0$ (g_0 the euclidean metric) is

$$\mathrm{Ric} = -(n-2)(Ddf - df \otimes df) + (\Delta f - (n-2)|df|^2) g_0 \tag{1.1}$$

https://doi.org/10.1112/S0010437X06002016 Published online by Cambridge University Press

which can be rewritten, in the case where $f = f(r)$ is a radial function and h is the unit round metric on $\mathbb{S}^{n-1}$, as

$$\begin{aligned}\mathrm{Ric} &= -(n-2)(f''dr^2 - (f')^2dr^2 + rf'h) + (-f'' - (n-1)r^{-1}f' - (n-2)(f')^2)(dr^2 + r^2h)\\ &= -(n-1)(f'' + r^{-1}f')dr^2 - (f'' + (2n-3)r^{-1}f' + (n-2)(f')^2)r^2h. \qquad (1.2)\end{aligned}$$

A short computation leads to the fact that if $\alpha \in]0, \frac{1}{2}[$ and $f(r) = -\alpha\log(1+r^2)$, then the Ricci tensor is then nonnegative and the metric is complete.

Optimality of the decay condition of Chen and Zhu [CZ02]

Let us analyse the examples above a little further. The same computation leads to an estimate of the Ricci curvature:

$$|\mathrm{Ric}| \leqslant 4\alpha(n-1)(n + (n-2)(1-\alpha))r^{-2+4\alpha}.$$

However, geodesic distance from the origin in this metric is $s \simeq r^{1-2\alpha}/(1-2\alpha)$ around infinity, hence

$$|\mathrm{Ric}| \leqslant O(s^{-2}).$$

In [CZ02], Chen and Zhu proved that a complete non-compact conformally flat manifold with nonnegative Ricci curvature is necessarily flat if the scalar curvature is bounded, and if, for geodesic balls centered at some fixed origin,

$$\frac{s^2}{\mathrm{vol}\, B(s)} \int_{B(s)} |\mathrm{Ric}| \longrightarrow 0 \quad \text{as } s \text{ goes to infinity.}$$

In particular, this is satisfied if the following simplest conditions hold true:

$$\mathrm{vol}\, B(s) \geqslant Cs^n, \quad s^{2+\delta}|\mathrm{Ric}| \longrightarrow 0 \quad \text{as } s \text{ goes to infinity, for some } \delta > 0.$$

As a result, we see that the assumptions in Chen and Zhu [CZ02] are close to optimal on manifolds conformally equivalent to $\mathbb{R}^n$.

2. The proof

The proof of Theorem A is divided into two parts: the first is a classification of the possible holonomy coverings of a complete (compact or not) conformally flat manifold (M, g) with nonnegative Ricci curvature. This is basically Zhu's result [Zhu94], but for the sake of completeness, and also because this is quite a short proof, we have provided below the arguments leading to the results we will precisely need. Our arguments are roughly similar to those of Noronha [Nor93] and Zhu [Zhu94], although they differ in some places. It then turns out that the classification of the holonomy coverings is a classification of the universal coverings, and it then remains to show that, in the presence of a fundamental group, one gets metric rigidity (unless we are in the spherical spaceform case).

The very first tool one needs to achieve the holonomy covering classification is Schoen and Yau's analysis [SY88] of conformally flat manifolds with nonnegative scalar curvature. This applies in dimensions $n \geqslant 7$ without any further restrictions, and to manifolds of dimensions $3 \leqslant n \leqslant 6$ provided that a very general positive mass holds true for manifolds with one asymptotically flat end and an arbitrary number of other complete ends.[1]

This last result is considered as highly probable, but has unfortunately remained unpublished so far (see [SY88, § 4] and Appendix A for details on this subtle point). Luckily enough, we only need Schoen–Yau's analysis to hold true in our setting where extra conditions on the Ricci curvature

[1] Here the positive mass theorem has to be applied to the manifold obtained by a stereographic blow-up of the original manifold; see Appendix A.

https://doi.org/10.1112/S0010437X06002016 Published online by Cambridge University Press

are available. It turns out that a direct proof of the holonomy covering classification for conformally flat manifolds with nonnegative scalar curvature can be obtained when one moreover assumes that the volume growth for the geodesic balls is sub-euclidean in dimensions $3 \leqslant n \leqslant 6$. From the Bishop–Gromov comparison theorem, this extra condition holds when the Ricci curvature is nonnegative.

For sake of completeness, we have included a description of the elements of that proof in Appendix A. Note also that Christ and Lohkamp recently announced a proof of a general positive mass theorem with new techniques that might lead to the desired result [CL].

We now come back to our main proof: Schoen–Yau's analysis then implies that the holonomy covering of M is a domain Ω included in the complement $\Omega(\Gamma)$ of the limit set of the fundamental group $\Gamma = \pi_1(M)$ of M in the sphere $\mathbb{S}^n$.

We now look at Ω: it is endowed with a complete metric conformal to the round metric and, due to the nonnegativity of Ricci curvature, it has sub-euclidean volume growth:

$$\operatorname{vol} B(r) \leqslant Cr^n.$$

One can now apply [CH02, Proposition 2.2], which shows that the complement of Ω in the sphere is of n-capacity zero, hence of zero Hausdorff dimension. Moreover, manifolds with nonnegative Ricci curvature have a finite number of ends [Liu92] and, arguing as in [CH02, Proof of Theorem 2.1], we can conclude that Ω must be the complement of a finite number of points in the sphere $\mathbb{S}^n$. Three cases can now occur.

(1) First of all, Ω might be the whole sphere. Then M is compact, with a metric g conformal to the round metric and with nonnegative Ricci curvature. As already noticed, there are plenty of such metrics, such as, for example, C^2-small conformal deformations of the round metric.

(2) Assume now that Ω has at least two ends. As it has nonnegative Ricci curvature, the Cheeger–Gromoll theorem [CG71] implies that Ω splits as a Riemannian product $\mathbb{R} \times Y$. Remarks due to Lafontaine [Laf82] assert that a conformally flat manifold that is a Riemannian product with a line is necessarily a Riemannian product of a constant curvature manifold with the line. Hence, Y must be of constant curvature, and as Ω is simply connected, $Y = \mathbb{R}^{n-1}$ or $\mathbb{S}^{n-1}$ (hyperbolic space would lead to negative Ricci curvature). As Ω has at least two ends, we end up with $Y = \mathbb{S}^{n-1}$ as the sole possibility in this case.

(3) If Ω has only one end, it forces it to be diffeomorphic to $\mathbb{R}^n$, endowed with a metric conformal to the euclidean metric and with nonnegative Ricci curvature. Either M itself is diffeomorphic to $\mathbb{R}^n$ (and we have seen in the previous section that the metric is not necessarily flat) or M is different from its universal covering Ω. From now on, we will assume that we are in that last case, and we will show that g is flat.

If $\Omega(\Gamma) = \mathbb{S}^n$, it implies that Γ is a group of conformal diffeomorphisms of the sphere acting properly discontinuously without fixed points on the whole sphere. It is then a finite group, and it must fix the missing point $\Omega(\Gamma) - \Omega$. Any conformal map of finite order has at least two fixed points (if not they would be parabolic, and this would imply both $\Omega(\Gamma) = \Omega$ and infinite order) and Γ cannot act without fixed points on Ω, hence this case is excluded.

We are left with the case where $\Omega(\Gamma) = \Omega$ is diffeomorphic to $\mathbb{R}^n$ and Γ acts properly discontinuously without fixed points as conformal diffeomorphisms of the flat space. However, the only possibility is then that Γ is a subgroup of the group of Euclidean isometries of flat space.

Let us note now that if M itself is compact, then it is (isometrically) a compact flat manifold. Indeed, M is diffeomorphic to a flat manifold, and is endowed with a conformal metric g with nonnegative Ricci curvature. By the Bieberbach theorem (see [Wol67]) it is covered by a torus, and the Bochner technique [GHL04, 4.37] (or, alternatively, the Gromov–Lawson scalar curvature obstruction [GL83]) applies to show that g itself is flat.

https://doi.org/10.1112/S0010437X06002016 Published online by Cambridge University Press

We now consider the case where M is non-compact. By the Bieberbach theorem again, there exists a finite cover $\widehat{M}$ of M, which is conformally equivalent to a product of a flat torus and a flat space:

$$\widehat{M} = \mathbb{T}^k \times \mathbb{R}^{n-k}.$$

Moreover, M is obtained from $\widehat{M}$ by a finite group acting isometrically on the flat metric. To conclude at this point, one could try to look at an extension of Gromov–Lawson's obstruction to nonnegative scalar curvature for general non-compact Riemannian manifolds of type $\mathbb{T}^k \times \mathbb{R}^{n-k}$. However, a careful reading of [GL83, pp. 95–96] shows that it is useless to hope for such an extension unless $k = n-1$. We shall then need an *ad hoc* proof in the conformally flat case.

Let us now study the different possible values for k (with $1 \leqslant k \leqslant n-1$ as the case $k = n$ has already been treated above and the case $k = 0$ bears no restriction, as we know from § 1).

(1) If $k = n-1$, then $\widehat{M}$ has two ends, and Cheeger–Gromoll's theorem [CG71] applied again together with Lafontaine results [Laf82] shows that M is isometric to a flat spaceform.

(2) Suppose now that $1 \leqslant k \leqslant n-2$. Using the computations done above, the Ricci curvature of the metric $g = \phi^{-2} g_0$ (g_0 the euclidean metric on $\mathbb{T}^k \times \mathbb{R}^{n-k}$) is

$$\mathrm{Ric}^g = (n-2)\frac{Dd\phi}{\phi} - \left(\frac{\Delta\phi}{\phi} + (n-1)\frac{|d\phi|^2}{\phi^2}\right) g_0.$$

Suppose now that $\widehat{M} = \mathbb{T}^k \times \mathbb{R}^{n-k}$ for $0 < k < n-1$, and use polar coordinates on the $\mathbb{R}^{n-k}$-factor to get

$$\begin{aligned}\phi\,\mathrm{Ric}^g(\partial_r, \partial_r) &= (n-2)(\partial_r^2\phi) \\ &\quad - \left(-(\partial_r^2\phi) - \frac{n-k-1}{r}\partial_r\phi + (n-1)\frac{|d\phi|^2}{\phi} + \frac{1}{r^2}\Delta_{\mathbb{S}^{n-k-1}}\phi + \Delta_{\mathbb{T}^k}\phi\right)\end{aligned}$$

so that

$$0 \leqslant (n-1)(\partial_r^2\phi) + \frac{n-k-1}{r}\partial_r\phi - (n-1)\frac{(\partial_r\phi)^2}{\phi} - \frac{1}{r^2}\Delta_{\mathbb{S}^{n-k-1}}\phi - \Delta_{\mathbb{T}^k}\phi. \tag{2.1}$$

Similarly,

$$\begin{aligned}\phi\,\mathrm{tr}_{\mathbb{T}^k}\mathrm{Ric}^g &= -(n-2)\Delta_{\mathbb{T}^k}\phi \\ &\quad - k\left(-(\partial_r^2\phi) - \frac{n-k-1}{r}\partial_r\phi + (n-1)\frac{|d\phi|^2}{\phi} + \frac{1}{r^2}\Delta_{\mathbb{S}^{n-k-1}}\phi + \Delta_{\mathbb{T}^k}\phi\right),\end{aligned}$$

so that

$$0 \leqslant (\partial_r^2\phi) + \frac{n-k-1}{r}\partial_r\phi - (n-1)\frac{(\partial_r\phi)^2}{\phi} - \frac{1}{r^2}\Delta_{\mathbb{S}^{n-k-1}}\phi - \left(1 + \frac{n-2}{k}\right)\Delta_{\mathbb{T}^k}\phi. \tag{2.2}$$

We now integrate (2.1) and (2.2) on $\mathbb{S}^{n-k-1} \times \mathbb{T}^k$, and we let

$$\bar{\phi} = \int_{\mathbb{S}^{n-k-1}\times\mathbb{T}^k} \phi.$$

This yields

$$\begin{aligned}0 &\leqslant (n-1)\partial_r^2\bar{\phi} + \frac{n-k-1}{r}\partial_r\bar{\phi} - (n-1)\int_{\mathbb{S}^{n-k-1}\times\mathbb{T}^k}\frac{(\partial_r\phi)^2}{\phi}; \\ 0 &\leqslant \partial_r^2\bar{\phi} + \frac{n-k-1}{r}\partial_r\bar{\phi} - (n-1)\int_{\mathbb{S}^{n-k-1}\times\mathbb{T}^k}\frac{(\partial_r\phi)^2}{\phi}.\end{aligned} \tag{2.3}$$

https://doi.org/10.1112/S0010437X06002016 Published online by Cambridge University Press

By Hölder inequality,

$$\left(\partial_r\bar\phi\right)^2 = \left(\int_{\mathbb{S}^{n-k-1}\times\mathbb{T}^k}\partial_r\phi\right)^2 = \left(\int_{\mathbb{S}^{n-k-1}\times\mathbb{T}^k}\phi^{1/2}\phi^{-1/2}\partial_r\phi\right)^2$$
$$\leqslant \left(\int_{\mathbb{S}^{n-k-1}\times\mathbb{T}^k}\phi\right)\left(\int_{\mathbb{S}^{n-k-1}\times\mathbb{T}^k}\frac{(\partial_r\phi)^2}{\phi}\right) = \bar\phi\left(\int_{\mathbb{S}^{n-k-1}\times\mathbb{T}^k}\frac{(\partial_r\phi)^2}{\phi}\right).$$

Hence, our pair of equations are transformed into the following:

$$\begin{aligned} 0 &\leqslant (n-1)\partial_r^2\bar\phi + \frac{n-k-1}{r}\partial_r\bar\phi - (n-1)\frac{(\partial_r\bar\phi)^2}{\bar\phi};\\ 0 &\leqslant \partial_r^2\bar\phi + \frac{n-k-1}{r}\partial_r\bar\phi - (n-1)\frac{(\partial_r\bar\phi)^2}{\bar\phi}. \end{aligned} \tag{2.4}$$

Using this and the change of variables

$$r = \mathrm{e}^t, \quad \text{that is, } r\partial_r = \partial_t,\ r^2\partial_r^2 = \partial_t^2 - \partial_t,$$

and denoting $\psi(t) = \bar\phi(\mathrm{e}^t)$ for $t\in\mathbb{R}$, one gets

$$(n-1)\partial_t^2\psi - k\partial_t\psi - (n-1)\frac{(\partial_t\psi)^2}{\psi} \geqslant 0, \tag{2.5}$$

and

$$\partial_t^2\psi + (n-k-2)\partial_t\psi - (n-1)\frac{(\partial_t\psi)^2}{\psi} \geqslant 0. \tag{2.6}$$

From (2.5) one gets, if $k>0$,

$$\partial_t\psi \leqslant \frac{n-1}{k}\partial_t^2\psi - \frac{n-1}{k}\frac{(\partial_t\psi)^2}{\psi}. \tag{2.7}$$

Note now that $k\leqslant n-2$, so that injecting (2.7) into (2.6) leads to

$$\partial_t^2\psi - \left(1 + \frac{k(n-2)}{k+(n-k-2)(n-1)}\right)\frac{(\partial_t\psi)^2}{\psi} \geqslant 0. \tag{2.8}$$

Define now

$$\alpha = -\frac{k(n-2)}{k+(n-1)(n-k-2)} < 0, \tag{2.9}$$

so that

$$\begin{aligned} \partial_t^2(\psi^\alpha) &= \partial_t(\alpha\psi^{\alpha-1}\partial_t\psi)\\ &= \alpha(\psi^{\alpha-1}\partial_t^2\psi + (\alpha-1)\psi^{\alpha-2}(\partial_t\psi)^2)\\ &= \alpha\psi^{\alpha-1}\left(\partial_t^2\psi - (1-\alpha)\frac{(\partial_t\psi)^2}{\psi}\right). \end{aligned}$$

As $\alpha<0$ and $\psi>0$, this implies that ψ^α is a positive concave function on $\mathbb{R}$. However, constants are the only positive concave functions on $\mathbb{R}$ that do not change signs, so that the function $\bar\phi$ is necessarily a constant in r.

We now define

$$\Psi : x\in\mathbb{R}^{n-k} \longmapsto \int_{\{x\}\times\mathbb{T}^k}\phi.$$

https://doi.org/10.1112/S0010437X06002016 Published online by Cambridge University Press

Using x in $\mathbb{R}^{n-k}$ as the origin, the work done above shows that, for each $r > 0$,

$$\begin{aligned}\Psi(x) &= \lim_{s\to 0} \frac{1}{\operatorname{vol}\mathbb{S}^{n-k-1}}\bar{\phi}(s)\\ &= \frac{1}{\operatorname{vol}\mathbb{S}^{n-k-1}}\bar{\phi}(r)\\ &= \frac{1}{\operatorname{vol}\mathbb{S}^{n-k-1}}\int_{\{r\}\times\mathbb{S}^{n-k-1}\times\mathbb{T}^k}\phi\\ &= \frac{1}{\operatorname{vol} S_x(r)}\int_{S_x(r)}\Psi,\end{aligned}$$

or, equivalently, the function Ψ is harmonic on $\mathbb{R}^{n-k}$.

We are now in a position to end the proof. The condition for the scalar curvature to be nonnegative is

$$0 \leqslant \operatorname{Scal}^g = -2(n-1)\Delta\phi - n(n-1)\frac{|d\phi|^2}{\phi}, \tag{2.10}$$

so that

$$\Delta\phi = \Delta_{\mathbb{R}^{n-k}}\phi + \Delta_{\mathbb{T}^k}\phi \leqslant 0. \tag{2.11}$$

Integrating on $\mathbb{T}^k$, the first term in (2.11) vanishes as Ψ is harmonic, whereas the second vanishes by the definition of the Laplacian on $\mathbb{T}^k$. As a result, ϕ itself must be harmonic and the sign of the scalar curvature (2.10) shows that $d\phi$ must vanish. Hence, g must be flat.

Remark 2.1. Note that the end of the proof given above works exactly when we expect the result to be true. Indeed, it fails if α in (2.9) vanishes, that is, if $n = 2$ (surface case, where an alternative proof must be given) or if $k = 0$ (no torus factor, where the metric might be non-flat, in agreement with the example given in the first section).

Remark 2.2. For some values of k and n (namely $k \geqslant n-2$), a more geometric proof can be obtained along the following lines. One has

$$M = (\mathbb{T}^k \times \mathbb{R}^{n-k})/\Delta$$

for some finite group Δ, so that $\Gamma = \pi_1(M)$ has polynomial growth of order k. Moreover, from Bishop–Gromov classical comparison theorem, volume growth is polynomial of order n at most. It was proved by Anderson [And90] that if $\widehat{M} = \mathbb{T}^k \times \mathbb{R}^{n-k}$ has volume growth

$$\operatorname{vol}(B(r)) \geqslant Cr^p$$

then the group $\widehat{\Gamma} = \Gamma/\Delta$ has polynomial growth $n-p$ at most, hence Γ itself. This implies $p \leqslant n-k$. Applying Anderson's result once more, polynomial growth k of $\widehat{\Gamma}$ and volume growth

$$\operatorname{vol}(B(r)) \leqslant C'r^n$$

on Ω implies that $\widehat{M}$ has volume growth

$$\operatorname{vol}(B(r)) \leqslant C'r^{n-k}.$$

Now, if the curvature is bounded, and if $n - k \leqslant 2$, Anderson has proved in the same paper using minimal surfaces that $\widehat{M}$ must split off a factor $\mathbb{R}$ at least. Lafontaine's result [Laf82] then ends this remark. Unfortunately, there seem to be no way to extend this conceptual argument to higher codimensions.

Appendix A

In the last proof, we have used the following result of Schoen and Yau.

https://doi.org/10.1112/S0010437X06002016 Published online by Cambridge University Press

Theorem A.1 (Schoen–Yau [SY88, Theorems 4.1–4.5]). *Suppose (M, g) is complete, locally conformally flat with nonnegative scalar curvature, then the developing map*

$$\Phi : \tilde{M} \to \mathbb{S}^n$$

is injective, the fundamental group $\pi_1(M)$ is isomorphic to a discrete subgroup of C_n (the conformal group of $\mathbb{S}^n$), and $\Phi(\tilde{M})$ is a domain Ω in the complement $\Omega(\Gamma)$ of the limit set of the action of $\pi_1(M)$ on the sphere $\mathbb{S}^n$.

The proof of this result is done in [SY88] in two steps: a first argument covers the case $n \geqslant 7$; another argument based on the positive mass theorem covers the case $3 \leqslant n \leqslant 6$. However, the authors remark that '*for this application it is necessary to extend the positive mass theorems to the case of complete manifolds; that is assuming that the manifold has an asymptotically flat end and other ends which are merely complete. This extension will be carried out in a future work*' [SY88, p. 65].[2] It is widely believed in the Riemannian geometry community that the needed extension of the positive mass theorem is true, but, unfortunately, no such generalisation has been published so far. Of course, when the manifold is assumed to be spin, the Witten proof of the positive mass theorem [Par85, PT82, Wit81] extends to this general framework, but the spin assumption plays an essential role there. Note, however, that every orientable 3-manifold is spin. Hence, Schoen–Yau's analysis of conformally flat manifolds and Kleinian groups [SY88] for sure holds in dimension 3 as we only need to apply the positive mass theorem to the universal covering (mass increases as one takes quotients).

To avoid in our setting the problems caused by the absence of any written proof of the relevant positive mass theorem, it then remains to prove the adequate result for conformally flat manifolds with nonnegative Ricci curvature and dimensions $4 \leqslant n \leqslant 6$.

According to the Bishop–Gromov inequality, we know that on any covering of such a manifold the volume of geodesic balls is less than the volume of the euclidean balls of the same radius. We will show in this appendix that the argument given in [SY88] by Schoen and Yau to prove their theorem in dimension larger than 7 works in dimensions 4, 5 and 6 under just that very assumption on volume growth that can be obtained from nonnegativity of the Ricci curvature. More precisely, we prove below the following version of Theorem 3.1 in [SY88].

Theorem A.2 (Schoen–Yau's [SY88, Theorem 3.1] revisited). *Assume that (M, g) is a complete Riemannian conformally flat manifold of dimension $n \in [4, 6]$ such that $\mathrm{Scal}_g \geqslant 0$ and, for some $x \in M$, there exist C such that $\mathrm{vol}\, B(x, r) \leqslant Cr^n$ for every $r > 0$.*

Then if $\Phi : M \to \mathbb{S}^n$ is a conformal immersion, then Φ is injective and yields a conformal diffeomorphism of M onto $\phi(M)$. Moreover, the boundary of $\Phi(M)$ has zero Newtonian capacity.

Our proof closely follows [SY88], differing only at the points where dimension arguments enter the picture. We then refer to [SY88, pp. 58–61] for the overall strategy, and focus below only on the steps where some modification is needed. In particular, we refer to [SY88] for the proof of the fact that Theorem A.2 leads to the following final result.

Corollary A.3. *Suppose that (M, g) is complete, locally conformally flat with nonnegative Ricci curvature then the developing map*

$$\Phi : \tilde{M} \to \mathbb{S}^n$$

is injective, $\pi_1(M)$ is isomorphic to a discrete subgroup of C_n (the conformal group of $\mathbb{S}^n$), and $\Phi(M)$ is a domain Ω included in the complement $\Omega(\Gamma)$ of the limit set of $\pi_1(M)$ in $\mathbb{S}^n$.

[2] As the Ricci curvature is nonnegative, the holonomy covering has at most two ends. To apply the analysis of [SY88], one should then have a version of the positive mass theorem that is valid for the holonomy covering *minus a point*, that is a complete manifold with one asymptotically flat end and one or two other ends.

https://doi.org/10.1112/S0010437X06002016 Published online by Cambridge University Press

Proof of Theorem A.2. We first show how the original proof of Schoen and Yau can be easily adapted for the case $n \geqslant 5$; then we examine the case $n = 4$, which needs a more elaborate argument.

The core of the proof of Theorem 3.1 in [SY88] lies in the following construction: let $o \in M$ be a point and G_o be the Green kernel of the conformal Laplacian of (M, g) with pole at o, and $\bar{G}_o$ be the pulled-back (by Φ) of the Green kernel of the conformal Laplacian on the sphere with pole at $\Phi(o)$, normalized so that $\bar{G}_o - G_o$ is smooth near o. We let $v = G_o/\bar{G}_o$. The metric $\bar{g} = G_o^{4/(n-2)} g$ is flat (and incomplete) and the function v is $\bar{g}$-harmonic and one has $0 < v \leqslant 1$. Then Schoen and Yau prove that their theorem is obtained if one is able to show that $v = 1$ everywhere. It is that careful study of v, where the positive mass theorem plays a role in small dimensions, that we shall replace.

When $n \geqslant 5$ and $\alpha = 2(n-2)/n$, Schoen and Yau show in [SY88, (3.4), p. 59] that for every σ large enough,

$$\int_{B(o,\sigma/2)} |\bar{\nabla} v|^{\alpha-2} |\bar{\nabla}|\bar{\nabla} v||^2 d\,\mathrm{vol}_{\bar{g}} \leqslant \frac{C}{\sigma^2} \int_{B(o,3\sigma)\setminus B(o,\sigma/3)} G_o^{\alpha} d\,\mathrm{vol}_g\,.$$

Moreover, they have also shown that for every $p > n/(n-2)$ (see [SY88, Corollary 2.3]):

$$\int_{M\setminus B(o,1)} G_o^p d\,\mathrm{vol}_g < \infty;$$

in particular, as $n > 4$ we have

$$\int_{M\setminus B(o,1)} G_o^2 d\,\mathrm{vol}_g < \infty.$$

Using Hölder inequality and our assumption on volume growth, we then have

$$\int_{B(o,3\sigma)\setminus B(o,\sigma/3)} G_o^{\alpha} d\,\mathrm{vol}_g \leqslant C\sigma^2 \left(\int_{B(o,3\sigma)\setminus B(o,\sigma/3)} G_o^2 d\,\mathrm{vol}_g \right)^{1-2/n}.$$

Hence, letting $\sigma \to \infty$, we obtain that v is the constant function equal to one. As already said, this is enough to continue the proof with the argument of [SY88].

It remains to consider the case of the dimension 4 where, unfortunately, $\alpha = 1$. The first results we need are the following.

Lemma A.4. *For $R > 0$ large enough and for all $\varepsilon > 0$,*

$$\int_{M\setminus B(o,R)} \frac{|\nabla G_o|^2}{G_o |\log G_o|^{1+\varepsilon}} d\,\mathrm{vol}_g + \int_{M\setminus B(o,R)} \frac{\mathrm{Scal}_g\, G_o}{|\log G_o|^{\varepsilon}}\,\mathrm{vol}_g < \infty.$$

Proof. This is inspired by [SY88, Proof of Proposition 2.4(iii)].

Let $M = \bigcup_i U_i$ an exhaustion of M by a sequence of nested bounded open subsets with smooth boundaries, such that $o \in U_0$. Let $\varphi \in C^\infty(M)$ such that $\varphi = 1$ outside $B(o, R)$ and the support of φ does not meet $B(o, R/2)$ and let G_o^i be the Green kernel of the confomal Laplacian on U_i (with the Dirichlet boundary condition) of pole o.

We know that $G_o^i - G_o \to 0$ uniformly on compact subsets of M and that $G_o^i \leqslant G_o$. Moreover, nonnegativity of the scalar curvature implies that

$$\Delta G_o^i + \mathrm{Scal}_g\, G_o^i/6 \leqslant 0. \tag{A.1}$$

Hence, we can integrate by parts the quantity

$$\int_{U_i} \varphi \langle \nabla G_o^i, \nabla(|\log G_o^i|^{-\varepsilon}) \rangle d\,\mathrm{vol}_g$$

https://doi.org/10.1112/S0010437X06002016 Published online by Cambridge University Press

with i large enough, so that $B(o,R) \subset U_i$. Note that it is shown in [SY88, Lemma 3.2] that

$$\lim_{x\to\infty} G_o(x) = 0,$$

so that one may take i and R large enough to ensure that $G_o^i \leqslant \frac{1}{2}$ on the support of φ. Using inequality (A.1), and letting $i \to \infty$, this leads to

$$\varepsilon \int_{U_i\setminus B(o,R)} \frac{|\nabla G_o^i|^2}{G_o^i|\log G_o^i|^{1+\varepsilon}} d\operatorname{vol}_g + \frac{1}{6}\int_{U_i\setminus B(o,R)} \frac{\operatorname{Scal}_g G_o}{|\log G_o|^{\varepsilon}} d\operatorname{vol}_g \leqslant C\int_{B(o,R)\setminus B(o,R/2)} |\nabla G_o^i||\log G_o^i|^{-\varepsilon} d\operatorname{vol}_g .$$

The result follows from letting $i \to \infty$, because the right-hand side is an integral on a domain that does not depend on i and the convergence of G_o^i towards G_o is in C^∞ on compact subsets; moreover, this is true for every $\varepsilon > 0$. □

LEMMA A.5. *For any $R > 0$,*

$$\int_{M\setminus B(o,R)} \frac{G_o^2}{|\log G_o|^4} d\operatorname{vol}_g < \infty.$$

Proof. We note that Schoen and Yau showed that the Yamabe invariant of M is positive (in fact, it is exactly that of the standard sphere, see [SY88, Proposition 2.2]), hence we have a constant $c > 0$ such that for any $f \in C_0^\infty(M)$,

$$c\left(\int_M f^4 d\operatorname{vol}_g\right)^{1/2} \leqslant \int_M \left[|df|^2 + \frac{1}{6}\operatorname{Scal}_g f^2\right] d\operatorname{vol}_g$$

In particular, we can apply this to $f = \varphi\sqrt{G_o^i}/\log G_o^i$ (with φ a cut-off function as above) and Lemma A.4 with adequate choices of ε shows the desired result. □

Now we can adapt the argument of [SY88] to the case $n = 4$: we start again from inequality (3.4) of that paper:

$$\forall \phi \in C_0^\infty(M), \quad \int_M \phi^2 |\bar\nabla v|^{-1} |\bar\nabla|\bar\nabla v||^2 d\operatorname{vol}_{\bar g} \leqslant \int_M |d\phi|^2 G_o |dv| d\operatorname{vol}_g .$$

Let k an integer large enough; we take the following as a test function:

$$\phi_k(x) = \begin{cases} 1 & \text{if } x \in B(o,2^k) \\ \dfrac{\log(2^{2k}/r(x))}{\log(2^k)} \leqslant 1 & \text{if } x \in B(o,2^{2k}) \setminus B(o,2^k) \\ 0 & \text{if } x \notin B(o,2^{2k}) \end{cases}$$

where $r(x) = \operatorname{dist}(x,o)$. As we have

$$G_o|dv| \leqslant |dG_o| + G_o|d\log \bar G_o|,$$

we obtain

$$\int_{B(o,2^k)} |\bar\nabla v|^{-1} |\bar\nabla|\bar\nabla v||^2 d\operatorname{vol}_{\bar g} \leqslant \frac{c}{k^2}\int_{B(o,2^{2k})\setminus B(o,2^k)} [|dG_o| + G_o|d\log \bar G_o|] \frac{d\operatorname{vol}_g}{r^2(x)}. \tag{A.2}$$

We first treat the first term in the right-hand side of the previous estimate. One has

$$\int_{B(o,2^{2k})\setminus B(o,2^k)} |dG_o| \frac{d\operatorname{vol}_g}{r^2(x)} \leqslant \left(\int_{B(o,2^{2k})\setminus B(o,2^k)} |dG_o|^2 d\operatorname{vol}_g\right)^{1/2} \left(\int_{B(o,2^{2k})\setminus B(o,2^k)} \frac{d\operatorname{vol}_g}{r^4(x)}\right)^{1/2}.$$

https://doi.org/10.1112/S0010437X06002016 Published online by Cambridge University Press

Now the same proof as that of Lemma A.4 above (see also [SY88, Proposition 2.4(iii-a)]) shows that

$$\int_{M\setminus B(o,1)} |dG_o|^2 d\operatorname{vol}_g < \infty,$$

and the upper bound on the volume growth yields

$$\int_{B(o,2^{2k})\setminus B(o,2^k)} \frac{d\operatorname{vol}_g}{r^4(x)} \leqslant Ck$$

for a constant C independent of k. Hence we get that the first term goes to zero when k goes to infinity.

Now for the second term in the right-hand side of (A.2):

$$\begin{aligned}&\int_{B(o,2^{2k})\setminus B(o,2^k)} G_o|d\log \bar{G}_o|\frac{d\operatorname{vol}_g}{r^2(x)}\\ &\quad\leqslant \left(\int_{B(o,2^{2k})\setminus B(o,2^k)} G_o|\log G_o|^2\frac{d\operatorname{vol}_g}{r^4(x)}\right)^{1/2}\left(\int_{B(o,2^{2k})\setminus B(o,2^k)} \frac{G_o}{|\log G_o|^2}|d\log \bar{G}_o|^2 d\operatorname{vol}_g\right)^{1/2}.\end{aligned}$$

The function G_0 tends to zero at infinity by [SY88, Lemma 3.2], hence we can again choose $R>0$ large enough so that $G_o \leqslant \frac{1}{2}$ outside the ball $B(o,R/2)$. As a result, we have

$$\int_{B(o,2^{2k})\setminus B(o,2^k)} G_o|\log G_o|^2\frac{d\operatorname{vol}_g}{r^4(x)} \leqslant Ck.$$

We now use another estimate due to Schoen and Yau that is valid in every dimension: they show at the end of p. 60 in [SY88] that for any $\psi \in C_0^\infty(M)$ vanishing near o, we have

$$\int_M \psi^2|\nabla \log \bar{G}_o|^2 d\operatorname{vol}_g \leqslant c\int_M \psi^2|\nabla \log G_o|^2 d\operatorname{vol}_g + c\int_M |d\psi|^2 d\operatorname{vol}_g\,.$$

Now let ϕ be a function with the following properties:

- it has compact support in $B(o,2^{2k+1})\setminus B(o,2^{k-1})$;
- it is identically 1 on $B(o,2^{2k})\setminus B(o,2^k)$;
- its gradient is bounded by $2^{-2k}c$ on $B(o,2^{2k+1})\setminus B(o,2^{2k})$; and
- and its gradient is bounded by $2^{-k}c$ on $B(o,2^k)\setminus B(o,2^{k-1})$.

This does exist and we can apply the last estimate to $\psi = (\phi\sqrt{G_o})/\log G_o$. We obtain

$$\begin{aligned}&\int_{B(o,2^{2k})\setminus B(o,2^k)} G_o|\log G_o|^{-2}|d\log \bar{G}_o|^2 d\operatorname{vol}_g\\ &\quad\leqslant C\int_{B(o,2^{2k+1})\setminus B(o,2^{k-1})} |d\phi|^2 G_o|\log G_o|^{-2} d\operatorname{vol}_g\\ &\qquad + C\int_{B(o,2^{2k+1})\setminus B(o,2^{k-1})} G_o|(\log G_o|^{-2} + |\log G_o|^{-4})|d\log G_o|^2 d\operatorname{vol}_g\,.\end{aligned}$$

From Lemma A.4 we know that the second term in the right-hand side of this last estimate goes to zero when k goes to infinity. For the first term, the Cauchy–Schwartz inequality leads to

$$\begin{aligned}&\int_{B(o,2^{2k+1})\setminus B(o,2^{k-1})} |d\phi|^2\frac{G_o}{|\log G_o|^2}d\operatorname{vol}_g\\ &\quad\leqslant \left(\int_{B(o,2^{2k+1})\setminus B(o,2^{k-1})} |d\phi|^4 d\operatorname{vol}_g\right)^{1/2}\left(\int_{B(o,2^{2k+1})\setminus B(o,2^{k-1})} G_o^2|\log G_o|^{-4} d\operatorname{vol}_g\right)^{1/2}.\end{aligned}$$

https://doi.org/10.1112/S0010437X06002016 Published online by Cambridge University Press

However, our volume growth assumptions imply that

$$\int_{B(o,2^{2k+1})\setminus B(o,2^{k-1})} |d\phi|^4 d\,\mathrm{vol}_g \leqslant C$$

where C is a constant independent of k. Using Lemma A.5, we end with

$$\lim_{k\to\infty} \int_{B(o,2^{2k+1})\setminus B(o,2^{k-1})} G_o^2 |\log G_o|^{-4} d\,\mathrm{vol}_g = 0.$$

Hence, we obtain again that v is the constant function equal to 1 everywhere. This is what is needed to resume the proof of [SY88]. □

Acknowledgements

The authors are grateful to Jérôme Droniou and Jacques Lafontaine for useful discussions. Many thanks are due to Jeff Viaclovsky who pointed out S. Zhu's paper to us.

References

And90 M. T. Anderson, *On the topology of complete manifolds of non-negative Ricci curvature*, Topology **29** (1990), 41–55.

Bes81 A. L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 10 (Springer, Berlin, 1981).

CH02 G. Carron and M. Herzlich, *The Huber theorem for non-compact conformally flat manifolds*, Comment. Math. Helv. **77** (2002), 192–220.

CG71 J. Cheeger and D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Differential Geom. **6** (1971), 119–128.

CZ02 B.-L. Chen and X.-P. Zhu, *A gap theorem for complete non-compact manifolds with nonnegative Ricci curvature*, Comm. Anal. Geom. **10** (2002), 217–239.

CZ03 B.-L. Chen and X.-P. Zhu, *On complete noncompact Kähler manifolds with positive bisectional curvature*, Math. Ann. **327** (2003), 1–23.

Che01 Q.-M. Cheng, *Compact locally conformally flat Riemannian manifolds*, Bull. London Math. Soc. **33** (2001), 459–465.

CL U. Christ and J. Lohkamp, personal communication, 2005.

GHL04 S. Gallot, D. Hulin, and J. Lafontaine, *Riemannian geometry*, third edition (Springer, Berlin, 2004).

GL83 M. Gromov and H. B. Lawson, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Publ. Math. Inst. Hautes Études Sci. **58** (1983), 83–196.

Laf82 J. Lafontaine, *Remarques sur les variétés conformément plates*, Math. Ann. **259** (1982), 313–319.

Liu92 Z.-D. Liu, *Ball covering on manifolds with nonnegative Ricci curvature near infinity*, Proc. Amer. Math. Soc. **115** (1992), 211–219.

MSY81 N. Mok, Y.-T. Siu and S.-T. Yau, *The Poincaré–Lelong equation on complete Kähler manifolds*, Compositio Math. **44** (1981), 183–218.

Ni98 L. Ni, *Vanishing theorems on complete Kähler manifolds and their applications*, J. Differential Geom. **50** (1998), 89–122.

Ni04 L. Ni, *A monotonicity formula on complete Kähler manifolds with nonnegative bisectional curvature*, J. Amer. Math. Soc. **17** (2004), 909–946.

NST01 L. Ni, Y. Shi, and L.-F. Tam, *Poisson equation, Poincaré–Lelong equation and curvature decay on complete Kähler manifolds*, J. Differential Geom. **57** (2001), 339–388.

NT03 L. Ni and L.-F. Tam, *Plurisubharmonic functions and the structure of complete Kähler manifolds with nonnegative curvature*, J. Differential Geom. **64** (2003), 457–524.

Nor93 M.-H. Noronha, *Some compact conformally flat manifolds with nonnegative scalar curvature*, Geom. Dedicata **47** (1993), 255–268.

Par85 T. H. Parker, *Gauge choice in Witten's energy expression*, Commun. Math. Phys. **100** (1985), 471–480.

PT82 T. H. Parker and C. H. Taubes, *On Witten's proof of the positive energy theorem*, Commun. Math. Phys. **84** (1982), 223–238.

SY88 R. Schoen and S. T. Yau, *Conformally flat manifolds, Kleinian groups and scalar curvature*, Invent. Math. **92** (1988), 47–71.

Wit81 E. Witten, *A new proof of the positive energy theorem*, Commun. Math. Phys. **80** (1981), 381–402.

Wol67 J. Wolf, *Spaces of constant curvature* (McGraw-Hill, New York, 1967).

Zhu94 S. Zhu, *The classification of complete locally conformally flat manifolds of nonnegative Ricci curvature*, Pacific J. Math. **163** (1994), 189–199.

Gilles Carron gilles.carron@math.sciences.univ-nantes.fr
Laboratoire Jean Leray, UMR 6629 du CNRS, Université de Nantes, 44322 Nantes cedex 3, France

Marc Herzlich herzlich@math.univ-montp2.fr
Institut de Mathématiques et Modélisation de Montpellier, UMR 5149 du CNRS,
Université Montpellier II, 34095 Montpellier cedex 5, France

https://doi.org/10.1112/S0010437X06002016 Published online by Cambridge University Press