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SOLUTIONS OF THE tt*-TODA EQUATIONS AND QUANTUM COHOMOLOGY OF MINUSCULE FLAG MANIFOLDS

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Abstract. We relate the quantum cohomology of minuscule flag manifolds to the tt*-Toda equations, a special case of the topological–antitopological fusion equations which were introduced by Cecotti and Vafa in their study of supersymmetric quantum field theories. To do this, we combine the Lie-theoretic treatment of the tt*-Toda equations of Guest–Ho with the Lie-theoretic description of the quantum cohomology of minuscule flag manifolds from Chaput–Manivel–Perrin and Golyshev–Manivel.

§1. Introduction

It is well known that solutions of the two-dimensional Toda equations correspond to primitive harmonic maps into flag manifolds. The tt*-Toda equations provide a special case of the Toda equations; here, the harmonic maps can be regarded as generalizations of variations of Hodge structure (VHSs). Certain special solutions illustrate the mirror symmetry phenomenon: for example, according to Cecotti and Vafa [CV], the generalized VHS for a solution may correspond to the quantum (orbifold) cohomology of a certain Kähler manifold.

To be more precise, there are three aspects of this result. First, it is necessary to establish a bijective correspondence between global solutions on $\mathbb{C}^* = \mathbb{C} - \{0\}$ and their *holomorphic data*. Second, these holomorphic data have to be identified with a flat connection of the type used by Dubrovin in the theory of Frobenius manifolds—we call it the Dubrovin connection. Finally, for certain specific solutions, this has to be identified with the Dubrovin connection associated with the (small) quantum cohomology of a specific Kähler manifold. Guest, Its, and Lin have investigated all three aspects in the case of the Lie group type A_n [GIL].

In [GH], the tt*-Toda equations are described for general complex simple Lie algebras. Guest and Ho obtained a correspondence between solutions and the fundamental Weyl alcove. It is expected (but not yet proved beyond the A_n case) that this gives a bijective correspondence between global solutions and points of (a subset of) the fundamental Weyl alcove.

This paper is a contribution to the second and third aspects of the generalization of [GIL] to the case of general complex simple Lie algebras. That is, we establish a correspondence between the holomorphic data of certain specific solutions of the tt*-Toda equations and the Dubrovin connections of minuscule flag manifolds, based on the Lie-theoretic approach of [GH]. Minuscule flag manifolds are the projectivized weight orbits of minuscule weights (see [CMP]).

The quantum cohomology of flag manifolds has been the subject of many articles, especially from the point of view of quantum Schubert calculus. For Lie-theoretic treatments, we mention in particular [FW]. The minuscule case has been studied in detail in [CMP].

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Golyshev and Manivel [GM] described the quantum cohomology of minuscule flag manifolds in the context of the Satake isomorphism. For geometers, the most familiar example of this is the relation between the cohomology of the Grassmannian and the exterior powers of the cohomology of projective space. A quantum version of this was established in [GM]. It depends on a description of the quantum cohomology of a minuscule flag manifold G/P_{λ_i} in terms of a family of Lie algebra elements denoted by $\sum_{j=1}^{n} e_{-\alpha_j} + qe_\psi$ of the Lie algebra $\mathfrak{g}$ (see §2). Namely, quantum multiplication by the generator of the second cohomology of G/P_{λ_i} coincides with the action of $\sum_{j=1}^{n} e_{-\alpha_j} + qe_\psi$ under the representation whose highest weight is λ_i.

Our main observation is that this element arises from a certain solution of the tt*-Toda equations. In the theory of [GH], this solution corresponds to the origin of the fundamental Weyl alcove. The Dubrovin connection is then $d + (1/\lambda)(\sum_{j=1}^{n} e_{-\alpha_j} + qe_\psi)dq/q$.

As this solution depends only on G, that is, it is independent of the choice of minuscule representation of G, we obtain a relation between the quantum cohomology rings of all minuscule flag manifold G/P_{λ_i} (for fixed G). For Lie groups of types A_n, D_n, and E_6, there are several minuscule weights; thus, in these cases, the same solution of the tt*-Toda equations corresponds to the quantum cohomology of several minuscule flag manifolds. In particular, this means that the tt*-Toda equation gives an explanation for the quantum Satake isomorphism of [GM].

In addition to these tt* aspects, we give more concrete statements and more details of the quantum cohomology results, based on the existing literature. We show directly how the above statement concerning the action of $\sum_{j=1}^{n} e_{-\alpha_j} + qe_\psi$ follows from the quantum Chevalley formula. Unlike the original proof in [GM], a case-by-case argument is not needed for this.

The following are the contents of this paper. First, we review some aspects of the tt*-Toda equations, quantum cohomology, and representation theory. In §2.1, we prepare notation and recall the tt*-Toda equations for general complex simple Lie groups. Then we describe the relationship between a solution and an element of the fundamental Weyl alcove. In §2.2, we review the relations between representations, homogeneous spaces, and cohomology, in particular in the minuscule case. In §2.3, we make some observations on minuscule weight orbits. In §2.4, we give the definition of the Dubrovin connection. In §3, we state the main theorem (Theorem 10) of this paper, which gives an explicit relation between the quantum cohomology of a minuscule flag manifold G/P_{λ_i} and a particular solution of the tt*-Toda equations for G. After that, we give the proof and make some comments on the quantum Satake isomorphism.

§2. Preliminaries

First of all, we prepare some aspects of the tt*-Toda equations. Then we review some representation theory. We discuss minuscule weights and irreducible representations. From the Bruhat decomposition, we can obtain a cell decomposition of the projectivized maximal weight orbit, its cohomology, and its quantum cohomology [FW].

2.1 The tt*-Toda equations

We explain some theory of the tt*-Toda equations. It is possible to obtain local solutions through the DPW construction, and a relationship between the space of local solutions and the fundamental Weyl alcove. For more details, we refer to the article by Guest and Ho [GH].

Let G be a complex, simple, simply connected Lie group of rank n, and let $\mathfrak{g}$ be its Lie algebra. We take a Cartan subalgebra $\mathfrak{h}$ and let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \triangle} \mathfrak{g}_\alpha$ be the root decomposition where $\triangle$ is the set of roots. We choose positive roots $\triangle^+$, and we obtain simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$. We denote the negative roots $-\triangle^+$ by $\triangle^-$. Let $(\, , \,)$ be the Killing form. This Killing form induces a nondegenerate invariant form on $\mathfrak{h}^*$. We also denote this by the same notation $(\, , \,)$. We denote the coroot of α by $\alpha^\vee \in \mathfrak{h}$. This $\alpha^\vee$ corresponds to $\frac{2}{(\alpha,\alpha)}\alpha$ in $\mathfrak{h}^*$. We define an ordering of the roots by $\alpha < \beta$ if $\beta - \alpha$ is positive.

We define H_α by $(H_\alpha, h) = (\alpha, h)$ for all h in $\mathfrak{h}$ where we denote the pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$ by the same notation. Then we obtain a basis $H_{\alpha_1}, \dots, H_{\alpha_n}$ of $\mathfrak{h}$. We may choose basis vectors $e_\alpha \in \mathfrak{g}_\alpha$ such that $(e_\alpha, e_{-\alpha}) = 1$ for all $\alpha \in \triangle$. Then we have

$$[e_\alpha, e_\beta] = \begin{cases} 0, & \text{if } \alpha + \beta \notin \triangle, \\ H_\alpha, & \text{if } \alpha + \beta = 0, \\ N_{\alpha+\beta} e_{\alpha+\beta}, & \text{if } \alpha + \beta \in \triangle - \{0\}, \end{cases}$$

where $N_{\alpha+\beta}$ is a nonzero complex number. We define ϵ_i as the basis of $\mathfrak{h}$ which is dual to α_i, that is, $(\alpha_i, \epsilon_j) = \delta_{i,j}$. We denote the highest root by $\psi := \sum_{j=1}^n q_j \alpha_j$ and the Coxeter number by $s := 1 + \sum_{j=1}^n q_j$.

Fix $d_0, \dots, d_n \in \mathbb{C}^\times$. Let w be a function $w : U \to \mathfrak{h}$ where U is an open subset of $\mathbb{C}$ with coordinate t. Then the Toda equations are

$$2w_{t\bar{t}} = -\sum_{j=1}^n d_j e^{-2\alpha_j(w)} H_{\alpha_j} - d_0 e^{2\psi(w)} H_{-\psi}.$$

If we consider the connection form α,

$$\alpha = (w_t + \frac{1}{\lambda}\tilde{E}_-)dt + (-w_{\bar{t}} + \lambda \tilde{E}_+)d\bar{t} =: \alpha' dt + \alpha'' d\bar{t},$$

where $\tilde{E}_\pm = Ad(e^w)(\sum_{j=1}^n c_j^\pm e_{\pm\alpha_j} + c_0^\pm e_{\mp\psi})$ for $c_i^\pm \in \mathbb{C}^\times$, then the curvature $d\alpha + \alpha \wedge \alpha$ is zero if and only if the Toda equations hold.

Given a real form of $\mathfrak{g}$, the corresponding real form of the Toda equations is defined by imposing two reality conditions: $\alpha_j(w) \in \mathbb{R}$ for all j, and $\alpha'(t, \bar{t}, \lambda) \mapsto \alpha''(t, \bar{t}, 1/\bar{\lambda})$ under the conjugation with respect to the real form.

We add further conditions motivated by the tt* equations. Following Kostant [K], we introduce $h_0 = \sum_{j=1}^n \epsilon_j = \sum_{j=1}^n r_j H_{\alpha_j}$, $e_0 = \sum_{j=1}^n a_j e_{\alpha_j}$, and $f_0 = \sum_{j=1}^n (r_j/a_j) e_{-\alpha_j}$, where $r_j \in \mathbb{R}^\times$ and $a_j \in \mathbb{C}^\times$. Since these generators satisfy the conditions $[h_0, e_0] = e_0$, $[h_0, f_0] = -f_0$, and $[e_0, f_0] = h_0$, this subalgebra is isomorphic to $\mathfrak{sl}_2\mathbb{C}$. We can decompose $\mathfrak{g}$ according to the adjoint action by this subalgebra, and then we obtain highest weight vectors u_j of irreducible subrepresentations V_j of $\mathfrak{g} = \bigoplus_j V_j$.

We use the standard compact real form ρ which satisfies

$$\rho(e_\alpha) = -e_{-\alpha}, \ \rho(H_\alpha) = -H_\alpha,$$

for all $\alpha \in \triangle$. By Hitchin [Hit], we have a $\mathbb{C}$-linear involution $\sigma : \mathfrak{g} \to \mathfrak{g}$ defined by

$$\sigma(u_j) = -u_j, \ \sigma(f_0) = -f_0 \ (0 \le j \le n).$$

Using ρ and σ, we define

$$\chi := \sigma\rho.$$

Then it can be shown that $\sigma\rho = \rho\sigma$ [Hit] and that this χ defines a split real form $\mathfrak{g}_{\mathbb{R}}$.

DEFINITION 1 (The tt*-Toda equations). The tt*-Toda equations are the Toda equations for $w : \mathbb{C}^\times \to \mathfrak{h}$ together with:

(R) the above reality conditions (with respect to χ),

(F) $\sigma(w) = w$ (Frobenius condition), and

(S) $w = w(|t|)$ (similarity condition).

From (R), it follows that w takes values in $\mathfrak{h}_\sharp = \bigoplus_{j=1}^n \mathbb{R}H_{\alpha_j}$.

REMARK 2. It is known that σ is the identity on $\mathfrak{h}$ unless $\mathfrak{g}$ is of type A_n, D_{2n+1}, or E_6. Thus, the Frobenius condition on w is nontrivial only for these three types.

EXAMPLE 1. The tt*-Toda equations of A_n type (see [GH, Example 3.11] or [GIL]) are

$$2(w_i)_{t\bar{t}} = -d_{i+1}e^{2(w_{i+1}-w_i)} + d_i e^{2(w_i - w_{i-1})},$$

for $i = 0, 1, \ldots, n$, where $w_{n+1} = w_0$ and we assume $\sum_{i=0}^n w_i = 0$ and where all $d_i > 0$ and $d_i = d_{n-i+1}$. The Frobenius condition is $w_i + w_{n-i} = 0$ for $i = 0, \ldots, n$. We consider $w_i = w_i(|t|)$.

By the well-known DPW construction (see [GH], [GIL]), it is possible to construct a local solution w near $t = 0$ from the connection form

$$\omega = \frac{1}{\lambda}\left(\sum_{j=1}^n z^{k_j} e_{-\alpha_j} + z^{k_0} e_\psi\right) dz$$

(i.e., from any $k_0, \ldots, k_n \geq -1$). Here, z is a complex variable related to t by

$$t = s z^{\frac{1}{s}}.$$

This solution satisfies

$$w(|t|) \sim -m \log|t|,$$

as $t \to 0$, where $m \in \mathfrak{h}_\sharp$ is defined by

$$\alpha_j(m) = \frac{s}{N}(k_j + 1) - 1,\ 1 \leq j \leq n,$$

where $N = s + \sum_{i=0}^n k_i$. In fact, the converse is true.

PROPOSITION 3 [GH]. *Let $m \in \mathfrak{h}_\sharp$. There exists a local solution near zero of the tt*-Toda equations such that $w(|t|) \sim -m \log|t|$ as $t \to 0$ if and only if $\alpha_j(m) \geq -1$ for $j = 0, \ldots, n$.*

The condition $\alpha_j(m) \geq -1$, for $j = 0, \ldots, n$, is equivalent to the condition defining the fundamental Weyl alcove $\mathfrak{A} = \{x \in \sqrt{-1}\mathfrak{h}_\sharp |\ \alpha_j^{\mathrm{real}}(x) \geq 0,\ \psi^{\mathrm{real}}(x) \leq 1\}$. This gives the following theorem.

THEOREM 4 [GH]. *We have a bijective map between:*

(a) the space of asymptotic data $\mathcal{A} = \{m \in \mathfrak{h}_\sharp |\ \alpha_j(m) \geq -1,\ j = 0, \ldots, n\}$ when $G \neq A_n, D_{2m+1}, E_6$ (or the set $\mathcal{A}^\sigma = \{m \in \mathcal{A} |\ \sigma(m) = m\}$ when $G = A_n, D_{2m+1}, E_6$) and

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(b) the fundamental Weyl alcove $\mathfrak{A}$ *(or* $\mathfrak{A}^\sigma = \{x \in \mathfrak{A} | \sigma(x) = x\}$*) defined by*

$$\mathcal{A} \to \mathfrak{A},\ m \mapsto \frac{2\pi\sqrt{-1}}{s}(m + h_0),\ \ (or\ \mathcal{A}^\sigma \to \mathfrak{A}^\sigma).$$

2.2 Minuscule weights and homogeneous spaces

We review some properties of minuscule weights. We refer to the article [CMP]. For a simple complex Lie algebra, we define the weight lattice I as the $\mathbb{Z}$-module spanned by $\lambda_1, \dots, \lambda_n$ where λ_i is defined by $(\lambda_i, \alpha_j^\vee) = \delta_{ij}$. These λ_i are called the fundamental weights.

DEFINITION 5. We call a weight λ a dominant weight if $(\lambda, \alpha_i^\vee) > 0$ for all $\alpha_i \in \Pi$. We call a dominant weight λ a minuscule weight if $(\lambda, \alpha^\vee) \leq 1$ for all $\alpha \in \triangle^+$.

It is well known that the set of the minuscule weights is a subset of the fundamental weights. We summarize the minuscule weights for each type of Lie group at the end of §2.2.

By the Borel–Weil theorem, we can obtain an irreducible representation V_{λ_i} from each fundamental weight λ_i. When we consider the projective representation $\mathbb{P}(V_{\lambda_i})$, we obtain the homogeneous space

$$G/P_{\lambda_i} \cong G \cdot [v_{\lambda_i}] \subset \mathbb{P}(V_{\lambda_i}),$$

where v_{λ_i} is a highest weight vector and P_{λ_i} is the stabilizer group of $[v_{\lambda_i}]$. Here, P_{λ_i} is a parabolic subgroup, and we denote P_{λ_i} by P_i.

We denote the weight orbit of λ_i by $W \cdot \lambda_i$. That is, $W \cdot \lambda_i = \{x(\lambda_i) | \, x \in W\}$. When we write x as a product of simple reflections, we denote by $\ell(x)$ the minimal number of such reflections. The following fact holds for any parabolic subgroup P of G. Let $\triangle_P$ be the subset of $\triangle$ such that $\mathrm{Lie}(P) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \triangle_P} \mathfrak{g}_\alpha$. We denote the subset of the simple roots which belong to $\triangle_P$ by Π_P. Let W_P be the subgroup of W generated by the corresponding simple reflections.

PROPOSITION 6 (See §1.10 in [Hu]). *For* $x \in W$*, there exist unique elements* $u \in W^P$ *and* $v \in W_P$ *such that*

$$x = uv,$$

where $W^P = \{x \in W | \, \ell(xs_\alpha) > \ell(x)\ {}^\forall \alpha \in \Pi_P\}$.

By this fact, u is a representative of $[x] \in W/W_P$. We have $W \cdot \lambda_i = W^{P_i} \cdot \lambda_i$.

We consider the cohomology ring of G/P_i. The following fact is well known.

THEOREM 7 ((Bruhat decomposition) [Hil]). *For a parabolic subgroup* P *of* G*, we have a decomposition*

$$G = \coprod_{u \in W^P} BuP.$$

Here, we regard the elements of W as the elements of G by the isomorphism $W \cong N(T)/T$ where T is a maximal torus. We define the Schubert varieties of G/P_i by $X_u := \overline{BuP_i/P_i}$. We also define the opposite Schubert varieties by $Y_u := \overline{x_0 B x_0 u P_i/P_i} = x_0 X_{x_0 u}$ where x_0 is the longest element of W. Then $[Y_u] \in H_{2n-2\ell(u)}(G/P_i)$, and these classes form an additive basis. By the Poincaré duality theorem, we have a basis of $H^{2\ell(u)}(G/P_i)$. We denote this generator by σ_u.

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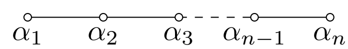
Now, we obtain the correspondence between $W^{P_i} \cdot \lambda_i$ and an additive basis of the cohomology $H^*(G/P_i)$ by

$$u(\lambda_i) \longleftrightarrow \sigma_u.$$

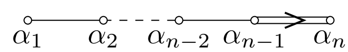
In the following table of fundamental weights, the minuscule weights are marked.

$A_n\ (n \geq 1):$



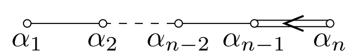
Fund. weight	λ_1	λ_2	λ_3		λ_{n-1}	λ_n
Minuscule	✓	✓	✓		✓	✓

$B_n\ (n \geq 2):$



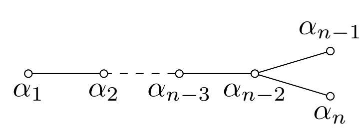
Fund. weight	λ_1	λ_2		λ_{n-2}	λ_{n-1}	λ_n
Minuscule						✓

$C_n\ (n \geq 3):$



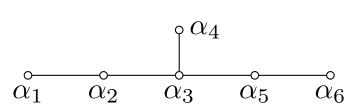
Fund. weight	λ_1	λ_2		λ_{n-2}	λ_{n-1}	λ_n
Minuscule	✓					

$D_n\ (n \geq 4):$



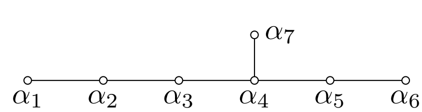
Fund. weight	λ_1	λ_2		λ_{n-3}	λ_{n-2}	λ_{n-1}	λ_n
Minuscule	✓					✓	✓

$E_6:$



Fund. weight	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
Minuscule	✓					✓

$E_7:$



Fund. weight	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
Minuscule	✓						

It is known that G_2, F_4, and E_8 have no minuscule weight. G/P_{λ_i} can be described conveniently as a quotient of compact groups as follows.

$$(A_n \text{ case})\ G/P_i \cong SU(n+1)/S(U(i)\times U(n+1-i)) \cong Gr(k,n+1).$$
$$(B_n \text{ case})\ G/P_n \cong SO(2n+1)/U(n) \cong OG(n,2n+1).$$
$$(C_n \text{ case})\ G/P_1 \cong Sp(n)/(U(1)\times Sp(n-1)) \cong \mathbb{C}P^{2n-1}.$$
$$(D_n \text{ case})\ G/P_1 \cong SO(2n)/(U(1)\times SO(2n-2)) \cong Q_{2n-2},$$
$$G/P_{n-1} \cong SO(2n)/U(n) \cong S_+, G/P_n \cong SO(2n)/U(n) \cong S_-.$$
$$(E_6 \text{ case})\ G/P_1 \cong G/P_6 \cong E_6/(SO(10)\times U(1)) \cong \mathbb{O}P^2.$$
$$(E_7 \text{ case})\ G/P_1 \cong E_7/(E_6\times U(1)).$$

Here, $OG(k,n)$ is the set of k-dimensional isotropic subspaces of n-dimensional complex vector space V with a nondegenerate quadratic form. This is called the orthogonal Grassmannian. For D_n, $OG(n,2n)$ has two components S_+ and S_-. These are called varieties of pure spinors (or spinor varieties), and these are isomorphic to each other [Ma]. For A_n, B_n, C_n, and D_n, the minuscule representations are familiar (see §6.5 in [BD]). For A_n, V_{λ_i} is the exterior power $\bigwedge^i V_{\lambda_1}$ $(1 \le i \le n)$ where V_{λ_1} is the standard representation on $\mathbb{C}^{n+1}$. For B_n, V_n is the half-spin representation. For C_n, V_{λ_1} is the standard representation on $\mathbb{C}^{2n}$. For D_n, V_{λ_1} is the standard representation on $\mathbb{C}^{2n}$. $V_{\lambda_{n-1}}$ and V_{λ_n} are the half-spin representations. We denote these two representations by Δ_+ and Δ_-. For exceptional groups, the minuscule representations are given in §5 of [Ge]. For E_6, V_{λ_1} and V_{λ_6} are 27-dimensional representations. For E_7, V_{λ_1} is a 56-dimensional representation.

2.3 Minuscule weight orbits and simple roots

In §2.3, we observe relationships between minuscule weight orbits and the simple roots. Let λ_i be a minuscule weight.

PROPOSITION 8. *The set of all weights of V_{λ_i} is the W-orbit of λ_i, and the multiplicities of all weights of V_{λ_i} are 1.*

Proof. It is obvious that $\sharp W/W_{P_i} \le \dim(W\cdot v_{\lambda_i}) \le \dim(V_{\lambda_i})$. If there is a weight which has multiplicity more than 1, then $\sharp W/W_{P_i} < \dim V_{\lambda_i}$. Therefore, by contraposition, when we show that $\sharp W/W_{P_i}$ coincides with $\dim_{\mathbb{C}} V_{\lambda_i}$, we obtain the statement of Proposition 8.

We justify the above claim in each case. We have the orders of all Weyl groups from Table 2 in §2.11 of [Hu]. For type A_n, we have $\dim_{\mathbb{C}} \bigwedge^i \mathbb{C}^{n+1} = \binom{n+1}{i}$ $(1 \le i \le n)$. On the other hand, for this representation, we have $W/W_{P_i} = \mathfrak{S}_{n+1}/(\mathfrak{S}_i \times \mathfrak{S}_{n+1-i})$. Therefore, we obtain $\sharp W/W_{P_i} = \frac{(n+1)!}{i!(n+1-i)!} = \binom{n+1}{i}$. For type B_n, a minuscule representation is the half-spin representation and its dimension is 2^n. Then $W/W_{P_n} = \mathfrak{S}_n \cdot (\mathbb{Z}_2)^n/\mathfrak{S}_n$. Hence, $\sharp W/W_{P_n} = 2^n \cdot n!/n! = 2^n$. For type C_n, a minuscule representation is the standard representation $\mathbb{C}^{2n}$ and its dimension is $2n$. The corresponding $W/W_{P_1} = \mathfrak{S}_n \cdot (\mathbb{Z}_2)^n/\mathfrak{S}_{n-1}\cdot(\mathbb{Z}_2)^{n-1}$. Hence, $\sharp W/W_{P_1} = 2^n \cdot n!/2^{n-1}\cdot(n-1)! = 2n$. For type D_n, there are three minuscule representations. These are the standard representations and the two half-spin representations. These dimensions are $2n$, 2^{n-1}, and 2^{n-1}, respectively. The corresponding W/W_{P_i} $(i = 1, n-1, n)$ are $\mathfrak{S}_n \cdot (\mathbb{Z}_2)^{n-1}/\mathfrak{S}_{n-1}\cdot(\mathbb{Z}_2)^{n-2}$, $\mathfrak{S}_n\cdot(\mathbb{Z}_2)^{n-1}/\mathfrak{S}_n$, and $\mathfrak{S}_n\cdot(\mathbb{Z}_2)^{n-1}/\mathfrak{S}_n$, and $\sharp W/W_{P_i}(i = 1, n-1, n)$ are $2n$, 2^{n-1}, and 2^{n-1}, respectively. For type E_6, there are two minuscule representations. These representations are both 27-dimensional representations.

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The corresponding W/W_{P_1} and W/W_{P_6} are both $W_{E_6}/\mathfrak{S}_5\cdot(\mathbb{Z}_2)^4$ where W_{E_6} is the Weyl group of E_6. Then $\sharp W_{E_6}/\mathfrak{S}_5\cdot(\mathbb{Z}_2)^4 = 2^7\cdot 3^4\cdot 5/2^4\cdot 5! = 27$. For type E_7, the minuscule representation is a 56-dimensional representation. The corresponding W/W_{P_1} is W_{E_7}/W_{E_6} where W_{E_7} is the Weyl group of E_7. Then $\sharp W/W_{P_1} = 2^{10}\cdot 3^4\cdot 5\cdot 7/2^7\cdot 3^4\cdot 5 = 56$. This completes the proof. □

From Proposition 8, we have the weights of V_{λ_i} as $\{v_{u(\lambda_i)} \mid u\in W^{P_i}\}$ and the multiplication of these weights are all one. In addition, we know that the Weyl group is generated by the simple reflections $\{s_{\alpha_j} \mid j\in\{1,\ldots,n\}\}$. Therefore, all weights can be obtained from λ_i by applying $\{s_{\alpha_j} \mid j\in\{1,\ldots,n\}\}$ to λ_i repeatedly. We use a canonical basis of V_{λ_i} from §5A.1 of the article [J] with the following properties:

$$e_{-\alpha_j}(v_{u(\lambda_i)}) = \begin{cases} v_{u(\lambda_i)-\alpha_j}, & (u(\lambda_i),\alpha_j^\vee)=1,\\ 0, & \text{otherwise,}\end{cases} \tag{2.1}$$

$$e_{\alpha_j}(v_{u(\lambda_i)}) = \begin{cases} v_{u(\lambda_i)+\alpha_j}, & (u(\lambda_i),\alpha_j^\vee)=-1,\\ 0, & \text{otherwise,}\end{cases} \tag{2.2}$$

$$H_{\alpha_j}(v_{u(\lambda_i)}) = (u(\lambda_i),\alpha_j^\vee)v_{u(\lambda_i)},$$

for all weights $u(\lambda_i)$ and all $j\in\{1,\ldots,n\}$. As a consequence of (2.2), we have

$$e_{\psi}(v_{u(\lambda_i)}) = \begin{cases} v_{u(\lambda_i)+\psi}, & (u(\lambda_i),\psi^\vee)=-1,\\ 0, & \text{otherwise.}\end{cases} \tag{2.3}$$

2.4 Dubrovin connection

We consider the minuscule flag manifolds G/P_i. Then $H^*(G/P_i)$ is given by the Bruhat decomposition (see Theorem 7). We have $\Pi\backslash\Pi_i = \{\alpha_i\}$, so there is only one element s_{α_i} which satisfies $\ell(u)=1$ in W^{P_i}. Therefore, $H^2(G/P_i)\cong\mathbb{C}$. We consider the quantum product by the second cohomology, that is, $\sigma_{s_{\alpha_i}}\circ_q$ where q is a nonzero complex number. Then we define the Dubrovin connection.

Definition 9. The Dubrovin connection on the trivial vector bundle $H^2(M;\mathbb{C})\times H^*(M;\mathbb{C})\to H^2(M;\mathbb{C})$ is defined by

$$\nabla = d + \frac{1}{\lambda}(\sigma_{s_{\alpha_i}}\circ_q)\frac{dq}{q}.$$

We seek flag manifolds whose Dubrovin connection forms are of the form $\omega = \frac{1}{\lambda}(\sum_{j=1}^n q^{k_j}e_{-\alpha_j} + q^{k_0}e_\psi)dq$. As we see in §3, we can use the quantum Chevalley formula to calculate $\sigma_{s_{\alpha_i}}$.

§3. Results

For any minuscule weight λ_i, the discussion in §2.2 establishes an isomorphism

$$V_{\lambda_i} = \bigoplus_{u(\lambda_i)\in W^P\cdot\lambda_i} V_{u(\lambda_i)} \cong H^*(G/P_i;\mathbb{C}).$$

We remark that, from §2.3, this isomorphism is given by

$$V_{u(\lambda_i)} \ni v_{u(\lambda_i)} \leftrightarrow \sigma_u \in H^{2\ell(u)}(G/P_i;\mathbb{C})$$

for all $u \in W^{P_i}$. From this, it can be seen that the cohomology grading on the right corresponds to the grading by simple roots on the left.

Now, we can state our main theorem.

THEOREM 10. *Fix $\mathfrak{g}$ and a minuscule weight λ_i. There is a natural correspondence between (i) the asymptotic data*

$$m = -h_0 = -\sum_{j=1}^{n} r_j H_{\alpha_j} \in \mathfrak{h}_\sharp$$

and (ii) the DPW data

$$\omega = \frac{1}{\lambda}\left(\sum_{j=1}^{n} e_{-\alpha_j} + q e_\psi\right)\frac{dq}{q}$$

for solutions of the tt-Toda equations. The asymptotic data correspond to a unique global solution when $\mathfrak{g}$ has type A_n (and conjecturally for any $\mathfrak{g}$). The holomorphic data correspond to the Dubrovin connection for the quantum cohomology of G/P_i, that is, the natural action of $\sum_{j=1}^{n} e_{-\alpha_j} + q e_\psi$ corresponds to quantum multiplication by a generator of $H^2(G/P_i, \mathbb{C})$.*

Proof. In the bijection of Theorem 4 (§2.1), we see that $m = -h_0$ corresponds to the origin of the fundamental Weyl alcove, and in this case, we have $k_0 = 0$ and $k_1 = \cdots = k_l = -1$. This gives the correspondence between (i) and (ii) (with $q = z$). For the statement concerning global solutions in the A_n case, we refer to [GIL], [Mo]. The identification of ω with the Dubrovin connection can be extracted from [GM], but we present a new[1] and more direct proof here, using the quantum Chevalley formula.

THEOREM 11 [FW]. *For $\beta \in \Pi \backslash \Pi_{P_i}$ and $u \in W^{P_i}$, we have the quantum product $\circ$ by σ_β as*

$$\begin{aligned}\sigma_{s_\beta} \circ \sigma_u = &\sum_{\ell(us_\alpha)=\ell(u)+1} (\lambda_\beta, \alpha^\vee)\sigma_{us_\alpha} \\ &+ \sum_{l(us_\alpha)=l(u)-n_\alpha+1} (\lambda_\beta, \alpha^\vee)\sigma_{us_\alpha} \cdot q^{d(\alpha)},\end{aligned}$$

where α ranges over $\triangle^+ \backslash \triangle^+_{P_i}$, λ_β is the fundamental weight corresponding to β,

$$n_\alpha = \left(\sum_{\gamma \in \triangle^+ \backslash \triangle^+_{P_i}} \gamma, \alpha^\vee\right),$$

and

$$d(\alpha) = \sum_{\beta \in \Pi \backslash \Pi_{P_i}} (\lambda_\beta, \alpha^\vee)\sigma(s_\beta),$$

and where $\sigma(s_\beta)$ is the homology class of $H_2(G/P_i)$ which is Poincaré dual to σ_{s_β}. ∎

[1] After finishing the first draft of this paper, we found essentially the same proof is given in [LT]. We note some differences between the proof given here and [LT] in Remark 15.

In our situation, $\Pi\backslash\Pi_{P_i} = \{\alpha_i\}$. Therefore, the generator of the second cohomology is only $\sigma_{s_{\alpha_i}}$ and $\lambda_\beta = \lambda_i$. We have $d(\alpha) = (\lambda_i, \alpha^\vee)\sigma(s_{\alpha_i}) = \sigma(s_{\alpha_i})$ for $\alpha \in \triangle^+\backslash\triangle^+_{P_i}$ because λ_i is a minuscule weight. We consider $q^{\sigma(s_\beta)}$ only as a complex parameter q in $\mathbb{C}$.

From Lemma 3.5 in [FW], the first Chern class of G/P_i is n_α times a generator of $H^2(G/P_i)$, and by [CMP], we know that n_α is the Coxeter number s. Explicitly, we have $n_\alpha = n+1$ (A_n type), $n_\alpha = 2n$ (B_n type), $n_\alpha = 2n$ (C_n type), $n_\alpha = 2n-2$ (D_n type), $n_\alpha = 12$ (E_6 type), and $n_\alpha = 18$ (E_7 type) for all $\alpha \in \triangle^+\backslash\triangle^+_{P_i}$.

Then we have the quantum Chevalley formula as follows:

$$\sigma_{s_{\alpha_i}} \circ \sigma_u = \sum_{\ell(us_\alpha)=\ell(u)+1} (\lambda_i, \alpha^\vee)\sigma_{us_\alpha} + \sum_{\ell(us_\alpha)=\ell(u)-(s-1)} (\lambda_i, \alpha^\vee)\sigma_{us_\alpha} \cdot q,$$

where $\alpha \in \triangle^+\backslash\triangle^+_{P_i}$.

To replace the conditions of these summations, the following lemma, corollary, and proposition are key ingredients.

LEMMA 12. *Let λ_i be a minuscule weight. For $u \in W^{P_i}$ and $\alpha \in \Pi$, we have the three following situations.*

(I) $(u(\lambda_i), \alpha^\vee) = 1 \Leftrightarrow \ell(s_\alpha u) = \ell(u)+1$.

(II) $(u(\lambda_i), \alpha^\vee) = 0 \Leftrightarrow \ell(s_\alpha u) = \ell(u)$.

(III) $(u(\lambda_i), \alpha^\vee) = -1 \Leftrightarrow \ell(s_\alpha u) = \ell(u)-1$.

Here, we consider the length function $l(u)$ in W^{P_i}.

Proof. (a) First, we show the implication ($\Rightarrow$), for each of (I)–(III).

(II) We assume $(u(\lambda_i), \alpha^\vee) = 0$. We show $s_{u^{-1}(\alpha)} \in W_{P_i}$. Let $u^{-1}(\alpha)^\vee = \sum_{i=1}^n b_i\alpha_i^\vee$ ($b_i \in \mathbb{R}$). Then we have

$$(\lambda_i, u^{-1}(\alpha)^\vee) = b_i = 0.$$

Therefore. $u^{-1}(\alpha) \in \triangle_{P_i}$ and $s_{u^{-1}(\alpha)} \in W_{P_i}$. We obtain

$$\ell(s_\alpha u) = \ell(us_{u^{-1}(\alpha)}) = \ell(u) \text{ in } W^{P_i}.$$

(I) and (III) Notice that $w \notin W^{P_i}$ if and only if there exists $\beta \in \Pi_{P_i}$ such that $w(\beta)$ is a negative root. Since $u \in W^{P_i}$, we have $u(\beta)$ is a positive root for all $\beta \in \Pi_{P_i}$. So, if $s_\alpha u(\beta)$ is a negative root, then $s_\alpha u(\beta) = -\alpha$ because s_α preserves $\triangle^-\backslash\{-\alpha\}$. Then it must hold that $\beta = u^{-1}(\alpha) \in \Pi_{P_i}$.

We assume $(u(\lambda_i), \alpha^\vee) \neq 0$. Then we have $b_i \neq 0$ for $u^{-1}(\alpha)^\vee = \sum_{i=1}^n b_i\alpha_i^\vee$ as the same way of (II). Thus, $u^{-1}(\alpha)$ is not in Π_{P_i}. Therefore, $s_\alpha u(\beta)$ is a positive root for all $\beta \in \Pi_{P_i}$. So we have $s_\alpha u \in W^{P_i}$.

If $(u(\lambda_i), \alpha^\vee) = 1$, then $(\lambda_i, u^{-1}(\alpha)^\vee) = 1$ and $u^{-1}(\alpha)$ is a positive root. Therefore, $\ell(s_\alpha u) = \ell(u)+1$ in W^{P_i} (see §1.6 in [Hu]). If $(u(\lambda_i), \alpha^\vee) = -1$, then $(\lambda_i, u^{-1}(\alpha)^\vee) = -1$. $u^{-1}(\alpha)$ is a negative root. Thus, we obtain $\ell(s_\alpha u) = \ell(u)-1$ in W^{P_i} (see also §1.6 in [Hu]).

(b) Next, we show the implication ($\Leftarrow$), for each of (I)–(III). For (I), we assume $\ell(s_\alpha u) = \ell(u)+1$. Since λ_i is minuscule, $(u(\lambda_i), \alpha^\vee)$ takes only the values $1, 0$, and -1. If $(u(\lambda_i), \alpha^\vee)$ is 0 or -1, we obtain a contradiction, by part (a). The proofs in the cases (II) and (III) are similar. ∎

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Now, we have the weights of V_{λ_i} as $\lambda_i - \sum_{j=1}^n n_j\alpha_j$ where $n_j \in \mathbb{Z}_{\geq 0}$. From Lemma 12, we obtain the following corollary.

Corollary 13. *For $u \in W^{P_i}$ such that $u(\lambda_i) = \lambda_i - \sum_{j=1}^n n_j\alpha_j$, we have $\ell(u) = \sum_{j=1}^n n_j$.*

Proof. We have

$$\begin{aligned}\ell(s_{\alpha_j}u) = \ell(u)+1 &\Leftrightarrow (u(\lambda_i),\alpha_j^\vee) = 1\\ &\Leftrightarrow s_{\alpha_j}(u(\lambda_i)) = u(\lambda_i) - \alpha_j\end{aligned}$$

by Lemma 12. The elements of W^{P_i} are described by a product of simple reflections. Thus, $\ell(u) = \sum_{j=1}^n n_j$. ∎

We have the following proposition.

Proposition 14. *(I) If there exist $\alpha \in \triangle^+$ such that $\ell(s_\alpha u) = \ell(u)+1$ for $u \in W^{P_i}$, then $\alpha \in \Pi$ and $(u(\lambda_i),\alpha^\vee) = 1$.*

(II) If there exist $\alpha \in \triangle^+$ such that $\ell(s_\alpha u) = \ell(u)-(s-1)$ for $u \in W^{P_i}$, then $\alpha = \psi$ and $(u(\lambda_i),\psi^\vee) = -1$.

Proof. (I) For $\alpha \in \triangle^+$ such that $\ell(s_\alpha u) = \ell(u)+1$, we have

$$s_\alpha u(\lambda_i) = u(\lambda_i) - (u(\lambda_i),\alpha^\vee)\alpha.$$

By the assumption that $\ell(s_\alpha u) > \ell(u)$, we have $(u(\lambda_i),\alpha^\vee) = 1$ and α must be a simple root by Corollary 13.

(II) For $\alpha \in \triangle^+$ such that $\ell(s_\alpha u) = \ell(u)-(s-1)$, we have

$$s_\alpha u(\lambda_i) = u(\lambda_i) - (u(\lambda_i),\alpha^\vee)\alpha.$$

By the assumption $\ell(s_\alpha u) < \ell(u)$, we have $(u(\lambda_i),\alpha^\vee) = -1$. When $\alpha = \sum_{j=1}^n q_j\alpha_j$, then α must be ψ because there is only one positive root which has the height $\sum_{j=1}^n q_j = s-1$. ∎

By using the relation $us_\alpha = s_{u(\alpha)}u = s_{-u(\alpha)}u$, Corollary 13, and Proposition 14, we can replace the conditions of the summation in the quantum Chevalley formula.

We show that we can simplify the first summation to

$$\sum_{(u(\lambda_i),\alpha'^\vee)=1,\alpha'\in\Pi} \sigma_{s_{\alpha'}u}$$

by setting $\alpha' = u(\alpha)$. Then we show that α' is a positive root. In fact, if α' is a negative root, then $(u(\lambda_i),\alpha'^\vee) = -1$ satisfies $\ell(s_{\alpha'}u) = \ell(u)+1$. However, this contradicts $\alpha \in \triangle^+\backslash\triangle^+_{P_i}$ because we have

$$(u(\lambda_i),\alpha'^\vee) = -1 \Leftrightarrow (u(\lambda_i),u(\alpha^\vee)) = -1 \Leftrightarrow (\lambda_i,\alpha^\vee) = -1.$$

Thus, α' is in $\triangle^+$. By Proposition 14, we have $\alpha' \in \Pi \subset \triangle^+$. Hence, we have

$$\sum_{\ell(us_\alpha)=\ell(u)+1} (\lambda_i,\alpha^\vee)\sigma_{us_\alpha} = \sum_{(u(\lambda_i),\alpha'^\vee)=1,\alpha'\in\Pi} \sigma_{s_{\alpha'}u}$$

as the first summation of $\sigma_{s_{\alpha_i}} \circ \sigma_u$.

For the second summation, let $\alpha' = -u(\alpha)$. Then we show that α' is also a positive root. In fact, if α' is a negative root, then $(u(\lambda_i),\alpha'^\vee) = 1$ satisfies $\ell(s_{\alpha'}u) = \ell(u)-(s-1)$.

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However, this contradicts $\alpha \in \triangle^+ \backslash \triangle^+_{P_i}$ because we have

$$(u(\lambda_i), \alpha'^\vee) = 1 \Leftrightarrow (u(\lambda_i), -u(\alpha^\vee)) = 1 \Leftrightarrow (\lambda_i, \alpha^\vee) = -1.$$

Thus, $\alpha' = -u(\alpha)$ is in $\triangle^+$ for $\alpha \in \triangle^+ \backslash \triangle^+_{P_i}$. By Proposition 14, we have $\alpha' = \psi$ and $(u(\lambda_i), \psi^\vee) = -1$. Hence, for the second summation of $\sigma_{s_{\alpha_i}} \circ \sigma_u$, we have

$$\begin{aligned}
&\sum_{\ell(us_\alpha) = \ell(u) - (s-1)} (\lambda_i, \alpha^\vee) \sigma_{us_\alpha} \cdot q \\
&= \sum_{\ell(s_{\alpha'} u) = \ell(u) - (s-1)} (\lambda_i, -u^{-1}(\alpha'^\vee)) \sigma_{s_{\alpha'} u} \cdot q \\
&= \begin{cases} q\sigma_{s_\psi u}, & (u(\lambda_i), \psi^\vee) = -1, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Thus, we obtain

$$\sigma_{s_{\alpha_i}} \circ \sigma_u = \begin{cases} \displaystyle\sum_{(u(\lambda_i), \alpha_j^\vee) = 1} \sigma_{s_{\alpha_j} u} + q\sigma_{s_\psi u}, & (u(\lambda_i), \psi^\vee) = -1, \\ \displaystyle\sum_{(u(\lambda_i), \alpha_j^\vee) = 1} \sigma_{s_{\alpha_j} u}, & \text{otherwise.} \end{cases}$$

On the other hand, for $v_{u(\lambda_i)}$, we have

$$\begin{aligned}
&(\sum_{j=1}^{n} e_{-\alpha_j} + q e_\psi) \cdot v_{u(\lambda_i)} \\
&= \begin{cases} \displaystyle\sum_{(u(\lambda_i), \alpha_j^\vee) = 1} v_{u(\lambda_i) - \alpha_j} + q v_{u(\lambda_i) + \psi}, & (u(\lambda_i), \psi^\vee) = -1, \\ \displaystyle\sum_{(u(\lambda_i), \alpha_j^\vee) = 1} v_{u(\lambda_i) - \alpha_j}, & \text{otherwise,} \end{cases} \\
&= \begin{cases} \displaystyle\sum_{(u(\lambda_i), \alpha_j^\vee) = 1} v_{s_{\alpha_j} u(\lambda_i)} + q v_{s_\psi u(\lambda_i)}, & (u(\lambda_i), \psi^\vee) = -1, \\ \displaystyle\sum_{(u(\lambda_i), \alpha_j^\vee) = 1} v_{s_{\alpha_j} u(\lambda_i)}, & \text{otherwise,} \end{cases}
\end{aligned}$$

by using the definitions of (2.1) and (2.3). Therefore, we obtain

$$\sum_{j=1}^{n} e_{-\alpha_j} + q e_\psi = \sigma_{s_{\alpha_i}} \circ .$$

This completes the proof of Theorem 10.

REMARK 15. We note some differences between the proof given here and Proposition 4.9 in [LT]. They mainly use Proposition 6.1 by Gross [Gr] and Lemma 5.3 by Stembridge [S]. However, Stembridge shows that there are only simple roots which satisfy the condition of the first summation of the quantum Chevalley formula by using the idea of *fully commutative elements*. We show this directly by using the minuscule condition and considering the length of $w \in W^{P_i}$.

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REMARK 16 (On the Satake isomorphism). When $\mathfrak{g}$ is of type A_n (or, conjecturally, of types D_n and E_6), the same global solution corresponds to the Dubrovin connection of any minuscule weight. This suggests a relation between the quantum cohomology algebras of the corresponding flag manifolds. In the A_n case, this can be stated as

$$\bigwedge^k QH^*(\mathbb{C}P^n) \cong QH^* Gr(k, n+1)$$

(see [GM] for further explanation).

In the D_n case, the analogous relation is

$$\bigwedge_{\pm}^{half} QH^*(Q_{2n-2}) \cong \mathrm{End}_{\mathbb{C}}(QH^*(S_\pm)). \tag{3.1}$$

This follows from Theorem 10 when we identify $H^*(Q_{2n-2};\mathbb{C})$ with $\mathbb{C}^{2n}$ and $H^*(S_\pm;\mathbb{C})$ with $\Delta_\pm$, because (3.1) corresponds to the well-known relation

$$\bigwedge_{\pm}^{half} \mathbb{C}^{2n} \cong \mathrm{End}_{\mathbb{C}}(\Delta_\pm).$$

This is an isomorphism of D_n-modules, and it preserves the operation of quantum product by the generator of the second cohomology (i.e., by the hyperplane class of the projectivized maximal weight orbit for each representation).

In order to explain the notation, we recall the relation here. We denote a positively oriented orthonormal basis of $\mathbb{C}^{2n}$ by $e_1, \dots, e_{2n}$. We define the isomorphism $\star : \bigwedge^i \mathbb{C}^{2n} \to \bigwedge^{2n-i} \mathbb{C}^{2n}$ by

$$\star(e_{\xi(1)} \wedge e_{\xi(2)} \wedge \cdots \wedge e_{\xi(i)}) = \mathrm{sign}(\xi) e_{\xi(i+1)} \wedge e_{\xi(i+2)} \wedge \cdots \wedge e_{\xi(2n)}$$

for any permutation ξ. Then we obtain $\star \cdot \star = (-1)^{i(2n-i)}\mathrm{id}$. We define $\iota := (-i)^n \star : \bigwedge^n \mathbb{C}^{2n} \to \bigwedge^n \mathbb{C}^{2n}$. Then $\iota \cdot \iota = \mathrm{id}$. Thus, we have the canonical eigenspace decomposition $\bigwedge^n \mathbb{C}^{2n} \cong \bigwedge_+^n \mathbb{C}^{2n} \oplus \bigwedge_-^n \mathbb{C}^{2n}$. If $n = 2m+1$, then we define $\bigwedge_{\pm}^{half} \mathbb{C}^{2n}$ by

$$\bigwedge^0 \mathbb{C}^{4m+2} \bigoplus \bigwedge^2 \mathbb{C}^{4m+2} \bigoplus \cdots \bigoplus \bigwedge^{2m} \mathbb{C}^{4m+2}.$$

If $n = 2m$, then we define $\bigwedge_{+}^{half} \mathbb{C}^{2n}$ by

$$\bigwedge^0 \mathbb{C}^{4m} \bigoplus \bigwedge^2 \mathbb{C}^{4m} \bigoplus \cdots \bigoplus \bigwedge_+^{2m} \mathbb{C}^{4m}$$

and $\bigwedge_{-}^{half} \mathbb{C}^{2n}$ by

$$\bigwedge^0 \mathbb{C}^{4m} \bigoplus \bigwedge^2 \mathbb{C}^{4m} \bigoplus \cdots \bigoplus \bigwedge_-^{2m} \mathbb{C}^{4m}.$$

From Theorem 6.2 of [BD], we have

$$\begin{aligned} \Delta_+ \otimes \Delta_+ &= \bigwedge_+^n + \bigwedge^{n-2} + \cdots, \\ \Delta_+ \otimes \Delta_- &= \bigwedge^{n-1} + \bigwedge^{n-3} + \cdots, \\ \Delta_- \otimes \Delta_- &= \bigwedge_-^n + \bigwedge^{n-2} + \cdots \end{aligned}$$

as $\mathfrak{spin}(2n)$ representations where the last terms are $\bigwedge^4 + \bigwedge^2 + \bigwedge^0$ or $\bigwedge^3 + \bigwedge^1$. If $n = 2m+1$, then we have

$$\begin{aligned} \mathrm{End}_{\mathbb{C}}(\Delta_+) &\cong \Delta_+^* \otimes \Delta_+ \cong \Delta_+ \otimes \Delta_- \\ &\cong \bigwedge^{2m} + \bigwedge^{2m-2} + \cdots + \bigwedge^2 + \bigwedge^0 \\ &= \bigwedge_{\pm}^{half} \mathbb{C}^{4m+2}. \end{aligned}$$

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If $n = 2m$, then we have

$$\begin{aligned}\mathrm{End}_{\mathbb{C}}(\Delta_+) &\cong \Delta_+^* \otimes \Delta_+ \cong \Delta_+ \otimes \Delta_+ \\ &\cong \textstyle\bigwedge_+^{2m} + \bigwedge^{2m-2} + \cdots + \bigwedge^2 + \bigwedge^0 \\ &= \textstyle\bigwedge_+^{half} \mathbb{C}^{4m}.\end{aligned}$$

When we consider the minuscule Δ_- and the corresponding homogeneous space S_-, we obtain

$$\textstyle\bigwedge_-^{half} QH^*(Q_{2n-2}) \cong \mathrm{End}_{\mathbb{C}}(QH^*(S_-))$$

as in the case of Δ_+.

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References

[BD] T. Bröcker and T. Dieck, *Representations of Compact Lie Groups*, Springer, Berlin–Heidelberg, 1985.

[CV] S. Cecotti and C. Vafa, *Topological–anti-topological fusion*, Nucl. Phys. **367** (1991), 359–461.

[CMP] P. E. Chaput, L. Manivel, and N. Perrin, *Quantum cohomology of minuscule homogeneous spaces. II. Hidden symmetries*, Int. Math. Res. Not. IMRN **2007**, 1–29.

[FW] W. Fulton and C. Woodward, *On the quantum product of Schubert classes*, J. Algebraic Geom. **13** (2004), 641–661.

[Ge] M. Geck, "Minuscule weights and Chevalley groups" in *Finite Simple Groups: Thirty Years of the Atlas and Beyond*, Contemp. Math. **694**, Amer. Math. Soc., Providence, RI, 2017, 159–176.

[GM] V. Golyshev and L. Manivel, *Quantum cohomology and the Satake isomorphism*, preprint, arxiv:1106.3120

[Gr] J. Gross, *On minuscule representations and the principal SL_2*, Represent. Theory **4** (2000), 225–244.

[GH] M. A. Guest and N.-K. Ho, *Kostant, Steinberg, and the Stokes matrices of the tt*-Toda equations*, Selecta Math. (N.S.) **25** (2019), 50.

[GIL] M. A. Guest, A. Its, and C. S. Lin, *Isomonodromy aspects of the tt* equations of Cecotti and Vafa III. Iwasawa factorization and asymptotics*, Commun. Math. Phys. **374** (2020), 923–973.

[Hil] H. Hiller, *Geometry of Coxeter Groups*, Res. Notes Math. **54**, Pitman, Boston, 1982.

[Hit] N. J. Hitchin, *Lie groups and Teichmüller space*, Topology **31** (1992), 449–473.

[Hu] J. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Univ. Press, Cambridge, 1990.

[J] J. C. Jantzen, *Lectures on Quantum Groups*, Grad. Stud. Math. **6**, Amer. Math. Soc., Providence, RI, 1996.

[K] B. Kostant, *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group*, Amer. J. Math. **81** (1959), 973–1032.

[LT] T. Lam and N. Templier, *The mirror conjecture for minuscule flag varieties*, preprint, arXiv:1705.00758

[Ma] L. Manivel, *Double spinor Calabi–Yau varieties*, Épijournal Géom. Algébrique **3** (2019), 14.

[Mo] T. Mochizuki, *Harmonic bundles and Toda lattices with opposite sign*, preprint, arXiv:1301.1718

[S] J. Stembridge, *On the fully commutative elements of Coxeter groups*, J. Algebraic Combin. **5** (1996), 353–385.

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