ON THE MONODROMY GROUPS OF LIFTED
EULER EQUATIONS

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In [13], Poincaré asked the following question: Which abstract groups can appear as monodromy groups [14] of second order, linear, homogeneous differential equations with meromorphic coefficients (which might depend on one or more parameters) on $\hat{\mathbb{C}}$? In the present paper, we initiate a classification of monodromy groups of differential equations on compact Riemann surfaces of genus 1. We proceed as follows: Let

$$y'' + \frac{\alpha}{z} y' + \frac{\beta}{z^2} y = 0 \quad \text{with} \quad \alpha, \beta \in \mathbb{C} \tag{1}$$

be the general Euler equation [1] which has two regular singular points at 0 and ∞ in the extended complex plane $\hat{\mathbb{C}}$. Further, let $y_\nu(z)$ ($\nu = 1, 2$) be an arbitrary but fixed pair of linearly independent solutions to (1) valid in a neighborhood of some ordinary point. Analytic continuation of each solution along a closed loop Λ in $\hat{\mathbb{C}} - \{0, \infty\}$, starting and ending at some fixed base point, produces a new solution $y_{\nu,\Lambda}$ ($\nu = 1, 2$) which can be expressed as

$$y_{\nu,\Lambda}(z) = a_{\nu 1} y_1(z) + a_{\nu 2} y_2(z)$$

where the constants $a_{\nu\mu}$ ($\nu, \mu = 1, 2$) in $\mathbb{C}$ depend on the homotopy class $[\Lambda]$ of Λ. Clearly, $y_{1,\Lambda}(z)/y_{2,\Lambda}(z) = T \circ (y_1(z)/y_2(z))$ where the Möbius transformation $T : w \mapsto (a_{11}w + a_{12})/(a_{21}w + a_{22})$ depends on $[\Lambda]$. The set of all Möbius transformations T belonging to every possible closed loop Λ in $\hat{\mathbb{C}} - \{0, \infty\}$ forms a group G, called the *monodromy group of the Euler equation* (1). G is generated by the Möbius transformation belonging to a simple, closed loop Λ_0 encircling 0. Hence, G is cyclic.

We will show that substitution of various meromorphic functions $z = g(w)$ into (1) results in equations (henceforth referred to as *untransformed lifted Euler equations*) which can be transformed to eliminate their first derivative terms to give equations (called *transformed lifted Euler equations*) having coefficients doubly periodic with respect to an arbitrary lattice [12] $L = \langle n\omega_1 + m\omega_2, \ n, m \in Z \times Z, \ \text{Im}(\omega_1/\omega_2) > 0 \rangle$. Consequently, the transforms can be viewed as equations on an arbitrary complex torus $\mathbb{C}/L$. Thus, each transformed lifted Euler equation has a monodromy group G^*. G^* is also the monodromy group of the untransformed equation because both lifted equations have the same ratio of linearly independent solutions.

The generators for G^* are obtained by the analytic continuation of $y_1(g(w))/y_2(g(w))$ along simple, closed loops about the singularities of the transformed equation as well as along any two generating loops for the homotopy group [10] of $\mathbb{C}/L$, where all loops avoid singularities. Equivalently, we find [7] these generators by analytically continuing $y_1(z)/y_2(z)$ along the images of the above mentioned loops under $g(w)$ in $\hat{\mathbb{C}}$. Our selected

substitutions g will not necessarily be doubly periodic so that these images of loops can be either open arcs or closed loops. These observations will ultimately permit us to determine G^* explicitly.

We now proceed with a detailed description of our findings.

THEOREM 1. *The substitutions* $z = t^\lambda(w)$, $\lambda \in \mathbb{C}^*$, *with* $t(w)$ *of the form* [11]

$$t(w) = \frac{\sqrt{(\mathscr{P}(w) - e_1)} + \sqrt{(\mathscr{P}(w) - e_2)}}{\sqrt{(e_2 - e_1)}} \quad \text{with} \quad e_i = \mathscr{P}\left(\frac{\omega_i}{2}\right), \quad i = 1, 2 \tag{2}$$

or

$$t(w) = \frac{\prod_{i=1}^{k} \sigma^{m_i}(w - a_i)}{\prod_{j=1}^{l} \sigma^{n_j}(w - b_j)} e^{cw} \quad \text{with} \quad k, l > 0; \quad m_i, n_j \in \mathbb{Z}^+; \quad \sum_{i=1}^{k} m_i - \sum_{j=1}^{l} n_j = 0 \tag{3}$$

into any Euler equation (1) *(with difference of indicial roots* $r = r_1 - r_2$*) on* $\hat{\mathbb{C}}$ *produce lifted Euler equations which can be transformed respectively into*

$$y'' + \frac{1}{2}\left[\frac{1 - (r\lambda)^2}{2}(\mathscr{P}(w) - e_3) - \frac{3}{2}\mathscr{P}\left(w - \frac{\omega_1 + \omega_2}{2}\right)\right]y = 0 \quad \text{where} \quad e_3 = \mathscr{P}\left(\frac{\omega_1 + \omega_2}{2}\right) \tag{4}$$

or

$$y'' + \frac{1}{2}\left[\frac{1 - (r\lambda)^2}{2}\left(\sum_{i=1}^{k} m_i \zeta(w - a_i) - \sum_{j=1}^{l} n_j \zeta(w - b_j) + c\right)^2 + \theta_2 t(w)\right]y = 0, \tag{5}$$

where $\mathscr{P}$, σ, ζ *are the Weierstrass functions* [12] *for an arbitrary lattice* L *and* θ_2 *is the Schwarzian derivative operator* [4]. *Equations* (4) *and* (5) *have coefficients doubly periodic for* L, *can be treated as equations on* $\mathbb{C}/L$ *and determine monodromy groups on* $\mathbb{C}/L$.

REMARK 1. It has been shown [11] that the mapping (2) represents a specific non-affine, branched projective structure [4] on an arbitrary unpunctured torus and is associated with a Schwarzian differential equation with non-affine monodromy group. Here we use the mapping (2) to construct differential equations on punctured tori as well as to determine the monodromy groups of these equations. Similarly, the mappings of type (3) represent classically known affine structures on unpunctured tori [3] and are used here to construct differential equations on tori with arbitrarily many punctures as well as to find the monodromy groups of these equations.

Proof. The Euler equation (1) with difference of indicial roots $r = \sqrt{((\alpha - 1)^2 - 4\beta)}$ (see p. 174–177 of [1] for all relevant facts concerning Euler's equation) can be lifted by the map $z = t^\lambda(w)$ to $\mathbb{C}$ by a two step process as follows: Let $z = f \circ t(w)$ with $f(t) = t^\lambda$. First, lift (1) by $z = f(t)$ to a new Euler equation

$$y''(t) + \frac{\alpha'}{t} y'(t) + \frac{\beta'}{t^2} y(t) = 0 \tag{6}$$

with $\alpha' = \lambda\alpha - \lambda + 1$ and $\beta' = \beta\lambda^2$ and with difference of indicial roots

$$r' = \sqrt{((\alpha' - 1)^2 - 4\beta')} = \lambda r. \tag{7}$$

Second, lift (6) to $\mathbb{C}$ by the meromorphic substitution $t = t(w)$ to obtain

$$Y''(w) + P(w)Y'(w) + Q(w)Y(w) = 0 \qquad (8)$$

where $P(w) = (-t''/t') + \alpha'(t'/t)$ and $Q(w) = \beta'(t'/t)^2$. Note that $\mathcal{P}$ and σ in (2) and (3) can belong to an arbitrary lattice L.

Since $t = t(w)$ is a meromorphic function on $\mathbb{C}$, the transformation [6]

$$Y(w) = e^{-1/2 \int^w P(s)\,ds} y(w)$$

can be used to transform (8) into

$$y''(w) + J(w)y(w) = 0 \qquad (9)$$

where

$$J(w) = Q(w) - \tfrac{1}{2}P'(w) - \tfrac{1}{4}P^2(w) = \beta'\left(\frac{t'}{t}\right)^2 - \frac{1}{2}\left(-\frac{t''}{t'} + \alpha'\frac{t'}{t}\right)' - \frac{1}{4}\left(-\frac{t''}{t'} + \alpha'\frac{t'}{t}\right)^2$$

$$= \left(\beta' + \frac{\alpha'}{2} - \frac{(\alpha')^2}{4}\right)\left(\frac{t'}{t}\right)^2 + \frac{\theta_2 t(w)}{2}.$$

Elementary calculations using (7) produce $(1 - (\lambda r)^2)/4 = \beta' + (\alpha'/2) - ((\alpha')^2/4)$ so that

$$J(w) = \frac{1}{2}\left[\frac{1 - (\lambda r)^2}{2}\left(\frac{t'}{t}\right)^2 + \theta_2 t(w)\right].$$

Equations (8) and (9) have the same ratio of linearly independent solutions.

When (2) is used for $t(w)$, we obtain [11]

$$\left(\frac{t'}{t}\right)^2 = \mathcal{P}(w) - e_3 \quad \text{and} \quad \theta_2 t(w) = -\frac{3}{2}\mathcal{P}\left(w - \frac{\omega_1 + \omega_2}{2}\right).$$

Hence, (9) assumes the form (4) which clearly has doubly periodic coefficients.

Alternatively, when (3) is used for $t(w)$, we obtain [12]

$$\frac{t'}{t} = \sum_{i=1}^{k} m_i \zeta(w - a_i) - \sum_{j=1}^{l} n_i \zeta(w - b_i) + c.$$

Hence, (9) assumes the form (5). We observe that t'/t is doubly periodic since $\sum_{i=1}^{k} m_i - \sum_{j=1}^{l} n_i = 0$. Also, $\theta_2 t(w)$ is doubly periodic since $t(w)$ can be viewed as a multiplicative multi-valued function on $\mathbb{C}/L$ for some character [3].

From the double periodicity of the coefficients of (4) and (5), we can conclude that these equations live on $\mathbb{C}/L$ and have monodromy groups there.

REMARK 2. The proof of this theorem implies that if the substitutions $z = t^\lambda(w)$ and $z = t(w)$, $t(w)$ fixed and of form (2) or (3), are made respectively into any two Euler equations with respective differences of indicial roots r and $r' = r\lambda$, then the same transformed lifted Euler equation results.

We are now in a position to state

THEOREM 2. *Equation* (4) *has monodromy group* G^* *given by*
(a) D_{2n} *if* $r\lambda = k/n$, $n \in Z^+$, $k \in \{\pm 1, \pm 3, \dots\}$ *and* $(|k|, n) = 1$
(b) D_n *if* $r\lambda = k/n$, $n \in Z^+$, $k \in \{\pm 2, \pm 4, \dots\}$ *and* $(|k|, n) = 1$
(c) $C_2 * C_2$ *if* $r\lambda \in (\mathbb{C} - Q) \cup \{0\}$.

REMARK 3. G^* *is the Klein 4-group if* $n = 1$ *and* $k \in \{\pm 1, \pm 3, \dots\}$ *while* G^* *is* C_2 *if* $n = 1$ *and* $k \in \{\pm 2, \pm 4, \dots\}$.

Proof. Remark 1 implies that there exists an equation (1) which lifts by map (2) to an equation which transforms into (4). Therefore, $\lambda = 1$ can be assumed with no loss of generality. Let $u(z)$ be some ratio of linearly independent solutions to (1) and $h(w) = u \circ t(w)$ the corresponding ratio of linearly independent solutions to (4). When viewed as an equation on $\mathbb{C}/L$, (4) has singularities only at the natural projections of 0 and $(\omega_1 + \omega_2)/2$ to $\mathbb{C}/L$. Let Λ_0' and $\Lambda_{(\omega_1+\omega_2)/2}'$ be simple loops starting at some fixed point and enclosing the respective natural projections of 0 and $(\omega_1 + \omega_2)/2$. Let Λ_{ω_1}' and Λ_{ω_2}' be non-contractable loops (based at the same fixed point as above) which together generate the homotopy group of $\mathbb{C}/L$. Here, all loops avoid singularities of (4). Lift [10] Λ_0', $\Lambda_{(\omega_1+\omega_2)/2}'$ and Λ_{ω_1}', Λ_{ω_2}' to $\mathbb{C}$ to unique loops Λ_0, $\Lambda_{(\omega_1+\omega_2)/2}$ and unique arcs Λ_{ω_1}, Λ_{ω_2}, all having the same initial point. We can assume that the terminal points of Λ_{ω_i} $(i = 1, 2)$ are translates of the common initial point by ω_i $(i = 1, 2)$. G^* is generated by the monodromy elements corresponding to the above loops and arcs (on $\mathbb{C}$).

Since $t(0) = \infty$ and $t((\omega_1 + \omega_2)/2) = (\sqrt{(e_3 - e_1)} + \sqrt{(e_3 - e_2)})/\sqrt{(e_2 - e_1)} = K \neq 0$ [11], we can find [7] the generators for G^* associated with Λ_0 and $\Lambda_{(\omega_1+\omega_2)/2}$ by continuing

$$u(z) = \begin{cases} z^r & \text{if} \quad r \in \mathbb{C}^* \\ \ln z & \text{if} \quad r = 0 \end{cases} \tag{10}$$

along the loops $\Gamma_\infty = t(\Lambda_0)$ and $\Gamma_K = t(\Lambda_{(\omega_1+\omega_2)/2})$ where Γ_∞ is homotopic in $\hat{\mathbb{C}} - \{0, \infty\}$ to a simple loop enclosing ∞ while Γ_K is homotopically trivial in this space. Thus, the generator T_K corresponding to $\Lambda_{(\omega_1+\omega_2)/2}$ or Γ_K is the identity. Furthermore, the generator T_∞ corresponding to Λ_0 or Γ_∞ is

$$T_\infty(z) = \begin{cases} e^{-2\pi i r} z & \text{if} \quad r \in \mathbb{C}^* \\ z - 2\pi i & \text{if} \quad r = 0. \end{cases}$$

Note that $t(w + \omega_1) = 1/t(w)$ and $t(w + \omega_2) = -1/t(w)$ [11]. Hence, we determine the generators T_{ω_i} $(i = 1, 2)$ for G^* arising from Λ_{ω_i} $(i = 1, 2)$ by continuing (10) along the arcs $\Gamma_{\omega_i} = t(\Lambda_{\omega_i})$ as follows:

$$T_{\omega_1}(z) = \begin{cases} \dfrac{e^{2l_1 \pi i r}}{z} & \text{if} \quad r \in \mathbb{C}^* \\ 2l_1 \pi i - z & \text{if} \quad r = 0 \end{cases}$$

and

$$
T_{\omega_2}(z) = \begin{cases} \dfrac{e^{(2l_2+1)\pi i r}}{z} & \text{if } r \in \mathbb{C}^* \\[2mm] (2l_2+1)\pi i - z & \text{if } r = 0 \end{cases}
$$

where $l_i \in Z$ $(i = 1, 2)$ are fixed and depend respectively on Λ_{ω_1} and Λ_{ω_2}.

Replace T_{ω_1}, T_{ω_2} and T_∞ by the new generators [9] for G^* given as

$$
A(z) = T_\infty^{l_1} \circ T_{\omega_1}(z) = \begin{cases} \dfrac{1}{z} & \text{if } r \in \mathbb{C}^* \\[2mm] -z & \text{if } r = 0 \end{cases}
$$

and

$$
B(z) = T_\infty^{l_2} \circ T_{\omega_2}(z) = \begin{cases} \dfrac{e^{\pi i r}}{z} & \text{if } r \in \mathbb{C}^* \\[2mm] -z + \pi i & \text{if } r = 0 \end{cases}
$$

observing that T_∞ is not needed as a generator since $T_\infty = (A \circ B)^2$.

At this point, the proof splits naturally into three cases depending on the value of r.

Case 1. $r = k/n$, $n \in Z^+$, $k \in \{\pm 1, \pm 3, \dots\}$ and $(|k|, n) = 1$: Let $C_1 = A$ and $C_2 = B \circ A$ with $A(z)$ as above and $C_2(z) = e^{\pi i(k/n)}z$. Clearly, $C_1^2 = \text{id}$ and $C_1 C_2 C_1 = C_2^{-1}$. Also, $C_2^{2n} = \text{id}$ where elementary calculations using the oddness of k show that no lower positive power of C_2 is the identity. Therefore, $G^* = D_{2n}$. Hence, (a) has been verified.

Case 2. $r = k/n$, $n \in Z^+$, $k \in \{\pm 2, \pm 4, \dots\}$ and $(|k|, n) = 1$: Let C_1 and C_2 be defined as in Case 1. As above, $C_1^2 = \text{id}$ and $C_1 C_2 C_1 = C_2^{-1}$. However, $C_2^n = \text{id}$ where elementary calculations using the evenness of k show that no lower positive power of C_2 is the identity. Therefore, $G^* = D_n$. Hence, (b) has been verified.

Case 3. $r \in (\mathbb{C} - Q) \cup \{0\}$: A and B satisfy $A^2 = B^2 = \text{id}$. Consequently, all elements of G^* other than A, B and the identity can be realized as $(B \circ A)^n$ and $(B \circ A)^n \circ B$ for $n \in Z - \{0\}$. None of these elements is the identity for the given r values. Therefore, $G^* = \langle A; A^2 \rangle * \langle B; B^2 \rangle$. Hence, (c) has been verified.

COROLLARY. *The monodromy group* G^* *of equation* (4) *is Kleinian iff* $r\lambda \in (\mathbb{C} - \text{Re}) \cup Q$.

Proof. As in the proof of Theorem 2, we can assume that $\lambda = 1$ so that G^* is generated by

$$
A(z) = \begin{cases} \dfrac{1}{z} & \text{if } r \in \mathbb{C}^* \\[2mm] -z & \text{if } r = 0 \end{cases}
$$

and

$$
B(z) = \begin{cases} \dfrac{e^{i\pi r}}{z} & \text{if } r \in \mathbb{C}^* \\[2mm] -z + \pi i & \text{if } r = 0. \end{cases}
$$

Case 1. $r \in Q - \{0\}$: By Theorem 2, $G^* = D_n$ for some $n \in Z^+$. Since G^* is finite, it is Kleinian.

Case 2. $r = 0$: G^* is generated by $(B \circ A)(z) = z + \pi i$ and $A(z) = -z$ and therefore has fundamental domain [**3**]

$$FD = \{z : \operatorname{Re} z > 0 \quad \text{and} \quad 0 < \operatorname{Im} z < \pi\}.$$

Hence, G^* is Kleinian.

Case 3. $r \in \mathbb{C} - \operatorname{Re}$: Now, $r = a + ib$, $a, b \in \operatorname{Re}$, $b \neq 0$ and G^* is generated by $A(z) = 1/z$ and $B(z) = e^{i\pi r}/z = e^{i\pi a - \pi b}/z$. Therefore, $a \in (0, 2]$ and $b > 0$ can be assumed with no loss of generality.

We will show that

$$D = \{z : e^{-\pi b} < |z| < 1, \qquad 0 < \operatorname{Arg} z < \phi\}$$

(with ϕ determined below) is an open subset of a fundamental domain for G^* by examining the action of the nontrivial elements of G^* on D. As in Case 3 in the proof of Theorem 2, we can list all elements of G^* as follows:

(i) id
(ii) A
(iii) B
(iv) $(BA)^n$, $n = \pm 1, \pm 2, \dots$
(v) $(BA)^n B$, $n = +1, \pm 2, \pm 3, \dots$.

Here, $(BA)^n(z) = e^{i\pi an - \pi nb} z$ and $(BA)^n B(z) = e^{(i\pi a - \pi b)(1+n)}/z$.

If $T \in G^*$ assumes forms (ii), (iv) or (v), then

$$T(D) = \{z : 1 < |z| < e^{\pi b}, -\phi < \operatorname{Arg} z < 0\}$$
$$T(D) = \{z : e^{-\pi b(n+1)} < |z| < e^{-\pi bn}, \pi an < \operatorname{Arg} z < \phi + \pi an, n \in \{\pm 1, \pm 2, \dots\}\}$$
$$T(D) = \{z : e^{-\pi b(n+1)} < |z| < e^{-\pi bn}, -\phi + \pi a(1 + n) < \operatorname{Arg} z < \pi a(1 + n), n \in \{+1, \pm 2, \pm 3, \dots\}\}$$

respectively. Thus, in each instance $T(D) \cap D = \varnothing$ independently of the selection of $\phi > 0$. If $T \in G^*$ assumes form (iii) and if $\phi = \pi a/2$, then

$$D = \left\{z : e^{-\pi b} < |z| < 1, 0 < \operatorname{Arg} z < \frac{\pi a}{2}\right\}$$

and

$$T(D) = \left\{z : e^{-\pi b} < |z| < 1, \frac{\pi a}{2} < \operatorname{Arg} z < \pi a\right\}.$$

$T(D) \cap D = \varnothing$ since $a \in (0, 2]$. Therefore, $T(D) \cap D \neq \varnothing$ only if T assumes form (i). Hence, D is an open subset of a fundamental domain for G^* and G^* is Kleinian.

Case 4. $r \in \operatorname{Re} - Q$: G^* contains $(B \circ A)(z) = e^{i\pi r} z$ which is an elliptic element of infinite order since r is irrational. Thus, G^* is not Kleinian [**8**].

THEOREM 3. *Each equation of form* (5) *has monodromy group* G^* *of one of the following types:* $\quad C_l, C_\infty \times C_l, C_\infty \times C_\infty \times C_l \quad \text{where} \quad l \in Z^+ \cup \{+\infty\}.$

All of these groups (for all $l \in Z^+ \cup \{+\infty\}$) are realized as $r\lambda \in \mathbb{C}$ and $t(w)$ of form (3) both vary.

Proof. Equation (5) has singularities at a_i $(i = 1, \ldots, k)$, b_j $(j = 1, \ldots, l)$ and c_p $(p = 1, \ldots, t)$, the poles of $\theta_2 t(w)$, as well as at all translates of these points by elements of the lattice L. Therefore, G^* is generated by the elements corresponding to simple loops Λ_{a_i} $(i = 1, \ldots, k)$, Λ_{b_j} $(j = 1, \ldots, l)$, Λ_{c_p} $(p = 1, \ldots, t)$ about the points a_i, b_j, c_p respectively as well as arcs Λ_{ω_i} $(i = 1, 2)$ defined as in the proof of Theorem 2. Since $t(a_i) = 0$, $t(b_j) = \infty$ and $t(c_p) = C_p \neq 0, \infty$, the monodromy elements T_{a_i}, T_{b_j}, T_{c_p} associated with the loops mentioned can be found [7] by continuing $u(z)$ in (10) along the closed loops $\Gamma_0^{m_i} = t(\Lambda_{a_i})$ $(i = 1, \ldots, k)$, $\Gamma_\infty^{n_j} = t(\Lambda_{b_j})$ $(j = 1, \ldots, l)$ and $\Gamma_{C_p}^{O_p} = t(\Lambda_{c_p})$ $(p = 1, \ldots, t)$, $O_p \in Z$, where Γ_0 is homotopic in $\hat{\mathbb{C}} - \{0, \infty\}$ to a simple loop enclosing 0 and Γ_∞ is defined similarly while Γ_{C_p} is homotopically trivial in $\hat{\mathbb{C}} - \{0, \infty\}$. Thus, the generators T_{c_p} corresponding to Λ_{c_p} or $\Gamma_{C_p}^{O_p}$ must all be the identity. Furthermore, the generators T_{a_i} corresponding to Λ_{a_i} or $\Gamma_0^{m_i}$ and the generators T_{b_j} corresponding to Λ_{b_j} or $\Gamma_\infty^{n_j}$ are respectively given by

$$T_{a_i}(z) = \begin{cases} e^{2\pi i m_i r} z & \text{if} \quad r \in \mathbb{C}^* \\ z + 2\pi i m_i & \text{if} \quad r = 0 \end{cases} \qquad (i = 1, \ldots, k)$$

and

$$T_{b_j}(z) = \begin{cases} e^{-2\pi i n_j r} z & \text{if} \quad r \in \mathbb{C}^* \\ z - 2\pi i n_j & \text{if} \quad r = 0 \end{cases} \qquad (j = 1, \ldots, l).$$

We now show that the subgroup of G^* generated by T_{a_i} and T_{b_j} can be generated by precisely one transformation T_0. We let d_i and e_j be integers satisfying

$$\sum_{i=1}^{k} d_i m_i - \sum_{j=1}^{l} e_j n_j = \gcd\left(m_i, n_j; \begin{matrix} i = 1, \ldots, k \\ j = 1, \ldots, l \end{matrix}\right).$$

Define

$$T_0(z) = \prod_{i=1}^{k} T_{a_i}^{d_i} \circ \prod_{j=1}^{l} T_{b_j}^{e_j}(z) = \begin{cases} e^{2\pi i r \gcd(m_i, n_j)} z & \text{if} \quad r \in \mathbb{C}^* \\ z + 2\pi i \gcd(m_i, n_j) & \text{if} \quad r = 0. \end{cases}$$

Since there exist integers $M_i = m_i/\gcd(m_i, n_j)$ and $N_j = n_j/\gcd(m_i, n_j)$, we conclude that $(T_0)^{M_i} = T_{a_i}$ $(i = 1, \ldots, k)$ and $(T_0)^{N_j} = T_{b_j}$ $(j = 1, \ldots, l)$. Therefore, T_0 generates this subgroup of G^* as claimed.

We can view any map $z = t(w)$ of form (3) as a multi-valued function on $\mathbb{C}/L$ belonging to a multiplicative character [3] generated by constants $A_i \in \mathbb{C}^*$ $(i = 1, 2)$ which are associated with the natural projections of the arcs Λ_{ω_i} $(i = 1, 2)$ in $\mathbb{C}$ to closed, non-contractable loops on $\mathbb{C}/L$. Equivalently, we note that $t(w)$ is a single-valued function on $\mathbb{C}$ with transformation laws $t(w + \omega_i) = A_i t(w)$ for all $w \in \mathbb{C}$ $(i = 1, 2)$. Hence, the generators T_{ω_i} $(i = 1, 2)$ corresponding to Λ_{ω_i} $(i = 1, 2)$ are found by continuing $u(z)$ in (10) along the arcs $\Gamma_{\omega_i} = t(\Lambda_{\omega_i})$, where Γ_{ω_i} is closed iff $A_i = 1$. We obtain

$$T_{\omega_i}(z) = \begin{cases} A_i^r e^{2\pi i r k_i} z & \text{if} \quad r \in \mathbb{C}^* \\ z + \ln A_i + 2\pi i k_i & \text{if} \quad r = 0 \end{cases} \qquad (i = 1, 2)$$

where A_i' or $\ln A_i$ are arbitrary but fixed branches and where $k_i \in Z$ depend on both Λ_{ω_i} and these branches.

G^* is generated by T_0, T_{ω_i} $(i = 1, 2)$ and is a group of affine mappings consisting entirely of multiplications if $r \in \mathbb{C}^*$ or of translations if $r = 0$. Hence, G^* is Abelian and a direct product of at most three cyclic groups [5].

We now show that G^* has at most one generator of finite order. If $r = 0$, then clearly G^* has no generators or nontrivial elements of finite order. If $r \in \mathbb{C}^*$, then assume that

$$C_1(z) = e^{2\pi i(K/M)}z, \qquad C_2(z) = e^{2\pi i(L/N)}z; \qquad K, L, M, N \in Z^*$$

are generators of G^* having finite order. Define

$$D(z) = e^{(2\pi i/MN)\, \gcd(KN,\, LM)}z.$$

Number theory shows that the subgroups of G^* generated by C_i $(i = 1, 2)$ and D are the same. We conclude that G^* has at most one generator of finite order. Therefore, G^* is one of the types claimed.

We now show that all of these types are realized as follows:

(A) $C_\infty \times C_\infty \times C_\infty$: Let $r = 0$ and

$$t(w) = \left(\sigma\!\left(w - \frac{\omega_1\sqrt{2} - \omega_2}{2\pi i} \right) \middle/ \sigma(w) \right) \cdot e^{((\eta_1\sqrt{2} - \eta_2)/\pi i)w}$$

where $\eta_i = \zeta(\omega_i/2)$ $(i = 1, 2)$. Then, $A_1 = e$, $A_2 = e^{\sqrt{2}}$ and $\gcd(m_i, n_j) = 1$. Thus, G^* is generated by

$$\begin{cases} T_0(z) = z + 2\pi i \\ T_{\omega_1}(z) = z + 1 + 2\pi i k_1 \\ T_{\omega_2}(z) = z + \sqrt{2} + 2\pi i k_2, \qquad k_i \in Z \qquad (i = 1, 2) \end{cases}$$

where the cyclic subgroups generated by T_0, T_{ω_1} and T_{ω_2} are disjoint. Hence, $G^* = C_\infty \times C_\infty \times C_\infty$.

(B) $C_\infty \times C_\infty \times C_l$, for all $l \in Z^+$: Let

$$r = \frac{1}{l} \quad \text{and} \quad t(w) = \left(\sigma\!\left(w - \frac{\omega_1\ln 3 - \omega_2\ln 2}{2\pi i} \right) \middle/ \sigma(w) \right) \cdot e^{((\eta_1\ln 3 - \eta_2\ln 2)/\pi i)w}.$$

Here, $A_1 = 2$, $A_2 = 3$ and $\gcd(m_i, n_j) = 1$. Thus, G^* is generated by

$$\begin{cases} T_0(z) = e^{2\pi i/l}z \\ T_{\omega_1}(z) = 2^{1/l}e^{(2\pi i/l)k_1}z \\ T_{\omega_2}(z) = 3^{1/l}e^{(2\pi i/l)k_2}z, \qquad k_i \in Z \qquad (i = 1, 2) \quad \text{and} \quad 2^{1/l}, 3^{1/l} \in \mathrm{Re}^+. \end{cases}$$

The disjoint cyclic subgroups generated by T_{ω_1} and T_{ω_2} are of infinite order and are both disjoint from the cyclic subgroup of order l generated by T_0. Hence, $G^* = C_\infty \times C_\infty \times C_l$ where $l \in Z^+$ is arbitrary. Note that when $l = 1$, we obtain $G^* C_\infty = C_\infty$.

(C) $C_\infty \times C_l$, for all $l \in Z^+$: Let

$$r = \frac{1}{l} \quad \text{and} \quad t(w) = \left(\sigma\left(w - \frac{\omega_1 \ln 3}{2\pi i}\right) \middle/ \sigma(w) \right) \cdot e^{((\eta_1 \ln 3)/\pi i)w}.$$

Here, $A_1 = 1$, $A_2 = 3$ and $\gcd(m_i, n_j) = 1$. Thus, G^* is generated by

$$\begin{cases} T_0(z) = e^{2\pi i/l} z \\ T_{\omega_1}(z) = e^{2\pi i(k_1/l)} z \\ T_{\omega_2}(z) = 3^{1/l} e^{2\pi i(k_2/l)} z, \quad k_i \in Z \quad (i = 1, 2) \quad \text{and} \quad 3^{1/l} \in \text{Re}^+. \end{cases}$$

$T_{\omega_l} = T_0^{k_1}$, $O(T_0) = l$ and $O(T_{\omega_2}) = \infty$ imply that $G^* = C_\infty \times C_l$. Note that when $l = 1$, we obtain $G^* = C_\infty$.

(D) C_l, for all $l \in Z^+$: Let

$$r = \frac{1}{l} \quad \text{and} \quad t(w) = \left(\sigma^2\left(w - \frac{\omega_1 + \omega_2}{4}\right) \middle/ \sigma\left(w - \frac{\omega_1}{2}\right) \sigma\left(w - \frac{\omega_2}{2}\right) \right).$$

Here, $A_i = 1$ $(i = 1, 2)$ and $\gcd(m_i, n_j) = 1$. Thus, G^* is generated by

$$\begin{cases} T_0(z) = e^{2\pi i/l} z \\ T_{\omega_i}(z) = e^{2\pi i(k_i/l)} z, \quad k_i \in Z \quad (i = 1, 2). \end{cases}$$

Since $T_{\omega_i} = (T_0)^{k_i}$ and $O(T_0) = l$, we conclude that $G^* = C_l$. Note that when $l = 1$, $G^* = id$.

REMARK 4. If $t(w)$ of form (3) is doubly periodic (or equivalently if $A_i = 1$ for $i = 1, 2$), then the arcs Λ_{ω_i} $(i = 1, 2)$ as well as all loops Λ_{a_i} and Λ_{b_i} project under t to closed loops about 0 (or ∞) in $\hat{C} - \{0, \infty\}$. Thus, G^* is a (cyclic) subgroup of the group G of (1) when $\lambda = 1$ and t is as above.

In fact, we can prove

THEOREM 4. *The group G^* of the equation (5) obtained by lifting an arbitrary equation (1) by any non-constant doubly periodic function $t(w)$ ($\lambda = 1$) is a proper subgroup of the monodromy group G of (1) if*

(I) $1 \neq \gcd\left(O(G), m_i, n_j; \begin{matrix} i = 1, \ldots, k \\ j = 1, \ldots, l \end{matrix}\right) = M_1$ *when $O(G)$ is finite or*

$1 \neq \gcd\left(m_i, n_j; \begin{matrix} i = 1, \ldots, k \\ j = 1, \ldots, l \end{matrix}\right) = M_2$ *when $O(G) = \infty$*

and if

(II) *there exists a doubly periodic root function $t^{1/M_2}(w)$.*
In fact, if (I) and (II) are satisfied, then $[G : G^*] = M_1$ or M_2 depending respectively on $O(G) \in Z^+$ or $O(G) = \infty$.

REMARK 5. For a given monodromy group G, mappings t can be constructed for which $[G : G^*]$ in Z^+ is arbitrary. Thus, every subgroup of G is the monodromy group of an equation lifted from (1) by some map t.

Proof. The existence of a doubly periodic root function $t^{1/M_2}(w)$ implies that $t(\Lambda_{\omega_i}) = \Gamma_0^{M_2 k_i}$, $k_i \in Z$ $(i = 1, 2)$, where Γ_0 is homotopic in $\hat{\mathbb{C}} - \{0, \infty\}$ to a simple, closed loop about 0 [7]. Therefore, the generators T_{ω_i} $(i = 1, 2)$ are given by

$$T_{\omega_i}(z) = \begin{cases} e^{2\pi i r M_2 k_i} z & \text{if} \quad r \in \mathbb{C}^* \\ z + 2\pi i M_2 k_i & \text{if} \quad r = 0 \end{cases} \qquad (i = 1, 2).$$

Since $T_{\omega_i} = (T_0)^{k_i}$ where T_0 is defined as in Theorem 3 by

$$T_0(z) = \begin{cases} e^{2\pi i r M_2} z & \text{if} \quad r \in \mathbb{C}^* \\ z + 2\pi i M_2 & \text{if} \quad r = 0, \end{cases}$$

it follows that G^* is generated by T_0 alone.

Furthermore, G is generated by

$$D_0(z) = \begin{cases} e^{2\pi i r} z & \text{if} \quad r \in \mathbb{C}^* \\ z + 2\pi i & \text{if} \quad r = 0. \end{cases}$$

We now determine the relationship between G^* and G by consideration of the following two cases.

Case 1. If $r \in (\mathbb{C} - Q) \cup \{0\}$, then $G = \langle D_0 \rangle$ is free and $G^* = \langle T_0 \rangle = \langle D_0^{M_2} \rangle$ is a free proper subgroup of index M_2 in G. We note that both G^* and G are purely parabolic, elliptic or loxodromic.

Case 2. If $r \in Q - \{0\}$, then $O(G) \in Z^+$ and $r = l/O(G)$ where $l \in Z - \{0\}$ and $\gcd(|l|, O(G)) = 1$. Note that $O(G) \neq 1$. Otherwise, hypothesis I would be contradicted. Therefore, $r \notin Z - \{0\}$ and $G = \langle D_0; D_0^{O(G)} = 1 \rangle$ is not trivial. Also, as in Case 1, G^* is generated by $T_0 = D_0^{M_2}$.

We now determine $[G : G^*]$ and show that G^* is a proper subgroup of G. Observe that $G^* = \langle D_0^{M_2}; D_0^{O(G)} = 1 \rangle$. Since $M_1 = \gcd(O(G), M_2)$, there exist $k_1, k_2 \in Z$ such that $k_1 O(G) + k_2 M_2 = M_1$. Thus, $D_0^{M_1} = D_0^{k_1 O(G) + k_2 M_2} = D_0^{k_2 M_2}$ is an element of G^* which must generate G^* since $M_1 | M_2$. Hence, $G^* = \langle D_0^{M_1}; D_0^{O(G)} = 1 \rangle$. Suppose that there exists $L \in Z - \{0\}$ such that $|L| < M_1$ with $D_0^L \in G^*$. Since $D_0^{M_2}$ generates G^*, there exists $k_3 \in Z - \{0\}$ with $D_0^L = D_0^{k_3 M_2}$. It follows that $L \equiv k_3 M_2 \bmod O(G)$ and there exists $k_4 \in Z$ with $L = k_3 M_2 + k_4 O(G)$. This statement is contradicted by the fact that $M_1 = \gcd(O(G), M_2)$. Therefore, if $D_0^L \in G^*$, then $M_1 | L$. This result together with the facts that $G^* = \langle D_0^{M_1}; D_0^{O(G)} = 1 \rangle$ and $M_1 | O(G)$ imply, by group theory, that $O(G^*) = O(G)/M_1$. Hence, $[G : G^*] = M_1$. Since $M_1 > 1$, G^* is a proper subgroup of G.

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