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ADMISSIBLE SOLUTIONS OF THE SCHWARZIAN DIFFERENTIAL EQUATION

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Abstract

Let $R(z, w)$ be a rational function of w with meromorphic coefficients. It is shown that if the Schwarzian equation

$$(*) \qquad \{w, z\}^m = R(z, w)$$

possesses an admissible solution, then $d + 2m\sum_{j=1}^{l} \delta(\alpha_j, w) \leq 4m$, where α_j are distinct complex constants. In particular, when $R(z, w)$ is independent of z, it is shown that if $(*)$ possesses an admissible solution $w(z)$, then by some Möbius transformation $u = (aw + b)/(cw + d)$ $(ad - bc \neq 0)$, the equation can be reduced to one of the following forms:

$$\{u, z\} = C\frac{(u - \sigma_1)(u - \sigma_2)(u = \sigma_3)(u - \sigma_4)}{(u - \tau_1)(u - \tau_2)(u - \tau_3)(u - \tau_4)},$$

$$\{u, z\}^3 = C\frac{[(u - \sigma_1)^3(u - \sigma_2)^3]}{[(u - \tau_1)^3(u - \tau_2)^2(u - \tau_3)]},$$

$$\{u, z\}^3 = C\frac{[(u - \sigma_1)^3(u - \sigma_2)^3]}{[(u - \tau_1)^2(u - \tau_2)^2(u - \tau_3)^2]},$$

$$\{u, z\}^2 = C\frac{[(u - \sigma_1)^2(u - \sigma_2)^2]}{[(u - \tau_1)^2(u - \tau_2)(u - \tau_3)]},$$

$$\{u, z\} = C\frac{[(u - \sigma_1)(u - \sigma_2)]}{[(u - \tau_1)(u - \tau_2)]},$$

$$\{u, z\} = C,$$

where τ_j $(j = 1, \dots, 4)$ are distinct constants, and σ_j $(j = 1, \dots, 4)$ are constants, not necessarily distinct.

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1. Introduction

Let $w(z)$ be a meromorphic function, and $\{w, z\}$ be its Schwarzian derivative:

$$\{w, z\} = \left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2. \tag{1.0}$$

Here we consider the differential equation

$$\{w, z\}^m = R(z, w) = P(z, w)/Q(z, w), \tag{1.1}$$

where $P(z, w)$ and $Q(z, w)$ are polynomials of w with meromorphic coefficients, with $\deg_w[P(z, w)] = p$ and $\deg_w[Q(z, w)] = q$, respectively:

$$\begin{cases} P(z, w) = \xi_p(z)w^p + \xi_{p-1}(z)w^{p-1} + \cdots + \xi_0(z), & \xi_p(z) \not\equiv 0, \\ Q(z, w) = \eta_q(z)w^q + \eta_{q-1}(z)w^{q-1} + \cdots + \eta_0(z), & \eta_q(z) \not\equiv 0, \end{cases} \tag{1.1'}$$

where $\xi_j(z)$, $\eta_k(z)$ are meromorphic functions. We suppose that $P(z, w)$ and $Q(z, w)$ are mutually prime. Sometimes we call

$$\xi_j(z)/\eta_q(z) \quad \text{and} \quad \eta_k(z)/\eta_q(z) \tag{1.1''}$$

the *reduced coefficients* of $R(z, w)$. Put

$$\max(p, q) = \deg_w[R(z, w)] = d. \tag{1.2}$$

We are concerned with the determination of the equations (1.1) which admit transcendental meromorphic solutions.

Steinmetz [11] treated the case $m = 1$ and $d = 0$ in (1.1), and the present author [4] investigated the case $m = 1$ and $d \geq 0$. Here we will consider the case $m \geq 1$ and $d \geq 0$.

We use standard notations in Nevanlinna theory.

Let $f(z)$ be a meromorphic function. As usual, $m(r, f)$, $N(r, f)$ and $T(r, f) = m(r, f) + N(r, f)$ denote the proximity function, the counting function, and the characteristic function of $f(z)$, respectively. For $\alpha \in \mathbb{C}$, put

$$m(r, \alpha; f) = m(r, 1/(f - \alpha)), \qquad N(r, \alpha; f) = N(r, 1/(f - \alpha)).$$

Sometimes, we write $m(r, f)$ or $N(r, f)$ as $m(r, \infty; f)$ and $N(r, \infty; f)$.

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Let $\overline{n}(r, f)$ be the number of distinct poles of $f(z)$ in $|z| \le r$, and put

$$\overline{N}(r, f) = \int_0^r \frac{\overline{n}(t, f) - \overline{n}(0, f)}{t}\, dt + \overline{n}(0, f) \log r,$$

$$\overline{N}(r, \alpha; f) = \overline{N}(r, 1/(f-\alpha)), \qquad \alpha \in \mathbb{C},$$

$$N_1(r, f) = N(r, f) - \overline{N}(r, f), \qquad N_1(r, \alpha; f) = N_1(r, 1/(f-\alpha)).$$

Let $\overline{n}^*(r, 0; f, g)$ be the number of distinct common zeros of $f(z)$ and $g(z)$ in $|z| \le r$, and put

$$\overline{N}^*(r, 0; f, g) = \int_0^r \frac{\overline{n}^*(t, 0; f, g) - \overline{n}^*(0, 0; f, g)}{t}\, dt + n^*(0, 0; f, g) \log r.$$

Further we put as usual [9, pages 226, 277, 280]

$$\delta(\alpha, f) = \varliminf_{r\to\infty} \frac{m(r, \alpha; f)}{T(r, f)} = 1 - \varlimsup_{r\to\infty} \frac{N(r, \alpha; f)}{T(r, f)} \qquad \text{(deficiency)},$$

$$\theta(\alpha, f) = \varliminf_{r\to\infty} \frac{N_1(r, f)}{T(r, f)} \qquad \text{(ramification index)},$$

$$\Theta(\alpha, f) = \varliminf_{r\to\infty} \frac{m(r, \alpha; f) + N_1(r, \alpha; f)}{T(r, f)} \qquad \text{(total ramification)}.$$

A function $\varphi(r)$, $0 \le r < \infty$, is said to be $S(r, f)$ if there is a set $E \subset \mathbb{R}^+$ of finite linear measure such that

$$\varphi(r) = o(T(r, f)) \quad \text{as } r \to \infty, \quad r \notin E.$$

A meromorphic function function $a(z)$ is called *small with respect to* $f(z)$, if $T(r, a) = S(r, f)$.

Let $a_1(z), \dots, a_n(z)$ be meromorphic functions. A transcendental meromorphic function $w(z)$ is called *admissible* with respect to $a_j(z)$. If

$$T(r, a_j) = S(r, w), \qquad j = 1, \dots, n.$$

We call $w(z)$ an *admissible solution* of (1.1), if $w(z)$ satisfies (1.1) and is admissible with respect to the reduced coefficients of $R(z, w)$ (see (1.1″)). *In this paper, "admissible" implies "transcendental".*

REMARK 1. Suppose (1.1) possesses an admissible solution $w = w(z)$. Then we have $\overline{N}^*(r, 0; P, Q) = S(r, w)$, where $P(z) = P(z, w(z))$ and $Q(z) = Q(z, w(z))$. For, since $P(z, w)$ and $Q(z, w)$ are mutually prime, there exist polynomials of w, $U(z, w)$ and $V(z, w)$ such that

$$(*) \qquad U(z, w)P(z, w) + V(z, w)Q(z, w) = s_{P,Q}(z) = s(z),$$

where $s(z)$ and coefficients of $U(z, w)$ and $V(z, w)$ are small functions

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with respect to $w(z)$. Suppose $\overline{N}(r, 0; P, Q) \neq S(r, w)$. Then $N(r, 1/s) \neq S(r, w)$, which is a contradiction. Hence if z_0 is a zero of $Q(z, w(z))$ which is neither a zero nor a pole of $s(z)$ or of the coefficients of $P(z, w)$ and $Q(z, w)$, then z_0 is a pole of $R(z) = P(z, w(z))/Q(z, w(z))$ (see [5, page 169]).

Our results are as follows:

THEOREM 1. *Let $\alpha_1, \alpha_2, \dots, \alpha_l$ be distinct constants. If (1.1) possesses an admissible solution, then we have*

$$d + 2m \sum_{j=1}^{l} \delta(\alpha_j, w) \leq 4m. \tag{1.3}$$

REMARK 2. Inequality (1.3) is a limitation for $d = \deg[R]$ by deficiencies of a solution. Mues [8] classified the algebraic Riccati equations by the number of Picard exceptional values of transcendental solutions. In this connection we classified [4] the Riccati equations (with meromorphic coefficients) by the number of Picard exceptional values of admissible solutions. We showed that, if $w(z)$ satisfies a Riccati equation, then $w(z)$ also satisfies a Schwarzian differential equation (1.1) with $m = 1$ for some $R(z, w)$, and that, if $w(z)$ has $l (= 1$ or $2)$ Picard exceptional values, then $\deg_w[R(z, w)] = 4 - 2l$. Further, if $w(z)$ has no Picard value, then $\deg_w[R(z, w)] = 2$ or 4. Theorem 1 is a generalization of these results.

THEOREM 2. *If (1.1) possesses an admissible solution, then the denominator $Q(z, w)$ of $R(z, w)$ must be one of the following*:

$$Q(z, w) = c(z)(w + b_1(z))^{2m}(w + b_2(z))^{2m}, \tag{1.4}$$

$$Q(z, w) = c(z)(w^2 + a_1(z)w + a_0(z))^{2m}, \tag{1.5}$$

$$Q(z, w) = c(z)(w + b(z))^{2m}, \tag{1.6}$$

$$Q(z, w) = c(z)(w + b(z))^{2m}(w - \tau_1)^m(w - \tau_2)^m, \tag{1.7}$$

$$Q(z, w) = c(z)(w + b(z))^{2m}(w - \tau_1)^{2m/n}, \quad n|(2m),\ n \geq 2, \tag{1.8}$$

$$Q(z, w) = c(z)(w - \tau_1)^m(w - \tau_2)^m(w - \tau_3)^m(w - \tau_4)^m, \tag{1.9}$$

$$Q(z, w) = c(z)(w - \tau_1)^m(w - \tau_2)^m(w - \tau_3)^{2m/n}, \quad n|(2m),\ n \geq 2, \tag{1.10}$$

$$Q(z, w) = c(z)(w - \tau_1)^m(w - \tau_2)^{2m/3}(w - \tau_3)^{2m/3}, \tag{1.11}$$

$$Q(z, w) = c(z)(w - \tau_1)^m(w - \tau_2)^{2m/3}(w - \tau_3)^{2m/4}, \tag{1.12}$$

$$Q(z, w) = c(z)(w - \tau_1)^m(w - \tau_2)^{2m/3}(w - \tau_3)^{2m/5}, \tag{1.13}$$

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$$(1.14)\qquad Q(z, w) = c(z)(w-\tau_1)^m(w-\tau_2)^{2m/3}(w-\tau_3)^{2m/6},$$

$$(1.15)\qquad Q(z, w) = c(z)(w-\tau_1)^{2m/3}(w-\tau_2)^{2m/3}(w-\tau_3)^{2m/3},$$

$$(1.16)\qquad Q(z, w) = c(z)(w-\tau_1)^m(w-\tau_2)^{2m/4}(w-\tau_3)^{2m/4},$$

$$(1.17)\qquad Q(z, w) = c(z)(w-\tau_1)^{2m/n_1}(w-\tau_2)^{2m/n_2}, \qquad n_j|(2m), n_j \geq 2,$$

$$(1.18)\qquad Q(z, w) = c(z)(w-\tau_1)^{2m/n}, \qquad n|2m, n \geq 2,$$

$$(1.19)\qquad Q(z, w) = c(z),$$

where $c(z)$, $a_1(z)$, $a_0(z)$ *are meromorphic functions,* $|a_1'| + |a_2'| \neq 0$, $b_1(z)$, $b_2(z)$, $b(z)$ *are nonconstant meromorphic functions, and* τ_j $(j = 1, 2, 3, 4)$ *are distinct constants.*

In particular, if $R(z, w)$ in (1.1) is independent of z, then we have

$$(1.20)\qquad \{w, z\}^m = P(w)/Q(w) = (w-\sigma_1)^{\lambda_1}\cdots(w-\sigma_n)^{\lambda_h}/Q(w),$$

where σ_j $(j = 1, \ldots, h)$ are distinct constants, and $Q(w)$ is one of the polynomials (1.9)–(1.19), with $c(z)$ constant.

THEOREM 3. *Suppose, in* (1.1), *that* $R(z, w)$ *is independent of* z. *If* (1.20) *possesses an admissible solution* $w(z)$, *then by some Möbius transformation* $u = (aw+b)/(cw+d)$, $ad - bc \neq 0$, *the equation can be reduced to one of the following forms*:

$$(1.21)\qquad \{u, z\} = C\frac{(u[\sigma_1)(u-\sigma_2)(u-\sigma_3)(u-\sigma_4)}{(u-\tau_1)(u-\tau_2)(u-\tau_3)(u-\tau_4)},$$

$$(1.22)\qquad \{u, z\} = C\frac{[(u-\sigma_1)^3(u-\sigma_2)^3]}{[(u-\tau_1)^3(u-\tau_2)^2(u-\tau_3)]},$$

$$(1.23)\qquad \{u, z\}^3 = C\frac{[(u-\sigma_1)^3(u-\sigma_2)^3]}{[(u-\tau_1)^2(u-\tau_2)^2(u-\tau_3)^2]},$$

$$(1.24)\qquad \{u, z\}^2 = C\frac{[(u-\sigma_1)^2(u-\sigma_2)^2]}{[(u-\tau_1)^2(u-\tau_2)(u-\tau_3)]},$$

$$(1.25)\qquad \{u, z\} = C\frac{[(u-\sigma_1)(u-\sigma_2)]}{[(u-\tau_1)(u-\tau_2)]},$$

$$(1.26)\qquad \{u, z\} = C,$$

where τ_j $(j = 1, \ldots, 4)$ *are distinct constants, and* σ_j $(j = 1, \ldots, 4)$ *are constants, not necessarily distinct.*

The equations (1.21)–(1.26) possess admissible solutions.

We will prove these theorems in Sections 3, 4, 5, respectively.

2. Preliminary material

We recall some well-known properties of the Schwarzian derivative [2].

Lemma A. *Let $w(z)$ be a meromorphic function.*

(a) *If z_0 is a simple pole of $w(z)$, then $\{w, z\}$ is regular at z_0.*

(b) *If z_0 is a multiple pole of $w(z)$ or a zero of $w'(z)$, then z_0 is a double pole of $\{w, z\}$. Further, if*

$$w(z) = c_m(z - z_0)^{-m} + c_{m+1}(z - z_0)^{-m+1} + \cdots$$

or

$$w(z) = c_0 + c_m(z - z_0)^m + c_{m+1}(z - z_0)^{m+1} + \cdots$$

with $c_m \neq 0$, $m \geq 2$, in a neighborhood of z_0, then we have

$$\{w, z\} = [(1 - m^2)/2](z - z_0)^{-2} + [(m^2 - 1)c_{m-1}/mc_m](z - z_0)^{-1} + \cdots$$

or

$$\{w, z\} = [(1 - m^2)/2](z - z_0)^{-2} + [(1 - m^2)c_{m+1}/mc_m](z - z_0)^{-1} + \cdots,$$

respectively.

(c) *$\{L(w), z\} = \{w, z\}$ for any Möbius transformation L.*

The following theorem was proved in [3].

Theorem B. *Let $f(z)$ be a transcendental meromorphic function and $Q(z, f)$ be a polynomial of f with small meromorphic coefficients with respect to f and $\deg[Q] \leq n - 2$. Let $a(z)$ be a small meromorphic function and $F(z) = A(z)f(z)^n - Q(z, f(z))$. If $Q(z, f(z)) \not\equiv 0$, then*

$$wT(r, f) \leq \overline{N}(r, f) + \overline{N}(r, 0; f) + \overline{N}(r, 0; F) + S(r, f). \tag{2.1}$$

Steinmetz and Rieth characterized differential equations of the form (2.2) below, which have admissible solutions:

Theorem C ([5], [10], [12]). *Let $R(z, w)$ be a rational function of w with meromorphic coefficients. Suppose the differential equation*

$$w'^m = R(z, w) \tag{2.2}$$

admits an admissible solution $w = w(z)$. Then, by a Möbius transformation $u = (aw + b)/(cw + d)$, $ad - bc \neq 0$, (2.2) is reduced to one of the following *equations*:

(R) $u' = a(z) + b(z)u + c(z)u^2$;

(H) $u'^2 = a(z)(u - b(z))^2(u - \tau_1)(u - \tau_2)$, $b(z) \neq \tau_1, \tau_2$;

(E_1) $u'^2 = a(z)(u - \tau_1)(u - \tau_2)(u - \tau_3)(u - \tau_4)$;

(E_2) $u'^3 = a(z)(u - \tau_1)^2(u - \tau_2)^2(u - \tau_3)^2$;

(E_3) $u'^4 = a(z)(u - \tau_1)^2(u - \tau_2)^3(u - \tau_3)^3$;

(E_4) $u'^6 = a(z)(u - \tau_1)^3(u - \tau_2)^4(u - \tau_3)^5$;

where τ_j *are distinct constants, and* $a(z)$, $b(z)$, $c(z)$ *are meromorphic.*

We further need the following lemmas:

LEMMA 1. *Suppose* $w = w(z)$ *is an admissible solution of* (1.1). *If we write* $Q(z, w(z))$ *as* $Q(z)$, *then*

$$qT(r, w) + S(r, w) \leq N(r, 1/Q). \tag{2.3}$$

LEMMA 2. *Let the polynomial* $Q(z, w)$ *be factored as follows*:

$$Q(z, w) = c(z)(V_1(z, w))^{\mu_1} \cdots (V_k(z, w))^{\mu_k}, \tag{2.4}$$

where $c(z)$ *is meromorphic and* $V_j(z, w)$, $j = 1, \dots, k$, *are polynomials of* w *with meromorphic coefficients, irreducible and mutually prime. Suppose* (1.1) *possesses an admissible solution* $w = w(z)$.

(i) *If* $V_{j_0}(z) = \frac{\partial}{\partial z} V_{j_0}(z, w)|_{w=w(z)} \not\equiv 0$, *then* $\mu_{j_0} = 2m$.

(ii) *If* $V_{j_1 z}(z) = \frac{\partial}{\partial z} V_{j_1}(z, w)|_{w=w(z)} \equiv 0$, *then* $\mu_{j_1} | (2m)$ *and* $\mu_{j_1} \leq m$.

Now we consider the case when $R(z, w)$ is independent of z:

$$\{w, z\}^m = P(w)/Q(w) = (w - \sigma_1)^{\lambda_1} \cdots (W - \sigma_h)^{\lambda_h}/Q(w). \tag{1.20}$$

Without loss of generality, we may assume below that $\deg[P(w)] = \deg[Q(w)]$, by applying a Möbius transformation L to w if necessary.

LEMMA 3. *Suppose* (1.20) *possesses an admissible solution* $w(z)$. *If* $w(z)$ *takes the value* σ_1, *then* $m | \lambda_i$.

LEMMA 4. *Suppose* (1.20), *with* $Q(w)$ *of the form* (1.18), *where* $c(z)$ *is constant, possesses an admissible solution. Then we have* $n = 2$.

LEMMA 5. *Suppose* (1.20) *possesses an admissible solution. Then* $m | \lambda_i$, $i = 1, \dots, h$.

LEMMA 6. *Let* $C \neq 0$, σ_i, $i = 1, 2, 3$, *by constants and* τ_j, $j = 1, 2, 3$, *be distinct constants. Then the differential equation*

$$\{w, z\} = C(w - \sigma_1)(w - \sigma_2)(w - \sigma_3)/[(w - \tau_1)(w - \tau_2)(w - \tau_3)] \tag{2.5}$$

possesses no admissible solution.

LEMMA 7. *Let* C *be a nonzero constant and* σ, τ *be distinct constants. Then the differential equation*

$$\{w, z\} = C(w - \sigma)/(w - \tau) \tag{2.6}$$

possesses no admissible solution.

LEMMA 8. *Let* $C \neq 0$, σ_1, *and* σ_2 *be constants and* τ_1, τ_2 *be distinct constants. Then the differential equation*

$$\{w, z\}^2 = C(w - \sigma_1)(w - \sigma_2)/[(w - \tau_1)(w - \tau_2)] \tag{2.7}$$

possesses no admissible solution.

Finally we define a usual symbol ω. For a meromorphic function $g(z)$, we define $\omega(z_0, g)$ as follows: if z_0 is a pole of order m (≥ 1) for $g(z)$, then $\omega(z_0, g) = m$; if $g(z_0) \neq \infty$, then $\omega(z_0, g) = 0$.

3. Proofs of Lemma 1 and Theorem 1

PROOF OF LEMMA 1. By the lemma on logarithmic derivatives [5], we have

$$m(r, R) = m(r, \{w, z\}^m) = S(r, w), \tag{3.1}$$

where R denotes $R(z, w(z))$. It is proved in [7] that

$$dT(r, w) + S(r, w) = T(r, R) = N(r, R) + m(r, R). \tag{3.2}$$

By (3.1) and (3.2), we have

$$dT(r, w) = N(r, R) + S(r, w). \tag{3.3}$$

If $p > q$ so that $d = p$, then

$$\begin{aligned} N(r, R) &\leq (p - q)N(r, w) + N(r, 1/Q) + S(r, w) \\ &\leq (p - q)T(r, w) + N(r, 1/Q) + S(r, w), \end{aligned} \tag{3.4}$$

where Q denotes $Q(z, w(z))$. By (3.3) and (3.4), $qT(r, w) + S(r, w) \leq N(r, 1/Q)$, which proves (2.3) for the case $p > q$. If $p \leq q = d$, then

$$N(r, R) \leq N(r, 1/Q) + S(r, w). \tag{3.5}$$

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By (3.3) and (3.5), we also obtain (2.3) in the case $p \le q$.

PROOF OF THEOREM 1. First we consider the case $p \le q$. By Remark 1, there exists a small (w.r.t. $w(z)$) function $s_{P,Q}(z) = s(z)$ such that common zeros of $P(z, w(z))$ and $Q(z, w(z))$ are zeros of $s(z)$. Let z_0 be a zero of $Q(z, w(z))$ such that, in (1.1′), $\xi_j(z_0) \ne 0, \infty$ and $\eta_k(z_0) \ne 0, \infty$ for $0 \le j \le p$, $0 \le k \le q$ and $s(z_0) \ne 0$. By (1.1) and Lemma A(b), z_0 must be a zero of $w'(z)$ and hence a double pole of $\{w, z\}$. Thus we have

$$2m\overline{N}(r, 1/w') = N(r, 1/Q) + S(r, w). \tag{3.6}$$

Hence

$$\frac{1}{2m} N(r, 1/Q) \le N(r, 1/w') + S(r, s). \tag{3.7}$$

By Lemma 1 and the second fundamental theorem we obtain, using (3.7), that

$$\frac{d}{2m} T(r, w) + \sum_{j=1}^{l} m(r, \alpha_j; w) \le 2T(r, w) + S(r, w),$$

and thus we obtain (1.3) in this case:

$$d + 2m \sum_{j=1}^{l} \delta(\alpha_j, w) \le 4m. \tag{3.8}$$

Next, suppose $p > q$. Choose $c \in \mathbb{C}$ such that $Q(z, c) \not\equiv 0$ and put $u = 1/(w - c)$ in (1.1). Then by Lemma A(c)

$$\{u, z\}^m = \frac{\xi_p(z)\left(\frac{1}{u}\right)^p +, \dots, +P(z, c)}{\eta_q(z)\left(\frac{1}{u}\right)^q +, \dots, +Q(z, c)} = P_1(z, u)/Q_1(z, u)$$

and $\deg_u[Q_1(z, u)] = d$, and hence we can apply the arguments for the case $p \le q$ and obtain (3.8) also.

EXAMPLE 1. Suppose $w(z)$ satisfies the Schwarzian differential equation $\{w, z\} = R(z, w)$. By Theorem 1, if $w(z)$ possesses j Picard exceptional values $(j = 1, 2)$, then $\deg_w[R(z, w)] \le 4 - 2j$. Solutions of the equations

$$w' = w^2 + \alpha w + \beta \qquad (\alpha, \beta \in \mathbb{C}, \alpha^2 - 4\beta \ne 0), \tag{3.9}$$

$$w' = (w - \alpha)(w + z) \qquad (\alpha \in \mathbb{C}), \tag{3.10}$$

$$w' = (w + z)^2, \tag{3.11}$$

$$w' = w^2 + z \tag{3.12}$$

possess two, one, no and again no Picard exceptional values, respectively, and satisfy the corresponding one of the following equations:

$$(3.9') \qquad \{w, z\} = 2(\beta - \alpha^2/4),$$

$$(3.10') \qquad \{w, z\} = \frac{-[(4+z^2)w^2 + 2z(z^2+2)w + z^4 + 3]}{[2(w+z)^2]},$$

$$(3.11') \qquad \{w, z\} = \frac{-4[(w+z)^2 + a]}{(w+z)^2},$$

$$(3.12') \qquad \{w, z\} = \frac{[4zw^4 - 8w^3 + 8z^2w^2 - 8zw + 4z^3 - 3]}{[(2(w^2+z)^2]}.$$

4. Proofs of Lemma 2 and Theorem 2

PROOF OF LEMMA 2. (i) Write $V_{j_0}(z, w(z))$ simply as $V(z)$. Since $V(z, w)$ is irreducible, $V(z, w)$ and $V_z(z, w) = \partial V(z, w)/\partial z$ are mutually prime as polynomials of w. Thus by Remark 1, there exists a $s_{V, V_z}(z)$, which is a small function with respect to $w(z)$, such that $\overline{N}^*(r, 0; V, V_z) \le N(r, 1/s_{V, V_z}) \le S(r, w)$ (see [5, pages 173–174]).

By Lemma 1, $V(z)$ has infinitely many zeros and $m(r, 0; V) = S(r, w)$. By Remark 1, $\overline{N}^*(r, 0; P, V) \le N(r, 1/s_{P, V}) \le S(r, w)$. Let z_0 be a zero of $V(z)$ which is neither a zero of $c(z)$ nor a zero of coefficients of $P(z, w)$ as well as coefficients of $V_j(z, w)$ $(j = 1, \dots, k)$ nor a zero of $s_{V, V_z}(z)$ and $s_{P, V}(z)$. By the proof of Theorem 1, $w'(z_0) = 0$ and hence $(V'(z) = dV(z, w(z))/dz$ and $V_z(z) = \partial V(z, w)/\partial|_{w=w(z)})$

$$V'(z_0) = V_z(z_0) + w'(z_0)V_w(z_0, w(z_0)) - V_z(z_0) \neq 0.$$

Thus z_0 is a simple zero of $V(z)$. By Lemma A(b), z_0 is a double pole of $\{w, z\}$. Thus by (1.1), $2m = \mu_{j_0}$.

(ii) We may write $V_{j_1}(z, w) = w - \tau$, $\tau \in \mathbb{C}$. By Lemma 1 and Remark 1, $m(r, \tau; w) = S(r, w)$ and $N(r, 0; P, V_{j_1}) \le N(r, 1/s_{P, V_{j_1}}) \le S(r, w)$. Let z_0 be a τ-point of w which is neither a pole nor a zero of $c(f)$, neither a pole nor a zero of coefficients of $P(z, w)$ and $V_j(z, w)$, $j = 1, \dots, k$, nor a zero of $s_{P, V_{j_1}}(z)$. Since $\{w, z\}$ has a pole at z_0, we must have $\omega(z_0, 1/(w - \tau)) = n \ge 2$. Thus z_0 is a double pole of $\{w, z\}$, and hence

$$(4.1) \qquad 2m = n\mu_{j_1},$$

which implies that $\mu_{j_1} \le m$ and $\mu_{j_1} | (2m)$.

PROOF OF THEOREM 2. The following four cases are to be considered.

I. There are $V_1(z, w) \neq V_2(z, w)$ such that $V_{1z}(z)V_{2z}(z) \neq 0$.

II. There is only one $V_1(z, w)$ for which $V_{1z}(z) \neq 0$. Further we suppose that $\deg_w[V_1(z, w)] \geq 2$.

III. There is only one $V_1(z, w)$ for which $V_{1z}(z) \not\equiv 0$. Further we suppose that $\deg_w[V_1(z, w)] = 1$.

IV. $V_{jz}(z) \equiv 0$ for any j.

We will treat these four cases in order.

CASE I. By Lemma 2 and Theorem 1, $Q(z, w)$ is of the form (1.4), since $d \leq 4m$.

CASE II. By Lemma 2 and Theorem 1, $Q(z, w)$ is of the form (1.5), since $d \leq 4m$.

CASE III. $Q(z, w)$ must be of the form

$$Q(z, w) = c(z)(w + b(z))^{2m}(w - \tau_1)^{\mu_1} \cdots (w - \tau_k)^{\mu_k},$$

where $\tau_1, \dots, \tau_k$ are distinct constants. By Lemma 1 and (3.6) we have

$$2m\overline{N}(r, 1/w') = 2mN\left(r, \frac{1}{w + b(z)}\right) + \sum_{j=1}^{k} \mu_j N(r, \tau_j; w) + S(r, w) = \left(2m + \sum_{j=1}^{k} \mu_j\right) T(r, w) + S(r, w). \tag{4.2}$$

Let z_j be a τ_j-point of $w(z)$ such that $\xi_t(z_j) \neq 0, \infty$, $\eta_i(z_j) \neq 0, \infty$, for $t = 0, \dots, p$, $i = 0, \dots, q$ and $s_{P,V}(z_j) \neq 0$, where $V_j = w - \tau_j$. Let $\omega(z_j, 1/(w - \tau_j)) = n_j \geq 2$. By (4.1)

$$2m = n_j \mu_j. \tag{4.3}$$

Since $\omega(z_j, 1/w') = n_j - 1$, we have by Lemma 1 that

$$N_1(r, 0; w') \geq \sum_{j=1}^{k} \left(\frac{n_j - 2}{n_j}\right) N(r, \tau_j; w) + S(r, w) \geq \left(k - 2\sum_{j=1}^{k} \frac{1}{n_j}\right) T(r, w) + S(r, w). \tag{4.4}$$

We have

$$2T(r, w) \geq N(r, 1/w') + S(r, w) \geq \overline{N}(r, 1/w') + N_1(r, 1/w') + S(r, w). \tag{4.5}$$

By (4.2)–(4.5) we have

$$2T(r, w) \geq \left(1 + \sum_{j=1}^{k} \frac{\mu_j}{2m}\right) T(r, w) + \left(k - 2\sum_{j=1}^{k} \frac{\mu_j}{2m}\right) T(r, w) + S(r, w),$$

that is,

$$\left(1 + \sum_{j=1}^{k} \frac{\mu_j}{2m}\right) T(r, w) \geq kT(r, w) + S(r, w),$$

and hence we obtain

$$1 + \sum_{j=1}^{k} \frac{\mu_j}{2m} \geq k. \tag{4.6}$$

By Theorem 1, $2m + \sum_{j=1}^{k} \mu_j \leq 4m$, $\sum_{j=1}^{k} \mu_j \leq 2m$. Therefore $k \leq 2$.

If $k = 2$, then $\mu_1 + \mu_2 = 2m$ and $\mu_1 = \mu_2 = m$ by Lemma 2(ii). Thus we obtain (1.7).

If $k = 1$, then we obtain (1.8). If $k = 0$, then we obtain (1.6).

CASE IV. $Q(z, w)$ must be of the form

$$Q(z, w) = c(z)(w - \tau_1)^{\mu_1} \cdots (w - \tau_k)^{\mu_k},$$

where $\tau_1, \ldots, \tau_k$ are distinct constants. By Lemma 1

$$2m\overline{N}(r, 1/w') = \sum_{j=1}^{k} \mu_j N(r, \tau_j; w) + S(r, w) = \sum_{j=1}^{k} \mu_j T(r, w) + S(r, w).$$

By (4.3)–(4.5) we obtain, as in the case III, that

$$2 + \sum_{j=1}^{k} \frac{\mu_j}{2m} \geq k. \tag{4.7}$$

By Theorem 1, $\sum_{j=1}^{k} \mu_j \leq 4m$. Hence we get $k \leq 4$.

If $k = 4$, then $\mu_1 = \mu_2 = \mu_3 = \mu_4 = m$ by Lemma 2(ii), and we have (1.9).

If $k = 3$, then from (4.7) and (4.3)

$$\frac{\mu_1 + \mu_2 + \mu_3}{2m} = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} \geq 1 \qquad (n_j \geq 2). \tag{4.8}$$

The only triplets (n_1, n_2, n_3) which satisfy (4.8) are as follows (we suppose $n_1 \le n_2 \le n_3$):

$$(4.9)\qquad \begin{cases} (2, 2, n), \text{ where either } n = 2, \text{ or } n \ge 3 \text{ and } n|m; \\ (2, 3, 3), (2, 3, 4), (2, 3, 5), (2, 3, 6); \\ (2, 4, 4), (3, 3, 3). \end{cases}$$

Therefore we get, by (4.9), that

$(\mu_1, \mu_2, \mu_3) = (m, m, 2m/n)$ (either $n = 2$, or $n \ge 3$ and $n|m$), which corresponds to (1.10);

$(\mu_1, \mu_2, \mu_3) = (m, 2m/3, 2m/3)$ which corresponds to (1.11);
$(\mu_1, \mu_2, \mu_3) = (m, 2m/3, 2m/4)$ which corresponds to (1.12);
$(\mu_1, \mu_2, \mu_3) = (m, 2m/3, 2m/5)$ which corresponds to (1.13);
$(\mu_1, \mu_2, \mu_3) = (m, 2m/3, 2m/6)$ which corresponds to (1.14);
$(\mu_1, \mu_2, \mu_3) = (m, 2m/4, 2m/4)$ which corresponds to (1.15);
$(\mu_1, \mu_2, \mu_3) = (2m/3, 2m/3, 2m/3)$ which corresponds to (1.16).

If $k = 2$, then $n_j|m$ and $n_j \ge 2$ $(j = 1, 2)$, and we get (1.17). If $k = 1$, then we have (1.18). If $k = 0$, we obtain (1.19).

5. Proofs of Lemma 3, 4, 5, 6, 7, 8, and Theorem 3

We suppose that the equation (1.1) is of the form (1.20).

Proof of Lemma 3. Let z_i be a σ_i point of $w(z)$. Then z_i is a zero of $\{w, z\}$, and hence by Lemma A(b), z_i is not a zero of $w'(z)$. Thus $\omega(z_i, 1/(w - \sigma_i)) = 1$. Put $\omega(z_i, 1/\{w, z\}) = n$. Then

$$(5.1)\qquad nm = \lambda_i, \text{ and hence } m|\lambda_i.$$

Proof of Lemma 4. Suppose that $n \ge 3$ in (1.18). We have that $h = 1$ in (1.20). In fact, suppose $h \ge 2$. Since $\deg[P(w)] = \deg[Q(w)] = 2m/n < m$, we have that σ_1 and σ_2 are Picard exceptional values of $w(z)$ by Lemma 3. Thus, by Theorem 1, $d = \deg[Q(w)] = 0$, which is a contradiction. Thus we may assume that the equation is of the following form:

$$(5.2)\qquad \{w, z\}^m = c_0 \left(\frac{w - \sigma}{w - \tau}\right)^{2m/n}, \qquad c_0(\ne 0) \in \mathbb{C}.$$

Put $u = (w - \sigma)/(w - \tau)$ in (5.2). Then by Lemma A(c),

$$(5.3)\qquad \{u, z\}^n = cu^2, \qquad c = c_0^{n/m}.$$

By the lemma on logarithmic derivatives,

$$2m(r, u) + O(1) = m(r, cu^2) = m(r, \{u, z\}^n) = S(r, u),$$

and hence

$$m(r, u) = S(r, u). \tag{5.4}$$

Let z_0 be a pole of $u(z)$ and $\omega(z_0, u) = \mu$. Suppose $\mu = 1$. Then by Lemma A(a), $\{u, z\}$ is regular at z_0, which contradicts (5.3). Thus $\mu \geq 2$. By Lemma A(b), $2n = 2\mu$. Therefore u has infinitely many poles of order n. By Lemma 3, u has no zeros since $n \geq 3$ and hence $n \nmid 2$. Also u' has no zeros as seen by Lemma A(b). Thus u'/u and u''/u' admit (simple) poles at poles of u only. Further, residues of u'/u and u''/u' are $-n$ and $-(n+1)$, respectively. If we put

$$n\phi = (n+1)u'/u - nu''/u', \tag{5.5}$$

then ϕ is an entire function. We have

$$m(r, \phi) = S(r, u), \tag{5.6}$$

and hence ϕ is a small function for u.

Write $u'/u = f$ and $u''/u' = g$. Then

$$f' = fg - f^2. \tag{5.7}$$

From (5.5) we have

$$g = af - \phi, \tag{5.8}$$

where $a = (n+1)/n$. From (5.7) and (5.8) we have

$$g' = af' - \phi' = (a^2 - a)f^2 - a\phi f - \phi'. \tag{5.9}$$

From (5.9) and (5.8) we obtain

$$\{u, z\} = g' - \frac{1}{2}g^2 = (a^2/2 - a)f^2 - \phi' - \frac{1}{2}\phi^2.$$

Since we supposed $n \geq 3$, we get $A = a^2/2 - a = a(a-2)/2 \neq 0$. Thus u satisfies the first order differential equation

$$(A(u'/u)^2 - \Phi)^n = cu^2, \tag{5.10}$$

where $\Phi = \phi' + \frac{1}{2}\phi^2$. Now $\Phi \not\equiv 0$ since, if $\Phi \equiv 0$, then equation (5.10) does not admit a transcendental solution. Put $F = Af^2 - \Phi = A(u'/u)^2 - \Phi$. Then F has no zeros as seen from (5.10), since u has no zeros. Applying Theorem B to $f = u'/u$, noting Φ is a small function with respect to $f(z)$, we obtain

$$\begin{aligned} 2T(r, f) &\leq \overline{N}(r, f) + \overline{N}(r, 0; f) + \overline{N}(r, 0; F) + S(r, f) \\ &\leq \overline{N}(r, f) + S(r, f) \leq T(r, f) + S(r, f), \end{aligned}$$

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a contradiction, which shows that $n < 3$.

PROOF OF LEMMA 5. Since we are assuming $\deg[P] = \deg[Q]$, the case (1.19), that is, $Q(w) = C$, a constant, need not to be considered. Since Q is independent of z, only (1.9)–(1.18) are to be considered. It suffices to show that the solution $w(z)$ takes any σ_i, as seen by Lemma 3. To the contrary we suppose that $w(z)$ has no σ_i points for some i. By Theorem 1, $d \le 2m$. This is impossible for (1.9)–(1.13), since $p = q$. If $Q(w)$ is of the form (1.14), by Lemma 1, we have $m(r, \tau_j; w) = S(r, w)$, $j = 1, 2, 3$. Let z_j be a τ_j point. Then by (4.3), $\omega(z_1, 1/w - \tau_1) = 2$, $\omega(z_2, 1/w - \tau_2) = 3$, $\omega(z_3, 1/w - \tau_3) = 6$. Hence we have $\sum_{j=1}^{3} \theta(\tau_j, w) = 1/2 + 2/3 + 5/6 = 2$, which contradicts Nevanlinna's theorem on total ramification, since $w(z)$ is supposed to omit σ_i. Similarly to the case (1.14), if $Q(w)$ is of the form (1.15) and (1.16), then we have $\sum_{j=1}^{3} \theta(\tau_j, w) = 2/3 + 2/3 + 2/3 = 2$, and $\sum_{j=1}^{3} \theta(\tau_j, w) = 1/2 + 3/4 + 3/4$, respectively, which are also contradictions. Thus for (1.9)–(1.13) and (1.14)–(1.16), $w(z)$ must take σ_i, $i = 1, 2, \dots, h$.

Suppose $Q(w)$ is of the form (1.17). If $n_1 > 2$ or $n_2 > 2$, then similarly to the case (1.14), we have $\theta(\tau_1, w) + \theta(\tau_2, w) = (n_1 - 1)/n_1 + (n_2 - 1)/n_2 \ge 7/6/$, which contradicts Nevanlinna's theorem, and since we have $n_1 = n_2 = 2$. Suppose $P(w)$ has a factor $(w - \sigma)^{\lambda}$, $m \nmid \lambda$. Since $p = q = 2m$ in (1.17), there is another factor $(w - \tilde{\sigma})^{\tilde{\lambda}}$, $m \nmid \tilde{\lambda}$. Then both σ and $\tilde{\sigma}$ are Picard values for $w(z)$, and we have $d = 0$ by Theorem 1, which is a contradiction.

Finally suppose $Q(w)$ is of the form (1.18). By Lemma 4, $\deg[Q] = m$. As in (1.17), we see that $P(w)$ must be of the form $(w - \sigma)^m$, which completes the proof of Lemma 5.

PROOF OF LEMMA 6. Suppose (2.5) possesses an admissible solution $w(z)$. Put $u = 1/(w - \tau_3)$. Then by Lemma A(c), we have

$$\{u, z\} = C(u - s_1)(u - s_2)(u - s_3)/[(u - t_1)(u - t_2)]. \tag{5.11}$$

By Lemma 1 and (4.3), $w(z)$ has infinitely many τ_3 points which are all of multiplicity 2. Therefore $u(z)$ has infinitely many poles which are all of order 2. Therefore $u(z)$ has infinitely many poles which are all of order 2. Let z_0 be a pole of $u(z)$, then

$$u(z) = \frac{R}{(z - z_0)^2} + \frac{\alpha}{(z - z_0)} + O(1), \qquad R \ne 0. \tag{5.12}$$

By Lemma A(b), we get

$$\{u, z\} = \frac{-3/2}{(z - z_0)^2} + \frac{3\alpha/2R}{(z - z_0)} + O(1). \tag{5.13}$$

On the other hand, the right-hand side of (5.11) can be written

$$(5.14)\qquad \begin{aligned} &C(u-s_1)(u-s_2)(u-s_3)/[(u-t_1)(u-t_2)] \\ &\quad = CR(z-z_0)^{-2} + C\alpha(z-z_0)^{-1} + O(1) \end{aligned}$$

near z_0. Thus, from (5.13) and (5.14), $-3/2 = CR$ and $3\alpha/2R = C\alpha$. Hence

$$(5.15)\qquad R = -\frac{3}{2C}, \qquad \alpha = 0.$$

Put

$$H(z) = u'(z)^2/[(u(z)-t_1)(u(z)-t_2)].$$

By (4.3), each t_j point ($j = 1, 2$) is of multiplicity 2. Thus $u'(z)$ has a simple zero there. Hence $H(z) \neq \infty$ at t_j points of $u(z)$. Thus, if z_0 is pole of $H(z)$, then z_0 is pole of $u(z)$, and $\omega(z_0, H) = 2$. By (5.12) and (5.15) we have

$$(5.16)\qquad H(z) = 4(z-z_0)^{-2} + O(1).$$

Put

$$(5.17)\qquad \varphi(z) = \frac{H(z)}{4} + \frac{2C}{3}u(z).$$

Then by (5.12), (5.15) and (5.16), $\varphi(z)$ is regular at z_0. Thus $\varphi(z)$ is an entire function. By Lemma 1, we have

$$m(r, u) = m(r, \tau_3; w) = S(r, w) = S(r, u).$$

Hence by the lemma on logarithmic derivatives

$$\begin{aligned} m(r, \varphi) &\le m(r, H) + m(r, u) + O(1) \\ &\le m(r, u'/(u-\tau_1)) + m(r, u'/(u-\tau_2)) \\ &\quad + m(r, u) + O(1) = S(r, u). \end{aligned}$$

Therefore $\varphi(z)$ is a small function for $u(z)$. From (5.17), we have

$$(5.18)\qquad u'^2 = C^*(u-t_1)(u-t_2)(u-\tilde{\varphi}), \qquad C^* = -8C/3, \qquad \tilde{\varphi} = -3\varphi/2C.$$

By Theorem C and (5.18), $\tilde{\varphi}$ is a constant.

Let z_* be a zero of $u'(z)$. By Lemma A(b), z_* is a pole of $\{u, z\}$, whence z_* is a t_1 or t_2 point of $u(z)$, as seen from (5.11). Thus, zeros of u' are t_j points of u, therefore by (5.18) we have that $\tilde{\varphi} = t_1$ or t_2, or $\tilde{\varphi}$ is Picard exceptional value. If $\tilde{\varphi} = t_1$, then

$$(5.18')\qquad u'^3 = C^*(u-t_1)^2(u-t_2).$$

Suppose $u(z_1) = t_1$ and put $\omega(z_1, 1/(u-t_1)) = l$. Then $\omega(z_1, 1/u'^2) = (2l-2) \neq \omega(z_1, 1/C^*(u-t_2)) = 2l$, whence $u(z)$ cannot take t_1. This

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contradicts Lemma 1, and hence $\tilde{\varphi} \neq t_1$. Similarly $\tilde{\varphi} \neq t_2$. If $\tilde{\varphi}$ is a Picard exceptional value, then $2 \geq d = 3$ by Theorem 1. Hence we obtain a contradiction. Therefore (5.11), and hence (2.5), cannot possess admissible solutions.

PROOF OF LEMMA 7. Suppose (2.6) possesses an admissible solution $w(z)$. Put $u = c(w-\sigma)/(w-\tau)$. Then

$$\{u, z\} = u. \tag{5.19}$$

By the lemma on the logarithmic derivative, $m(r, u) = m(r, \{u, z\}) = S(r, u)$. Hence $u(z)$ has infinitely many poles, which are of multiplicity 2 by (5.19) and Lemma A(b). Let z_0 be a pole of $u(z)$. Then

$$u(z) = \frac{R}{(z-z_0)^2} + \frac{\alpha}{(z-z_0)} + O(1) \qquad (R \neq 0). \tag{5.20}$$

Arguing as in the proof of Lemma 6, we obtain

$$R = 3/2, \qquad \alpha = 0. \tag{5.21}$$

Put $g = u''/u'$ in (5.19). Then

$$g' - \frac{1}{2}g^2 = u. \tag{5.22}$$

By (5.19) and Lemma A(b), we see that u' has no zeros. Thus if $\tilde{z}$ is a pole of $g(z)$, then $\tilde{z}$ is a pole of $u(z)$. From (5.20) and $\alpha = 0$, we get

$$g(z) = -3/(z-z_0) + O(z-z_0). \tag{5.23}$$

Put

$$\varphi = g' - \frac{1}{3}g^2. \tag{5.24}$$

Then $\varphi(z)$ is regular at z_0. Thus $\varphi(z)$ is entire. On the other hand

$$\begin{aligned} T(r, g) &= m(r, g) + N(r, g) = m(r, u''/u') + \overline{N}(r, u) \\ &= \frac{1}{2}T(r, u) + S(r, u), \end{aligned} \tag{5.25}$$

which shows that $S(r, u) = S(r, g)$. Thus

$$m(r, \varphi) \leq m(r, \{u, z\}) + m(r, (u''/u')^2) + O(1) = S(r, g).$$

Thus $\varphi(z)$ is a small function for $g(z)$. From (5.22) and (5.24), we have

$$\varphi(z) - \frac{1}{6}g(z)^2 = u(z). \tag{5.26}$$

By (5.26) and (5.24),

$$u' = \varphi' - \frac{1}{3}gg' = \varphi' - \frac{1}{9}g^3 - \frac{1}{3}\varphi g,$$
$$u'' = \varphi'' - \frac{1}{3}g'^2 - \frac{1}{3}gg''$$
$$= \varphi'' - \frac{1}{3}g'^2 - \frac{1}{3}g\left(\varphi' + \frac{2}{3}gg'\right)$$
$$= \varphi'' - \frac{1}{3}\varphi^2 - \frac{1}{9}g^4 - \frac{4}{9}\varphi g^2 - \frac{1}{3}\varphi' g.$$

Hence

$$g = u''/u' = \frac{9\varphi'' - 3\varphi^2 - g^4 - 4\varphi g^2 - 3\varphi' g}{9\varphi' - g^3 - 3\varphi g},$$

that is,

$$\varphi g^2 + 12\varphi' g - 9\varphi'' + 3\varphi^2 = 0. \tag{5.27}$$

if $\varphi \not\equiv 0$, g must be small for g, as seen by solving the quadratic equation (5.27) for g. Thus is impossible, and hence $\varphi \equiv 0$. Therefore by (5.24), $g(z)$ cannot be transcendental. This shows that u and hence w is not transcendental, contrary to our hypothesis.

PROOF OF LEMMA 8. Suppose (2.7) possesses an admissible solution $w(z)$. By Lemma 5, we have $\sigma_1 = \sigma_2$. Let L be a Möbius transformation which maps σ_1, τ_1, τ_2 to $\infty, 1, -1$, respectively. Put $u = L(w)$. Then

$$\{u, z\}^2 = 1/(1-u^2). \tag{5.28}$$

Put $V(z)^2 = v(z) = 1/(1-u(z)^2)$. Then $V(z)$ is meromorphic by (5.28) and by a simple calculation, we obtain

$$\left[\{v, z\} + \frac{3}{8}\left(\frac{v'}{v(v-1)}\right)^2\right]^2 = V^2 = v. \tag{5.29}$$

By Lemma 1 and (5.28), $u(z) \pm 1$ have infinitely many zeros, which are of multiplicity 4 by (4.3). Thus poles of $v(z)$ are infinite in number and of order 4. Let z_0 be a pole of $v(z)$. Then

$$v(z) = \frac{R}{(z-z_0)^4} + \frac{\alpha}{(z-z_0)^4} + O((z-z_0)^{-2}), \qquad R \neq 0. \tag{5.30}$$

Also $\frac{v'}{v(v-1)}$ is regular at z_0, and thus we get by Lemma A(b)

$$\{v, z\} + \frac{3}{8}\left(\frac{v'}{v(v-1)}\right)^2 = \frac{-15/2}{(z-z_0)^2} + \frac{15\alpha/4R}{(z-z_0)} + O(1). \tag{5.31}$$

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From (5.30) and (5.31), $R = (-15/2)^2$, $\alpha = -(-15/2)^2\alpha/R$. Hence we have

$$R = 225/4, \qquad \alpha = 0. \tag{5.32}$$

From (5.29) we see that, if $v'(\tilde{z}) = 0$, then $v(\tilde{z}) = 0$ or 1, since v is regular and $\{v, z\} = \infty$ at $z = \tilde{z}$. If $v(\tilde{z}) = 0$, then $u(\tilde{z}) = \infty$. By (5.28) and Lemma A(a), $\tilde{z}$ is a simple pole of u and hence is a double zero of v. If $v(\tilde{z}) = 1$, then $u(\tilde{z}) = 0$. By (5.28), $u'(\tilde{z}) \neq 0$. Thus $\tilde{z}$ is a simple zero of u and hence a double zero of $v(z) - 1$. Hence $\omega(\tilde{z}, 1/v') = 1$. If we put

$$h = v'^2/[v(v-1)], \quad \text{and} \quad \phi = h'/h, \tag{5.33}$$

then $h(z)$ has no zeros and infinitely many poles only at poles of v. From (5.33), poles of h are of order 2. Hence by (5.30) and (5.32) we have

$$h(z) = \frac{16}{(z - z_0)^2} + O(1). \tag{5.34}$$

Since $h(z)$ has no zeros, $\phi = \infty$ only at poles of h, and hence at poles of v. Thus by (5.34), we can write ϕ as

$$\phi(z) = \frac{-2}{(z - z_0)} + O(z - z_0). \tag{5.35}$$

Put

$$\tau(z) = \phi'(z) - \frac{1}{2}\phi(z)^2 \quad \text{and} \quad \sigma(z) = \phi'(z) - \frac{1}{8}h(z). \tag{5.36}$$

Then $\tau(z)$ and $\sigma(z)$ are regular at z_0. Thus $\tau(z)$ and $\sigma(z)$ are entire. By Lemma 1 and (5.29), (5.33), we have

$$\begin{aligned} 2T(r, V) + O(1) = T(r, v) &= m(r, v) + N(r, v) \\ &= N(r, v) + S(r, u) \end{aligned}$$

and

$$\begin{aligned} T(r, h) = m(r, h) + N(r, h) &= S(r, v) + \frac{1}{2}N(r, v) \\ &= \frac{1}{2}T(r, v) + S(r, v). \end{aligned}$$

Hence $S(r, u) = S(r, v) = S(r, V) = S(r, h)$. From (5.36) and (5.33),

$$m(r, \tau) \leq m(r, (h''/h) - (h'/h)^2) + m(r, (h'/h)^2) + O(1) = S(r, V),$$

and

$$m(r, \sigma) \leq m(r, (h''/h) - (h'/h)^2) + m(r, h) + O(1) = S(r, V).$$

Thus, $\tau(z)$ and $\sigma(z)$ are small functions for $V(z)$. From (5.29) and (5.33), we have by simple calculation

$$V = \frac{1}{2}\phi' - \frac{1}{8}\phi^2 - \frac{1}{2}h. \tag{5.37}$$

From (5.37) and (5.36), we have

$$V = (-15/32)h + \kappa, \tag{5.38}$$

where $\kappa(z) = \frac{1}{4}(\tau(z) + \sigma(z))$. On the other hand, from (5.29) and (5.33) we have

$$h = (2VV')^2/[V^2(V^2 - 1)] = 4V'^2/(V^2 - 1). \tag{5.39}$$

Hence, from (5.38) and (5.39), $V(z)$ satisfies the equation

$$V'^2 = -\frac{8}{15}(V - \kappa)(V - 1)(V + 1). \tag{5.40}$$

By Theorem C and (5.40), $\kappa(z)$ is a constant. If $\kappa \neq \pm 1$, then $V(z) - \kappa$ has infinitely many double zeros, and hence $v'(z) = 0$ at some (infinitely many) zeros of $v - \kappa^2$, which is impossible by (5.29). Therefore $\kappa = 1$ or -1.

By (5.40), $V'' = -\frac{4}{15}(3V^2 - 2\kappa V - 1)$. Since $V^2 = 1/(1 - u^2)$, we get $(u'/u)^2 = V'^2[V(V^2 - 1)]^2 = -\frac{8}{15}(V - \kappa)/[V^2(V^2 - 1)]$ and $u''/u = -2V'/V + \frac{1}{2}V'/(V - \kappa)$. thus we obtain

$$\{u, z\} = -2V''/V + \frac{1}{2}V''/(V - \kappa) - \frac{5}{8}V'^2/(V - \kappa)^2 + V'^2/[V(V - \kappa)].$$

Using (5.40), we have that

(5.41)

$$\{u, z\} = V + \frac{13}{15}, \quad \left(\{u, z\} - \frac{13}{15}\right)^2 = V^2 = 1/(1 - u^2) \quad \text{if } \kappa = 1,$$

$$\{u, z\} = V - \frac{13}{15}, \quad \left(\{u, z\} - \frac{13}{15}\right)^2 = V^2 = 1/(1 - u^2) \quad \text{if } \kappa = 1,$$

which contradicts (5.28). Therefore, (2.7) cannot possess any admissible solutions.

PROOF OF THEOREM 3. Suppose the equation (1.20) possesses an admissible solution $w = w(z)$. We assume that $p = q = d$, by applying a Möbius transformation if necessary. By Lemma 5, $q = p$ is a multiple of m. Thus $Q(w)$ cannot be the form of (1.11)–(1.13). If $Q(w)$ is of the form of (1.10), then $n = 2$ by the same reason. In this case, the equation (1.20) must be the form of (2.5), which is impossible by Lemma 6. If $Q(w)$ is the form of (1.18), then (1.20) is the form of (2.6), which is also rejected by Lemma 7. If $Q(w)$ is the form of (1.17), then by Lemma 5 we have

$$2m/n_1 + 2m/n_2 = 2m \text{ or } m \qquad (n_1, n_2 \geq 2). \tag{5.42}$$

Thus $(n_1, n_2) = (2, 2)$ or (4.4). Therefore we get the form of (1.25) or (2.7), respectively. By Lemma 8, the case (4, 4) is rejected. If $Q(w)$ is of the form of (1.9), (1.14), (1.15), (1.16) or (1.19), then we obtain the equation of the form (1.21), (1.22), (1.23), (1.24) or (1.26), respectively, which proves our assertion.

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