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REPRESENTATIONS OF FOUNDATION SEMIGROUPS AND THEIR ALGEBRAS

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Introduction. The aim of this paper is to extend to a suitable class of topological semigroups parts of well-defined theory of representations of topological groups. In attempting to obtain these results it was soon realized that no general theory was likely to be obtainable for all locally compact semigroups. The reason for this is the absence of any analogue of the group algebra $L^1(G)$. So the theory in this paper is restricted to a certain family of topological semigroups. In this account we shall only give the details of those parts of proofs which depart from the standard proofs of analogous theorems for groups.

On a locally compact semigroup S the algebra $\tilde{L}(S)$ of all $\mu \in M(S)$ for which the mapping $x \to \bar{x} * |\mu|$ and $x \to |\mu| * \bar{x}$ of S to $M(S)$ (where $\bar{x}$ denotes the point mass at x) are continuous when $M(S)$ has the weak topology was first studied in the sequence of papers **[1, 2, 3]** by A. C. and J. W. Baker. A locally compact topological semigroup S is said to be *foundation* if S coincides with the closure of $\cup \{\text{supp}(\mu):\mu \in \tilde{L}(S)\}$.

In the first three sections of the present paper we investigate the relationship between the representations of S and the representations of $\tilde{L}(S)$ by bounded operators on reflexive Banach (or Hilbert) spaces, whenever S is a foundation topological semigroup with identity. The techniques yield results about automatic continuity of representations of S and the *-semisimplicity of the Banach *-algebras $M(S)$ and $\tilde{L}(S)$. In Section 4, we study the weighted measure algebras $M(S, w)$ and $\tilde{L}(S, w)$ for a topological semigroup S with a Borel measurable weight function w. (We have been unable to find, in the literature, a full investigation of the basic properties of algebra $M(S, w)$ for arbitrary w.) Finally, in Section 5, we extend the major results of the earlier sections to foundation topological semigroups with a Borel measurable weight function. Although many of the results of Sections 2 and 3 are special cases of those given in Section 5, it was felt preferable, for the reason of clarity, to give proofs (for the case $w = 1$) in the earlier sections, and explain in Section 5 how these proofs are modified to deal with more general weight functions w.

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1. Definitions and notations. Let S be a topological semigroup and E be a normed linear space. A *representation* V *of* S *by bounded operators on* E is a homomorphism $x \to V_x$ of S into $B(E)$, the space of all bounded operators on E. That is; for each $x \in S$,

$$V_x \in B(E) \quad \text{and} \quad V_{xy} = V_x V_y \quad \text{for all } x, y \in S.$$

If there exists a positive constant k such that

$$\|V_x \xi\| \leqq k \, \|\xi\| \quad \text{for all } x \in S \text{ and all } \xi \in E,$$

then V is said to be *bounded*. The infimum of all such k will be denoted by $\|V\|$. A subspace E_1 of E is said to be *invariant for* V if $V_x(E_1) \subseteq E_1$ for all $x \in S$, and V is said to be *topologically irreducible* if $\{0\}$ and E are the only closed invariant subspaces for V. For notational reasons (as in the group case) whenever we are concerned with reflexive Banach spaces we shall consider the representations of S by bounded operators on E^*, the dual space of E. If E is a Hilbert space, the distinction vanishes.

Let E be a reflexive Banach space and V be a representation of S by bounded operators on E^*. If the function

$$x \to \langle V_x \xi, \eta \rangle$$

is continuous [Borel measurable, μ-measurable] for all $\xi \in E^*$ and $\eta \in E$, then V is said to be (*weakly*) *continuous* [*weakly Borel measurable*, *weakly* μ-*measurable*]. If S has an involution * (a map $*: S \to S$ such that $x^{**} = x$ and $(xy)^* = y^* x^*$ for all $x, y \in S$), and V is a representation of S by bounded operators on a Hilbert space H such that $V_{x^*} = V_x^*$ for all $x \in S$, where V_x^* is the adjoint operator of V_x on H, then V is called a *-*representation*. We denote by $\mathscr{R}(S)$ the space of all bounded and continuous *-representations of S by bounded operators on Hilbert spaces. We also denote by $[\mathscr{R}(S)]$ the subspace of $C(S)$ (the space of all bounded, complex-valued continuous functions on S) generated by all functions of the form

$$x \to \langle V_x \xi, \xi \rangle,$$

where $V \in \mathscr{R}(S)$ and ξ belongs to the representation Hilbert space of V. The representation V is said to be *faithful* if for each $x_1, x_2 \in S$ with $x_1 \neq x_2$, we have $V_{x_1} \neq V_{x_2}$. Given a nonvoid family of $\{H_\gamma\}_{\gamma \in \Gamma}$ of Hilbert spaces, and, for every $\gamma \in \Gamma$, a representation V^γ of S by bounded operators on H_γ such that

$$\sup\{\|V^\gamma\| : \gamma \in \Gamma\} < \infty,$$

the mapping $x \to V_x$ which is given by

$$V_x = \sum_{\gamma \in \Gamma} V_x^\gamma$$

defines a bounded representation V of S by bounded operators on

$$H = \bigoplus_{\gamma \in \Gamma} H_\gamma.$$

It is easy to see that V is a *-representation if and only if for each $\gamma \in \Gamma$, V^γ is a *-representation.

We assume that the reader is familiar with the representation theory of normed algebras. We recall that a representation T of an algebra A by bounded operators on a normed space E is said to be *topologically cyclic* if there exists a vector $\zeta \in E$ such that the linear subspace $\{T_x\zeta : x \in A\}$ is dense in E. Such a vector ζ is called a *cyclic vector*.

2. Relations between representations of $\tilde{L}(S)$ and representations of S for a foundation topological semigroup S. We begin with the following theorem. The proof is similar to that given for Theorem 22.3 of [7] in the group case, so we omit it.

THEOREM 2.1. *Let S be a topological semigroup and E be a reflexive Banach space. Let A be a subalgebra of $M(S)$ and V be a bounded representation of S by bounded operators on E^*. Suppose that for every $\mu, \nu \in A$, V is weakly $|\mu|$-measurable and weakly $|\mu| * |\nu|$-measurable. Then the formula*

$$\langle T_\mu \xi, \eta \rangle = \int_S \langle V_x \xi_\eta \rangle d\mu(x) \quad (\mu \in A, \xi \in E^*, \eta \in E)$$

defines a bounded representation T of A by bounded operators on E^* with $\|T\| \leqq \|V\|$.

The proof of the next lemma is straightforward.

LEMMA 2.2. *Let S be a topological semigroup and let f be a Borel measurable function on S such that*

$$\int_S f(x) d\mu(x) = 0$$

for all $\mu \in \tilde{L}(S)$. Then $f = 0$, μ-almost everywhere, for each $\mu \in \tilde{L}(S)$. In particular, if S is foundation and f is continuous then f vanishes identically on S.

We now state and prove the basic theorem of this section.

THEOREM 2.3. *Let S be a foundation topological semigroup with identity and let E be a reflexive Banach space. Suppose that T is a bounded and cyclic representation of $\tilde{L}(S)$ by bounded operators on E^*. Then there exists a unique continuous and bounded representation V of S by bounded operators on E^* with $\|V\| = \|T\|$, $V_1 = I$, and*

$$(1) \qquad \langle T_\mu \xi, \eta \rangle = \int_S \langle V_x \xi, \eta \rangle d\mu(x) \quad (\mu \in \tilde{L}(S), \xi \in E^*, \eta \in E).$$

Furthermore, $V_x T_\mu = T_{\bar{x}*\mu}$ *and* $T_\mu V_x = T_{\mu * \bar{x}}$ *for every* $\mu \in \tilde{L}(S)$ *and every* $x \in S$. *The representations* T *and* V *have the same closed invariant subspaces. If* T *is faithful then* V *is also faithful, and in this case* $V_x \neq 0$ *for all* $x \in S$.

Proof. Let $\zeta \in E^*$ be a fixed cyclic vector for T; thus the linear subspace

$$E^*_\zeta = \{T_\mu \zeta : \mu \in \tilde{L}(S)\}$$

of E^* is dense in E^*. Let (ν_α) be a fixed approximate identity for $\tilde{L}(S)$ with $\|\nu_\alpha\| = 1$ for all α (see [**10**, Theorem 3.13]). For every $x \in S$ and $\mu \in \tilde{L}(S)$ we have

$$(2) \qquad \|T_{\bar{x}*\nu_\alpha*\mu} - T_{\bar{x}*\mu}\| \leqq \|T\| \, \|\bar{x} * (\nu_\alpha * \mu - \mu)\| \\ \leqq \|T\| \, \|\nu_\alpha * \mu - \mu\| \to 0,$$

as α increases. Let $\xi \in E^*_\zeta$. Then $\xi = T_\mu \zeta$ for some $\mu \in \tilde{L}(S)$. From (2) it follows that

$$\lim_\alpha T_{\bar{x}*\nu_\alpha*\mu}(\zeta) = T_{\bar{x}*\mu}(\zeta),$$

for every $x \in S$. For each $x \in S$ we denote this limit by $V'_x \xi$, so

$$(3) \qquad V'_x \xi = \lim_\alpha T_{\bar{x}*\nu_\alpha*\mu}(\zeta) = \lim_\alpha T_{\bar{x}*\nu_\alpha}(\xi) = T_{\bar{x}*\mu}(\zeta).$$

We now prove that for every $x \in S$ and every $\xi \in E^*_\zeta$, $V'_x \xi$ is well defined. To prove this we take $\xi \in E^*_\zeta$ and we suppose that

$$\xi = T_{\mu_1} \zeta = T_{\mu_2} \zeta, \quad \text{for some } \mu_1, \mu_2 \in \tilde{L}(S).$$

Then for each α we have

$$T_{\bar{x}*\nu_\alpha}(T_{\mu_1} \zeta) = T_{\bar{x}*\nu_\alpha}(T_{\mu_2} \zeta) \quad (x \in S).$$

Thus

$$T_{\bar{x}*\nu_\alpha*\mu_1}(\zeta) = T_{\bar{x}*\nu_\alpha*\mu_2}(\zeta) \quad \text{for all } \alpha.$$

By (3), we have

$$\lim T_{\bar{x}*\nu_\alpha*\mu_1}(\zeta) = \lim T_{\bar{x}*\nu_\alpha*\mu_2}(\zeta).$$

This proves that V' is well defined. By the use of (3), one can easily prove that V' defines a bounded representation of S by bounded operators on E^*_ζ with $\|V'\| \leqq \|T\|$ and $V'_1 = I$, the identity operator on E^*_ζ. Let (x_β) be a net which converges to $x \in S$, and suppose that $\xi \in E_\zeta$ with $\xi = T_\mu \zeta$ for some $\mu \in \tilde{L}(S)$. Then we have

$$\begin{aligned}\|V'_{x_\beta}(\xi) - V'_x(\xi)\| &= \|T_{\bar{x}_\beta * \mu}(\zeta) - T_{\bar{x} * \mu}(\zeta)\| \\ &\leqq \|T\| \, \|\bar{x}_\beta * \mu - \bar{x} * \mu\| \, \|\zeta\| \to 0,\end{aligned}$$

by the norm continuity of the mapping $x \to \bar{x} * \mu$ (see [**10**; Theorem 3.13]). Therefore V' is continuous. Now, since for each $x \in S$, V'_x is a bounded operator on E^*, we can extend V'_x uniquely to a bounded operator V_x on E^*_ξ with $\|V_x\| = \|V'_x\|$. It is easily seen that V defines a bounded representation of S by bounded operators on E^* with $\|V\| \leqq \|T\|$. Using the continuity of V' and the fact that V is bounded we can also prove that V is continuous.

We now proceed to the proof of formula (1). For each fixed $\eta \in E$ the mapping

$$\mu \to \langle T_\mu \zeta, \eta \rangle \quad (\mu \in \tilde{L}(S))$$

defines a bounded linear functional on $\tilde{L}(S)$. So, by Lemma 2.2 of [**3**] we have

$$(4) \quad \langle T_{\mu * \nu} \zeta, \eta \rangle = \int_S \langle T_{\bar{x} * \nu} \zeta, \eta \rangle d\mu(x)$$

for all $\mu, \nu \in \tilde{L}(S)$. Let $\xi \in E^*_\xi$ with $\xi = T_\nu \zeta$ for some $\nu \in \tilde{L}(S)$. Then for every $\mu \in \tilde{L}(S)$ we have

$$\begin{aligned}\langle T_\mu \xi, \eta \rangle &= \langle T_{\mu * \nu} \zeta, \eta \rangle \\ &= \int_S \langle T_{\bar{x} * \nu} \zeta, \eta \rangle d\mu(x) \quad \text{(by (4))} \\ &= \int_S \langle V_x \xi, \eta \rangle d\mu(x) \quad \text{(by (3)).}\end{aligned}$$

Since both functions

$$\xi \to \langle T_\mu \xi, \eta \rangle \quad \text{and} \quad \xi \to \int_S \langle V_x \xi, \eta \rangle d\mu(x)$$

are linear in ξ and bounded by $\|T\| \, \|\mu\| \, \|\eta\|$ on E^*, and E^*_ξ is dense in E^*, we infer that

$$(5) \quad \langle T_\mu \xi, \eta \rangle = \int_S \langle V_x \xi, \eta \rangle d\mu(x)$$

for all $\xi \in E^*$, $\eta \in E$, and $\mu \in \tilde{L}(S)$. This establishes the formula (1). From (5) it follows that

$$| \langle T_\mu \xi, \eta \rangle | \leqq \|V\| \, \|\xi\| \, \|\eta\| \, \|\mu\|$$

($\mu \in \tilde{L}(S)$, $\xi \in E^*$, $\eta \in E$). Hence $\|T\| \leqq \|V\|$. Therefore $\|V\| = \|T\|$. Lemma 2.2 together with the formula (5) imply the uniqueness of V.

We now suppose the M is a closed invariant subspace for T. If M is not invariant for V, then there exists a $\xi_0 \in M$ and $x_0 \in S$ such that

$$V_{x_0}\xi_0 \notin M.$$

Since E is reflexive, there exists $\eta \in E$ such that

$$\eta(M) = 0 \quad \text{and} \quad \langle V_{x_0}\xi_0, \eta \rangle = 1.$$

From (5) it follows that

$$0 = \langle T_\mu \xi_0, \eta \rangle = \int_S \langle V_x \xi_0, \eta \rangle d\mu(x), \quad (\mu \in \widetilde{L}(S)).$$

Therefore by Lemma 2.2 we have

$$\langle V_x \xi_0, \eta \rangle = 0 \quad \text{for all } x \in S.$$

This is a contradiction. So, M is also invariant for V. The converse is trivial. That

$$V_x T_\mu = T_{\overline{x}*\mu} \quad \text{and} \quad T_\mu V_x = T_{\mu * \overline{x}} \quad (x \in S, \mu \in \widetilde{L}(S)).$$

are easy consequences of (5). Finally, we suppose that T is faithful and let $x_1, x_2 \in \widetilde{L}(S)$ with $x_1 \neq x_2$. Choose a measure $\mu \in \widetilde{L}(S)$ (with $1 \in \operatorname{supp}(\mu)$) such that $\overline{x}_1 * \mu \neq \overline{x}_2 * \mu$. Since T is faithful, we have

$$T_{\overline{x}_1 * \mu} \neq T_{\overline{x}_2 * \mu}.$$

Hence

$$V_{x_1} T_\mu \neq V_{x_2} T_\mu.$$

Thus $V_{x_1} \neq V_{x_2}$. The last assertion can be proved similarly. So the theorem is established.

In the next theorem we extend the above result to (not necessarily cyclic) *-representations of the Banach *-algebra $\widetilde{L}(S)$ by bounded operators on a Hilbert space H and the bounded continuous *-representations of S by bounded operators on H, where S is a foundation topological semigroup with identity and with a continuous involution *.

THEOREM 2.4. *Let S be a foundation topological semigroup with identity and with a continuous involution *. Suppose that T is a *-representation of the Banach *-algebra $\widetilde{L}(S)$ by bounded operators on a Hilbert space H such that for every $0 \neq \xi \in H$ there exists a $\mu \in \widetilde{L}(S)$ with $T_\mu \xi \neq 0$. Then there exists a unique bounded and continuous *-representation V of S by bounded operators on H with $\|V\| \leqq 1$ such that*

$$(6) \quad \langle T_\mu \xi, \eta \rangle = \int_S \langle V_x \xi, \eta \rangle d\mu(x) \quad (\mu \in \widetilde{L}(S), \xi, \eta \in H).$$

Moreover, T and V have the same closed invariant subspaces. For every $\mu \in \widetilde{L}(S)$ and $x \in S$,

$$V_x T_\mu = T_{\bar{x}*\mu} \quad \textit{and} \quad T_\mu V_x = T_{\mu*\bar{x}}.$$

If T is faithful then V is also faithful and $v_x \neq 0$ for all $x \in S$.

Proof. The proof of the formula (6) is essentially the same as that for groups (see [7], p. 339, (II) of Theorem 22.7). The rest of the theorem is an easy consequence of the formula (6).

As a consequence of the above theorem we obtain the following corollary.

COROLLARY 2.5. *Let S be a foundation topological semigroup with identity and with a continuous involution *. Let V be a bounded and weakly Borel measurable *-representation of S by bounded operators on a Hilbert space H such that $V_1\xi \neq 0$ for every $0 \neq \xi \in H$. Then V is continuous if and only if for every $\xi, \eta \in H$ the set*

$$\{x \in S: \langle V_x\xi, \eta\rangle = 0\}$$

is closed.

Proof. If V is continuous, then it is evident that for every $\xi, \eta \in H$ the set

$$\{x \in S: \langle V_x\xi, \eta\rangle = 0\}$$

is closed. To prove the converse we define T by

$$(7) \quad \langle T_\mu\xi, \eta\rangle = \int_S \langle V_x\xi, \eta\rangle d\mu(x) \quad (\mu \in \tilde{L}(S), \xi, \eta \in H).$$

By Theorem 2.1, T defines a bounded representation of $\tilde{L}(S)$ by bounded operators on H, and it is also easy to see that T is a *-representation. Moreover, for every $0 \neq \xi \in H$ there exists a $\mu \in \tilde{L}(S)$ such that $T_\mu\xi \neq 0$. For if not, then there exists a $0 \neq \xi \in H$ with $T_\mu\xi = 0$ for all $\mu \in \tilde{L}(S)$. So,

$$\langle T_\mu\xi, \eta\rangle = 0 \quad \text{for all } \mu \in \tilde{L}(S),$$

and hence

$$\int_S \langle V_x\xi, V_1\xi\rangle d\mu(x) = 0$$

for all $\mu \in \tilde{L}(S)$. Therefore by Lemma 2.2

$$\langle V_x\xi, V_1\xi\rangle = 0 \text{ a.e. } \mu,$$

for all $\mu \in \tilde{L}(S)$. This is a contradiction, because

$$U = \{x \in S: \langle V_x\xi, V_1\xi\rangle \neq 0\}$$

is an open set which contains 1, and S is foundation. By Theorem 2.4, there exists a bounded-continuous *-representation V' of S by bounded operators on H such that

(8) $\langle T_\mu \xi, \eta \rangle = \int_S \langle V'_x \xi, \eta \rangle d\mu(x) \quad (\mu \in \tilde{L}(S), \xi, \eta \in H).$

From (7), (8), and the fact that V and V' are *-representations it follows that

(9) $\langle T_\mu \xi, V'_y \eta \rangle = \langle T_\mu \xi, V_y \eta \rangle \quad (\mu \in \tilde{L}(S), \xi, \eta \in H).$

Using the same argument given on page 339 for part (ii) of the proof of Theorem 22.7 of [7], we can write H as a direct sum $\bigoplus_{\gamma \in \Gamma} H_\gamma$ of subspaces $\{H_\gamma\}_{\gamma \in \Gamma}$, which are closed, pairwise orthogonal, and invariant, such that for each $\gamma \in \Gamma$, T^γ (the restriction of T to H_γ) is a cyclic and bounded *-representation of $\tilde{L}(S)$ by bounded operators on H_γ, and V'^γ (the restriction of V' to H_γ) is a bounded and continuous *-representation of S by bounded operators on H_γ which is related to T^γ according to the formula

$$\langle T^\gamma_\mu \xi_\gamma, \eta_\gamma \rangle = \int_S \langle V'^\gamma_x \xi_\gamma, \eta_\gamma \rangle d\mu(x) \quad (\mu \in \tilde{L}(S), \xi_\gamma, \eta_\gamma \in H_\gamma).$$

For each $\gamma \in \Gamma$ let $\zeta_\gamma \in H_\gamma$ be a cyclic vector for T^γ. Now, if δ is a fixed element of Γ, then by (9) for every $\eta_\gamma \in H_\gamma$ we have

(10) $\langle T^\delta_\mu \zeta_\delta, V'^\gamma_y \eta_\gamma \rangle = \langle T^\delta_\mu \zeta_\delta, V_y \eta_\gamma \rangle$

for all $\mu \in \tilde{L}(S)$ and all $y \in S$. Since H_γ is invariant for V'^γ, and for every $\delta \in \Gamma$ the set

$$\{T^\delta_\mu \zeta_\delta : \mu \in \tilde{L}(S)\}$$

is dense in H_δ, it follows from (10) that H_γ is also invariant under V^γ (the restriction of V to H_γ) and V^γ defines a bounded representation of S by bounded operators on H_γ, with $V^\gamma = V'^\gamma$, for every $\gamma \in \Gamma$. Therefore V can be written as $\bigoplus_{\gamma \in \Gamma} V^\gamma$. Since

$$V' = \bigoplus_{\gamma \in \Gamma} V'^\gamma \quad \text{and} \quad V'^\gamma = V^\gamma \quad \text{for every } \gamma \in \Gamma,$$

it follows that $V' = V$. This completes the proof.

Remark. One can easily see that on the non-foundation topological semigroup $S = ([0, 1], \min)$ (with the usual topology and the natural involution $x^* = x$ for all $x \in S$) the function χ which is given by $\chi(x) = 0$ if $x \leqq 1/2$, and $\chi(x) = 1$ if $x > 1/2$, defines a Borel measurable one-dimensional representation which is discontinuous at $x = 1/2$.

Before proceeding any further we recall that a nonzero function $\chi: S \to \mathbf{C}$ is said to be a *semicharacter* on S if

$$\chi(xy) = \chi(x)\chi(y) \quad \text{for all } x, y \in S.$$

(For reasons that will become apparent, we shall not assume χ is bounded.) If S has an involution $*$, and χ is a semicharacter on S such that $\chi(x^*) = \overline{\chi(x)}$ $(x \in S)$, then χ is called a *$*$-semicharacter*.

PROPOSITION 2.6. *Let S be a commutative foundation topological semigroup with a continuous involution $*$. A nonzero, continuous and bounded $*$-representation of S by bounded operators on a Hilbert space is irreducible if and only if it is equivalent to a nonzero continuous and bounded $*$-semicharacter on S.*

Proof. The proof is essentially the same as for groups (see the proof of Theorem 22.17 of [7]).

3. The $*$-semisimplicity of $M(S)$ and $\tilde{L}(S)$. Our starting point is the following lemma. Throughout its proof we assume that the reader is familiar with the elementary properties of tensor products of Hilbert spaces as given on page 48 of [**9**].

LEMMA 3.1. *Let S be a topological semigroup with a continuous involution $*$. Then the following are satisfied;*

(i) *$[\mathscr{R}(S)]$ is a self-conjugate subalgebra of $C(S)$ and contains a nonzero constant function;*

(ii) *$[\mathscr{R}(S)]$ separates the points of S if and only if $\mathscr{R}(S)$ separates the points of S.*

Proof. (i) It is obvious that $[\mathscr{R}(S)]$ is a subspace of $C(S)$ which is closed under complex-conjugation of its elements and contains a nonzero constant function. To prove that $[\mathscr{R}(S)]$ is closed under multiplication we assume that $x \to \langle V_x\xi, \xi\rangle$ and $x \to \langle V'_x\xi', \xi'\rangle$ belong to $[\mathscr{R}(S)]$ with $\xi \in H$ and $\xi' \in H'$, the representation Hilbert spaces for V and V', respectively. It is easy to see that $V \otimes V'$ which is given by

$$(V \otimes V')_x = V_x \otimes V'_x \quad (x \in S)$$

defines a bounded and continuous $*$-representation of S by bounded operators on $H \otimes H'$, the Hilbert space tensor product of H and H'. Since

$$\langle V_x\xi, \xi\rangle\langle V'_x\xi', \xi'\rangle = \langle (V \otimes V')_x(\xi \otimes \xi'), \xi \otimes \xi'\rangle$$

it follows that the mapping

$$x \to \langle V_x\xi, \xi\rangle\langle V'_x\xi', \xi'\rangle$$

also belongs to $[\mathscr{R}(S)]$. This proves that $[\mathscr{R}(S)]$ is a subalgebra of $C(S)$.

(ii) The proof is standard and hence is omitted.

Before we proceed to the next theorem we recall that for a Banach $*$-algebra A, the $*$-radical, $*$-rad(A), is defined to be the intersection of the kernels of all irreducible *-representations of A by bounded operators on Hilbert spaces. If *-rad$(A) = 0$, then A is said to be **-semisimple.*

THEOREM 3.2. *Let S be a topological semigroup with a continuous involution *. If $\mathscr{R}(S)$ separates the points of S, then $M(S)$ is *-semisimple.*

Proof. By using the arguments given on page 223 of [**4**], we only need to show that for every $0 \neq \mu \in M(S)$ there exists a *-representation T of $M(S)$ by bounded operators on a Hilbert space with $T_\mu \neq 0$. Since $\mathscr{R}(S)$ separates the points of S, it follows from Lemma 3.1 that $[\mathscr{R}(S)]$ is a self-conjugate subalgebra of $C(S)$ which separates the points of S and contains a nonzero constant function. Therefore by Lemma 3.5 of [**3**], $[\mathscr{R}(S)]$ is dense (L^1-norm) in $L^1(S, |\mu|)$. Let f be a Borel measurable function on S with $|f| = 1$, $d\mu = fd|\mu|$ and $d|\mu| = \bar{f}d\mu$. *Put* $\epsilon = 1/2\,\|\mu\|$, and choose g in $[\mathscr{R}(S)]$ such that

$$\int_S |g(x) - \overline{f(x)}\,|d|\mu|\,(x) < \epsilon.$$

Then we have

$$\left| \int_S [g(x) - \overline{f(x)}\,]d\mu(x) \right| \leqq \int_S |g(x) - f(x)\,|d|\mu|\,(x) < \epsilon.$$

Hence

$$\left| \int_S g(x)d\mu(x) \right| \geqq \left| \int_S \overline{f(x)}d\mu(x) \right| - \epsilon$$

$$= \int_S 1d|\mu|\,(x) - \epsilon \geqq \epsilon > 0.$$

So there must exist a function $x \to \langle V_x\xi_0, \xi_0\rangle$ in $[\mathscr{R}(S)]$, where ξ_0 belongs to the representation Hilbert space H of V such that

$$\int_S \langle V_x\xi_0, \xi_0\rangle d\mu(x) \neq 0.$$

Now we define T on $M(S)$ by

$$\langle T_\nu\xi, \eta\rangle = \int_S \langle V_x\xi, \eta\rangle d\nu(x) \quad (\nu \in M(S), \xi, \eta \in H).$$

By Theorem 2.1, T defines a bounded representation of the Banach *-algebra $M(S)$ by bounded operators on H which is also a *-representation. Since

$$\langle T_\mu\xi_0, \xi_0\rangle \neq 0,$$

we infer that $T_\mu \neq 0$. This is what we wished to prove.

LEMMA 3.3. *Let S be a foundation topological semigroup with identity and with a continuous involution *. If $\tilde{L}(S)$ is *-semisimple then $\mathscr{R}(S)$ separates the points of S.*

Proof. Let x_1, x_2 be two distinct points of S. Then we can find a measure $\mu \in \widetilde{L}(S)$ $(1 \in \operatorname{supp}(\mu))$ such that

$$\mu * \overline{x_1} \neq \mu * \overline{x_2}.$$

By the *-semisimplicity of $\widetilde{L}(S)$, there is an irreducible *-representation T of $\widetilde{L}(S)$ by bounded operators on a Hilbert space H such that

$$T_{\mu * \bar{x}_1} \neq T_{\mu * \bar{x}_2}.$$

Since by (ii) of Theorem 21.30 of [7] every nonzero vector in H is a cyclic vector for T, we infer that for every $0 \neq \xi \in H$ there exists a $\nu \in \widetilde{L}(S)$ with $T_\nu \xi \neq 0$. Therefore, by Theorem 2.4, there exists a continuous and bounded *-representation V of S by bounded operators on H such that

$$(11) \quad \langle T_\nu \xi, \eta \rangle = \int_S \langle V_x \xi, \eta \rangle d\nu(x) \quad (\nu \in \widetilde{L}(S), \xi, \eta \in H).$$

From (11) it follows that

$$T_\mu V_{x_1} = T_{\mu * \bar{x}_1} \quad \text{and} \quad T_\mu V_{x_2} = T_{\mu * \bar{x}_2},$$

and therefore $V_{x_1} \neq V_{x_2}$. Hence $\mathscr{R}(S)$ separates the points of S.

A combination of Theorem 3.2 and Lemma 3.3 leads us to the following theorem.

THEOREM 3.4. *Let S be a foundation topological semigroup with identity and with a continuous involution *. The following are equivalent*:

(i) $M(S)$ *is *-semisimple*;

(ii) $\widetilde{L}(S)$ *is *-semisimple*;

(iii) $\mathscr{R}(S)$ *separates the points of* S.

Remark. We do not know whether the above result is true for non-foundation topological semigroups.

4. Representations of weighted topological semigroups. Our main goal in this section is to extend some of the results of previous sections to the situation of foundation topological semigroups which have positive Borel measurable weight functions. We first set out the theory of weighted measure algebras, as this does not seem to have been done before in the generality we required. We begin with the following definition.

Definition 4.1. A complex-valued function f on a topological space X is said to be *locally bounded* if for every $x \in X$ there exists a neighbourhood of x on which f is bounded.

Note that if X is locally compact then f is locally bounded if and only if it is bounded on each compact subset of X.

Definition 4.2. Let S be a semigroup. Then a function $w:S \to \mathbf{R}$ is said to be a *weight function* if $w(x) \geqq 0$, and

$$w(xy) \leqq w(x)w(y) \quad \text{for all } x, y \in S.$$

Definition 4.3. Let S be a semigroup and w be a weight function on S. A complex-valued function f on S is said to be *w-bounded* if there exists a constant $k \geqq 0$ such that

$$|f(x)| \leqq kw(x) \quad (x \in S).$$

The next lemma will be needed to prove the first theorem of this section.

LEMMA 4.4. *Let μ be a positive regular Borel measure on a locally compact Hausdorff space X. Then for every locally bounded Borel measurable function w with $0 \leqq w < \infty$ on X, $w\mu$ is a regular Borel measure.*

Proof. The result follows from Theorem E [**8**, p. 248].

Definition 4.5. Let μ and ν be regular Borel measures on a locally compact Hausdorff space X. Then ν is said to be *locally absolute continuous with respect to* (L.A.C.) μ if $|\nu|(F) = 0$, whenever F is a compact subset of X with $|\mu|(F) = 0$.

The following theorem is essential throughout this section.

THEOREM 4.6. *Let S be a topological semigroup and suppose that $w > 0$ is a Borel measurable weight function on S such that w and $1/w$ are locally bounded. Then*

(i) $C_0(S, w)$, *the space of all Borel measurable functions f on S such that $f/w \in C_0(S, w)$ under the w-norm given by*

$$\|f\|_w = \sup\{\,|f/w\,(x)|:x \in S\}$$

is a Banach space.

(ii) $M(S, w)$, *the space of all regular Borel measures μ on S such that $w|\mu| \in M(S)$ with the w-norm*

$$\|\mu\|_w = \int_S wd|\mu|$$

can be identified with $C_0(S, w)^*$, the dual of $C_0(S, w)$, via the pairing

$$\langle \mu, f\rangle = \mu(f) = \int_S f(x)d\mu(x) \quad (\mu \in M(S, w), f \in C_0(S, w)).$$

Given $\mu, \nu \in M(S, w)$, let $\mu * \nu$ be the measure in $M(S, w)$ defined by

$$(12) \quad (\mu * \nu)(f) = \int_S f(x)d\mu * \nu(\chi)$$
$$= \int_S \int_S f(xy)d\mu(x)d\nu(y) \quad (f \in C_0(S, w)).$$

Then the Banach space $M(S, w)$ with the convolution product $*$ is a convolution Banach algebra in which $M_K(S)$, the space of all measures in

$M(S)$ with compact supports, is w-norm dense.

(iii) *For every w-bounded Borel measurable function f on S we have*

$$(13)\quad \int_S f(x)d\mu * \nu(x) = \int_S \int_S f(xy)d\mu(x)d\nu(y) \quad (\mu, \nu \in M(S, w)).$$

Moreover, for each compact subset F of S and every $\mu, \nu \in M(S, w)$, we have

$$(14)\quad (\mu * \nu)(F) = \int_S \int_S \chi_F(xy)d\mu(x)d\nu(y)$$
$$= \int_S \nu(x^{-1}F)d\mu(x) = \int_S \mu(Fy^{-1})d\nu(y).$$

(iv) *$\widetilde{L}(S, w)$, the space of all regular Borel measures μ on S such that $w|\mu| \in \widetilde{L}(S)$ is a closed two-sided ideal of $M(S, w)$ and is solid in $M(S, w)$ in the sense that given $\mu \in \widetilde{L}(S, w)$ and $\nu \in M(S, w)$ with ν* L.A.C. *μ, then $\nu \in \widetilde{L}(S, w)$.*

(We have been unable to prove the formula (13) for each $|\mu| * |\nu|$-integrable function f, and even in the group case the proof given for part (4) of Theorem 4.19.5 of [**5**] does not seem to be correct.)

Proof. (i) This is clear.

(ii) This follows from Lemma 4.4 and the fact that w and $1/w$ are locally bounded.

(iii) Let τ denote the mapping $(x, y) \rightarrow xy$ of $S \times S$ into S. For each subset A of S, and every complex-valued function f on S, we denote $\tau^{-1}(A)$ and $f \circ \tau$ by $\widetilde{A}$ and $\widetilde{f}$, respectively. Suppose that f is a w-bounded Borel measurable function on S. It is clear that for each $\mu, \nu \in M(S, w)$, $|f|$ is $|\mu| * |\nu|$-integrable, and $|\widetilde{f}|$ is $|\mu| \times |\nu|$-integrable. Therefore, in view of Theorem 4.3.2 of [**5**] to prove (13) we only need to establish this formula for the case when μ, ν are positive measures in $M(S, w)$ and f is a Borel measurable function on S with $0 \leqq f(x) \leqq w(x)$ for all $x \in S$. For simplicity we denote $\mu * \nu$ by γ and $\mu \times \nu$ by π. By using an argument similar to that of the first part of the proof of Theorem 19.10 of [7] and the fact that w and $1/w$ are locally bounded we can easily prove that for every γ-null set A, $\widetilde{A}$ is also π-null. Now, for each $n \in \mathbf{N}$, we denote the set

$$\left\{ x \in S : f(x) > \frac{1}{n} \right\} \text{ by } A_n.$$

Then

$$\gamma(A_n) \leqq n\gamma(f) < \infty.$$

Since γ is regular, we can choose an increasing sequence (K_n) of compact sets such that

$$K_n \subseteq A_n \quad \text{and} \quad \gamma(A_n \setminus K_n) < \frac{1}{n} \quad (n \in \mathbf{N}).$$

It is easy to see that $(f\chi_{K_n})$ increases (pointwise) to f on $S \setminus A$, where

$$A = \bigcup_{n=1}^{\infty} (A_n \setminus \bigcup_{m=1}^{\infty} K_m).$$

Since $(A_n \setminus \bigcup_{m=1}^{\infty} K_m)$ is an increasing sequence, we have

$$0 \leqq \gamma(A) = \lim_{n\to\infty} \gamma(A_n \setminus \bigcup_{m=1}^{\infty} K_m) \leqq \lim_{n\to\infty} \gamma(A_n \setminus K_n) = 0.$$

It follows that $(\widetilde{f\chi_{K_n}})$ increases to $\tilde{f}$ π-almost everywhere. Therefore, in view of the monotone convergence theorem, we may assume that f vanishes identically outside a compact set K. So $\gamma(K) < \infty$, by the regularity of γ. Since $0 \leqq f/w \leqq 1$, and f/w vanishes identically outside K with $\gamma(K) < \infty$, it follows from a corollary to Lusin's theorem (see [**8**, p. 53]) that there exists a sequence (g_n) in $C_{00}(S)$ such that

$$|g_n| \leqq 1 \quad \text{and} \quad (f/w)(x) = \lim_{n\to\infty} g_n(x) \text{ a.e. } \gamma.$$

Since $\gamma \in M(S)$, w is γ-integrable. It is also clear that $\tilde{w}$ is π-integrable. By virtue of the dominated convergence theorem, we see that

$$\gamma(f) = \lim_{n\to\infty} \gamma(g_n w) \quad \text{and} \quad \pi(\tilde{f}) = \lim_{n\to\infty} \pi(\widetilde{g_n w}).$$

Since for every $n \in \mathbf{N}$, $g_n w \in C_0(S, w)$, from (12) it follows that

$$\gamma(g_n w) = \pi(\widetilde{g_n w}).$$

Therefore $\gamma(f) = \pi(\tilde{f})$. This establishes the formula (13). The formula (14) now follows from (13) by using the fact that for each compact subset F of S, χ_F is a w-bounded Borel measurable function on S.

(iv) This easily follows from the locally boundedness of w and $1/w$, and the fact that $\tilde{L}(S)$ is a solid ideal of $M(S)$.

Remark. For the rest of this paper we assume that $w > 0$ is a Borel measurable weight function on S such that w and $1/w$ are locally bounded.

PROPOSITION 4.7. *Let S be a foundation topological semigroup with identity and with a weight function w. Then*

(i) *for every $\mu \in \tilde{L}(S, w)$ the maps $x \to \mu * \bar{x}$, and $\bar{x} \to \bar{x} * \mu$ are w-norm continuous*;

(ii) *for each compact neighbourhood U of the identity there exists a bounded approximate identity $(\mu_\lambda)_{\lambda\in\Lambda}$ for $\tilde{L}(S, w)$ of positive measures such* *that* $\mathrm{supp}(\mu_\lambda) \subseteq U$; $\|\mu_\lambda\| = 1$, *and* $\|\mu_\lambda\|_w \leqq k$ *for all* $\lambda \in \Lambda$, *where* k *is any positive constant with* $w(x) \leqq k$ *for all* $x \in U$.

Proof. Both (i) and (ii) are easy consequences of Theorem 3.13 of [**10**], the locally boundedness of w, and the w-norm density of $\widetilde{L}_K(S)$ $(= \widetilde{L}(S) \cap M_K(S))$ in $\widetilde{L}(S)$.

LEMMA 4.8. *Let S be a foundation topological semigroup with a Borel measurable weight function w. Let f be a continuous complex-valued, and w-bounded function on S such that*

$$\int_S f d\mu = 0 \quad \text{for all } \mu \in \widetilde{L}(S, w).$$

Then f vanishes identically on S.

Proof. The result follows from the w-norm density of $\widetilde{L}_K(S)$ in $\widetilde{L}(S, w)$, and the fact that

$$S = \overline{\cup \{\mathrm{supp}(\mu) : \mu \in L_K(S)\}}.$$

We now establish a lemma which is essential for the next section.

LEMMA 4.9. *Let S be a topological semigroup with a Borel measurable weight function w such that the maps $x \to \mu * \bar{x}$ and $x \to \bar{x} * \mu$ $(\mu \in \widetilde{L}(S, w))$ are w-norm continuous. Let $\mu \in \widetilde{L}(S, w)$, $\nu \in M(S, w)$, and $f \in (\widetilde{L}(S, w))^*$. Then the maps $x \to f(\mu * \bar{x})$ and $x \to f(\bar{x} * \mu)$ define a w-bounded continuous function on S with*

$$f(\mu * \nu) = \int_S f(\mu * \bar{x}) d\nu(x), \quad f(\nu * \mu) = \int_S f(\bar{x} * \mu) d\nu(x).$$

Proof. It is clear that $x \to f(\mu * \bar{x})$ and $x \to f(\bar{x} * \mu)$ are continuous and w-bounded. From the w-norm continuity of the map $x \to \mu * \bar{x}$ and the locally boundedness of w it follows that for each compact subset F of S the map $x \to \mu * \bar{x}(F)$ is continuous on S. Suppose that F is a compact subset of S such that

$$|\mu| * |\nu|\,(F) = 0.$$

Therefore by (14) we have

$$0 = |\mu| * |\nu|\,(F) = \int_S |\mu|\,(x^{-1}F) d|\nu|\,(x).$$

Hence

$$|\mu| * \bar{x}(F) = 0 \text{ a.e. } |\nu|.$$

From the continuity of the function $x \to |\mu| * \bar{x}(F)$ it follows that

$$|\mu| * \bar{x}(F) = 0 \quad \text{for all } x \in \mathrm{supp}(\nu).$$

Let us denote by $L(S, |\mu| * |\nu|)$ the spaces of all measures in $\widetilde{L}(S, w)$ which are L.A.C. $|\mu| * |\nu|$. Then the map: $\lambda \to w\lambda$ defines a linear isometry of $L(S, |\mu| * |\nu|)$ into $L^1(S, w(|\mu| * |\nu|))$. For each $\lambda \in L(S, |\mu| * |\nu|)$ we set $\Phi(w\lambda) = f(\lambda)$. It is easy to see that Φ defines a bounded linear functional on the subspace

$$\{w\lambda : \lambda \in L(S, |\mu| * |\nu|)\}$$

of $L^1(S, w(|\mu| * |\nu|))$. By the Hahn-Banach theorem we may assume that Φ is a bounded linear functional on $L'(S, w(|\mu| * |\nu|))$. Therefore there exists a bounded Borel measurable function θ_1 on S such that

$$\Phi(\gamma) = \int_S \theta_1(x) d\gamma(x) \quad (\gamma \in L^1(S, w(|\mu| * |\nu|))).$$

Therefore

$$(15) \quad f(\lambda) = \Phi(w\lambda) = \int_S \theta_1(x) d(w\lambda)(x) = \int_S (\theta_1 w) d\lambda(x),$$

for each $\lambda \in L(S, |\mu| * |\nu|)$. Since $\theta = \theta_1 w$ is w-bounded, by (15) we have

$$\begin{aligned} f(\mu * \nu) &= \int_S \theta d\mu * \nu \\ &= \int_S \int_S \theta(yx) d\mu(x) d\nu(y) \quad (\text{by } (13)) \\ &= \int_S d\nu(x) \int_S \theta(y) d\mu * \bar{x}(y) \\ &= \int_S f(\mu * \bar{x}) d\nu(x). \end{aligned}$$

The proof of the other formula is similar. So the lemma is proven.

5. Relationship between the bounded representations of $\widetilde{L}(S, w)$ and the w-bounded representations of S. We commence this section with the following definition.

Definition 5.1. Let S be a semigroup with a weight function w, and let E be a reflexive Banach space. Then a representation V of S by bounded operators on E^* is said to be *w-bounded* if there exists a positive constant k such that

$$|\langle V_x \xi, \eta \rangle| \leqq k w(x) \|\xi\| \|\eta\|, \quad (x \in S, \xi \in E^*, \text{ and } \eta \in E).$$

Using the formula (13) of Theorem 4.6 with the techniques of the proof of [**7**, Theorem 22.3] we can easily obtain the following generalization of Theorem 2.1.

THEOREM 5.2. *Let S be a topological semigroup with a Borel measurable weight function w. Let E be a reflexive Banach space and V be a w-bounded*

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Borel measurable representation of S by bounded operators on E^*. Then for every subalgebra A of $M(S, w)$ the formula

$$\langle T_\mu \xi, \eta\rangle = \int_S \langle V_x \xi, \eta\rangle d\mu(x) \quad (\mu \in A, \xi \in E^*, \eta \in E)$$

defines a bounded representation of A by bounded operators on E^* with $\|T\| \leqq \|V\|_w$.

We shall now give a generalization of Theorem 2.3.

THEOREM 5.3. *Let S be a foundation topological semigroup with identity and with a Borel measurable weight function w. Suppose that T is a bounded cyclic representation of $\tilde{L}(S, w)$ by bounded operators on E^*, where E is a reflexive Banach space. Then there exists a unique, w-bounded, and continuous representation V of S by bounded operators on E^* with $V_1 = 1$, $\|T\| \leqq \|V\|_w$, and $\|V\|_w \leqq k\|T\|$ for some positive constant k, with the property that the representations T and V are related to each other according to the formula*

$$\langle T_\mu \xi, \eta\rangle = \int_S \langle V_x \xi, \eta\rangle d\mu(x) \quad (\mu \in \tilde{L}(S, w), \xi \in E^*, \eta \in E). \tag{16}$$

Moreover, T and V have the same closed invariant subspaces. Furthermore,

$$V_x T_\mu = T_{\bar{x}*\mu} \quad \text{and} \quad T_\mu V_x = T_{\mu*\bar{x}}$$

for every $x \in S$ and $\mu \in \tilde{L}(S, w)$. If T is faithful, then V is faithful, and in this case $V_x \neq 0$ for all $x \in S$. Finally, if w is continuous at identity and $w(1) = 1$, then $\|V\|_w = \|T\|$.

Proof. Let $\zeta \in E^*$ be a fixed cyclic vector for T. Then the linear subspace

$$E^*_\zeta = \{T_\mu \zeta : \mu \in \tilde{L}(S, w)\}$$

of E^* is dense in E^*. Let (ν_α) be a fixed approximate identity for $\tilde{L}(S, w)$ and suppose that k is a positive constant such that

$$\|\nu_\alpha\|_w \leqq k \quad \text{for all } \alpha.$$

By a method similar to that of Theorem 3.2, we can prove that there exists a (unique) continuous representation V of S by bounded operators on E^* such that

$$V_x \xi = \lim_\alpha T_{\bar{x}*\nu_\alpha*\mu}(\zeta) = \lim_\alpha T_{\bar{x}*\nu_\alpha}(\xi) = T_{\bar{x}*\mu}(\zeta), \tag{17}$$

for every $x \in S$ and $\xi = T_\mu \zeta \in E^*_\zeta$. From (17) it follows that

$$\|V\|_w \leqq k\|T\|.$$

Now, if, in the proof of Theorem 2.3, we apply Lemma 4.9 instead of Lemma 2.2 of [**3**], then we establish the formula (16). The rest of the proof is similar to the proof of the corresponding part of Theorem 2.3. To prove the final assertion we suppose that $\epsilon > 0$ is given. Then there exists a compact neighbourhood U_ϵ of the identity such that

$$|w(x) - 1| < \epsilon \quad \text{for all } x \in U_\epsilon.$$

Let $(\nu_\alpha^{(\epsilon)})$ be a bounded approximate identity for $\tilde{L}(S, w)$ as in part (ii) of Proposition 4.7. Then we have

$$\|\nu_\alpha^{(\epsilon)}\|_w - 1 = \int_{U_\epsilon} [w(x) - 1] d\nu_\alpha^{(\epsilon)}(x) \leqq \epsilon \nu_\alpha^{(\epsilon)}(U) = \epsilon.$$

Therefore

$$\|\nu_\alpha^{(\epsilon)}\|_w \leqq 1 + \epsilon.$$

By virtue of the equalities in (17) we see that the definition of V is independent of the choice of the approximate identity for $\tilde{L}(S, w)$. Replacement of ν_α by $\nu_\alpha^{(\epsilon)}$ in (17) results in the equation

$$V_x(\xi) = \lim_\alpha T_{\bar{x} * \nu_\alpha^{(\epsilon)}(\xi)} \quad (\xi \in E_\xi^*).$$

Therefore

$$\|V\|_w \leqq \|T\| (1 + \epsilon).$$

Now the arbitrariness of $\epsilon > 0$ implies that $\|V\|_w \leqq \|T\|$. This completes the proof of the theorem.

Remark. It is not true in general that a Borel measurable weight function on a foundation topological semigroup which is continuous at the identity is also continuous everywhere. For example on the foundation topological semigroup $(\mathbf{R}_+, +)$, the additive semigroup of nonnegative real numbers with the usual topology, the function $w(x) = e^x$ if $0 \leqq x \leqq 1$ and $w(x) = e^{x/2}$ if $x > 1$, defines a Borel measurable weight function which is continuous at 0, the identity, and is discontinuous at $x = 1$.

Using the techniques of the proof of Theorem 2.4 with the aid of Theorem 5.3, we can easily establish the following generalization of Theorem 2.4, and for that reason the proof is omitted.

THEOREM 5.4. *Let S be a foundation topological semigroup with identity and with a continuous involution *. Suppose that w is a Borel measurable weight function on S such that $w(x^*) = w(x)$ for all $x \in S$. Let T be a bounded *-representation of the Banach *-algebra $\tilde{L}(S, w)$ by bounded operators on a Hilbert space H such that for every $0 \neq \xi \in H$ there exists a measure $\mu \in \tilde{L}(S, w)$ with $T_\mu \xi \neq 0$. Then there exists a unique w-bounded continuous *-representation V of S by bounded operators on H such that*

$1 \leqq \|V\|_w \leqq k$ *for some positive constant k. The representations T and V are related to each other according to the formula.*

$$\langle T_\mu \xi, \eta \rangle = \int_S \langle V_x \xi, \eta \rangle d\mu(x) \quad (\mu \in \tilde{L}(S, w), \xi, \eta \in H).$$

Furthermore, T and V satisfy all the other conditions of Theorem 2.4. *Moreover, if w is continuous at the identity and* $w(1) = 1$, *then* $\|V\|_w = 1$.

Conjecture. Let S and w be as in the hypotheses of Theorem 5.4. Then Theorem 5.4 easily implies that if the Banach $*$-algebra $\tilde{L}(S, w)$ is $*$-semisimple, then $\mathscr{R}(S, w)$, the space of all w-bounded continuous $*$-representations of S by bounded operators on Hilbert spaces, separates the points of S. We conjecture that the converse is also true, as it is for the case when $w = 1$ by Theorem 3.4.

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