BULL. AUSTRAL. MATH. SOC. 22A05, 54H11, 54E15
VOL. 51 (1995) [309–335]

ON GROUP UNIFORMITIES ON THE SQUARE OF A SPACE AND EXTENDING PSEUDOMETRICS

MICHAEL G. TKAČENKO

We give some conditions under which, for a given pair (d_1, d_2) of continuous pseudometrics respectively on X and X^2, there exists a continuous semi-norm N on the free topological group $F(X)$ such that $N(x \cdot y^{-1}) = d_1(x, y)$ and $N(x \cdot y \cdot t^{-1} \cdot z^{-1}) \geqslant d_2((x, y), (z, t))$ for all $x, y, z, t \in X$. The "extension" results are applied to characterise thin subsets of free topological groups and obtain some relationships between natural uniformities on X^2 and those induced by the group uniformities ${}^*\mathcal{V}$, $\mathcal{V}^*$ and ${}^*\mathcal{V}^*$ of $F(X)$.

0. INTRODUCTION

By a theorem of Nummela [7] and Pestov [8], the two-sided uniformity ${}^*\mathcal{V}^*$ of the free topological group $F(X)$ induces the finest possible uniformity on X compatible with its topology, that is, ${}^*\mathcal{V}^*|_X = \mathcal{U}_X$, where $\mathcal{U}_X$ is the universal uniformity of X. This important result is the starting point of our investigation of uniformities on X^2 induced by ${}^*\mathcal{V}$, $\mathcal{V}^*$ and ${}^*\mathcal{V}^*$, the left, right and two-sided group uniformities of $F(X)$. There are at least three natural problems in this area:

A. What are the relations between the uniformities ${}^*\mathcal{V}|_{X^2}, \mathcal{V}^*|_{X^2}$ and ${}^*\mathcal{V}^*|_{X^2}$ on one hand and $\mathcal{U}_X \times \mathcal{U}_X$, $\mathcal{U}_{X^2}$ on the other hand ($\mathcal{U}_{X^2}$ stands for the universal uniformity of X^2)?

B. When does the equality ${}^*\mathcal{V}^*|_{X^2} = \mathcal{U}_X \times \mathcal{U}_X$ hold?

C. For which spaces X does the equality ${}^*\mathcal{V}^*|_{X^2} = \mathcal{U}_{X^2}$ hold?

One can as well replace ${}^*\mathcal{V}^*$ by ${}^*\mathcal{V}$ or $\mathcal{V}^*$ in Problems B and C, thus obtaining four more problems. To settle these problems we elaborate a method of simultaneous "extension" of a pair (d_1, d_2) of continuous concordant pseudometrics from X and X^2 respectively to a continuous semi-norm N on $F(X)$ (to a semi-norm on the open subgroup $G(X)$ of $F(X)$, to be precise). Theorem 1.4 and Theorem 2.1 are the main results of the paper going in this direction. However, we postpone treating Problems A–C till the forthcoming paper (with the same title) because of the length of the present one.

Received 25th May, 1994

https://doi.org/10.1017/S0004972700014143 Published online by Cambridge University Press

Graev [**3**] was the first to apply an extension of continuous pseudometrics from a set X of generators to the free topological group $F(X)$ over X. Graev's method of extension of pseudometrics made possible a proof of the equality ${}^*\mathcal{V}^*|_X = \mathcal{U}_X$ (see [**7**, **8**]). Various special extensions of continuous pseudometrics from a space X to different free topological algebras generated by X were considered in [**12**, **6**, **13**, **11**]. Graev extension was also used in [**11**] to show that if X admits a one-to-one continuous mapping onto a metrisable space, then $F(X)$ is a NSS-group, that is, has no small subgroups. Unfortunately, we can not apply any of those constructions for our purpose because they all produce invariant pseudometrics on $F(X)$, and hence can not distinguish the left and right group uniformities on $F(X)$. The use of certain pseudometrics on X^2 and their extensions to $F(X)$ enables us to do that.

The principal idea of our construction is to produce continuous semi-norms on $F(X)$ which are "sensitive" to inner automorphisms of $F(X)$ generated by elements of X. More precisely, let N be a semi-norm on $F(X)$ which right-induces a pseudometric d on X, that is, $d(a,b) = N(a \cdot b^{-1})$ for all $a, b \in X$. Suppose that there exists a function f on X such that $N(x \cdot a \cdot b^{-1} \cdot x^{-1}) = f(x) \cdot d(a,b)$ for all $a, b \in X$. We can say that f is a rate of sensibility of the semi-norm N. How fast can the function f grow? The inequality

$$\begin{aligned} N(yab^{-1}y^{-1}) &= N(yx^{-1} \cdot xab^{-1}x^{-1} \cdot xy^{-1}) \\ &\leqslant N(yx^{-1}) + N(xab^{-1}x^{-1}) + N(xy^{-1}) = N(xab^{-1}x^{-1}) + 2d(x,y) \end{aligned}$$

shows that the pseudometric d is a natural regulator for N and f. In particular, if d is reasonably non-vanished, that is, there exist points $a, b \in X$ with $d(a,b) = 1$, then the above inequality implies that $|f(x) - f(y)| \leqslant 2d(x,y)$ for all $x, y \in X$. This explains our special attention in the second part of the paper to the case when the pseudometric d satisfies the condition $d(x,y) = |f(x) - f(y)|$ for all $x, y \in X$.

In the first part of the paper we define the notion of right-concordant pseudometrics (Definition 1.3) and prove that if continuous pseudometrics d_1 and d_2 on X and X^2 respectively are right-concordant, then there exists a continuous semi-norm N on $F(X)$ such that $N(ab^{-1}) = d_1(a,b)$, $N(axy^{-1}a^{-1}) = d_2((a,x),(a,y))$ and $N(axy^{-1}b^{-1}) \geqslant d_2((a,x),(b,y))$ for all $a, b, x, y \in X$ (Theorem 1.4).

One natural way of construction of right-concordant pseudometrics is given in the second part of the paper (see Theorem 2.1). In the forthcoming paper this special method will be applied to solve Problems A–C for various classes of spaces. We apply this method here only once to give an alternative and short proof of Theorem 3 of [**15**], characterising subspaces of a space X which are thin in $F(X)$.

The results of the paper on extension of concordant pseudometrics were announced (without proofs) in [**16**].

All spaces are assumed Tikhonov. The free topological group over a space X is denoted by $F(X)$. The set of positive integers is denoted by N^+.

1. Extension of concordant pseudometrics

Let X be a space. Every element g of the group $F(X)$ has the form $g = x_1^{\varepsilon_1} \cdot \ldots \cdot x_n^{\varepsilon_n}$, where $x_1, \ldots, x_n \in X$ and $\varepsilon_1, \ldots, \varepsilon_n = \pm 1$. Denote by $l_+(g)$ the number of indices $i \leqslant n$ with $\varepsilon_i = 1$, and $l_-(g)$ the number of indices $i \leqslant n$ with $\varepsilon_i = -1$. We put

$$G(X) = \{g \in F(X) : l_+(g) = l_-(g)\}.$$

It is easy to see that $G(X)$ is an open subgroup of $F(X)$. Indeed, let f be a mapping of X to the discrete group Z of integers, $f(x) = 1$ for each $x \in X$. Extend f to a continuous homomorphism $\widehat{f} : F(X) \to Z$. Then $G(X)$ is the kernel of $\widehat{f}$, and hence is open in $F(X)$.

Thus, a study of properties of the group $F(X)$ can practically be reduced to a study of corresponding properties of $G(X)$. We give here only one result showing the difference between $F(X)$ and $G(X)$.

Assertion 1.1. *The group $G(X)$ is connected if and only if X is connected.*

Proof: Assume that the space X is connected. For every integer $n \in N^+$ and $\overline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n) \in \{1, -1\}^n$ denote by $i_{\overline{\varepsilon}}$ the mapping of X^n to $F(X)$ defined by $i_{\overline{\varepsilon}}(x_1, \ldots, x_n) = x_1^{\varepsilon_1} \cdot \ldots \cdot x_n^{\varepsilon_n}$. The mapping $i_{\overline{\varepsilon}}$ is continuous, and since X^n is connected, so is $i_{\overline{\varepsilon}}(X^n)$. Let $S(\overline{\varepsilon}) = \sum\limits_{i=1}^{n} \varepsilon_i$. Note that if $S(\overline{\varepsilon}) = 0$, then the identity e of $F(X)$ belongs to $i_{\overline{\varepsilon}}(X^n)$. So, the connectedness of $G(X)$ follows from the equality $G(X) = \bigcup\{i_{\overline{\varepsilon}}(X^{2n}) : n \in N^+, \overline{\varepsilon} \in \{1, -1\}^{2n} \text{ and } S(\overline{\varepsilon}) = 0\}$.

Assume then that X is disconnected. We can write $X = U \cup V$ for some disjoint open non-empty subsets U, V of X. Let G be a discrete free group with two generators a and b. Denote by φ a mapping from X to G defined by $\varphi(x) = a$ for each $x \in U$ and $\varphi(x) = b$ for each $x \in V$. Extend φ to a continuous homomorphism $\widehat{\varphi} : F(X) \to G$. Then the kernel of $\widehat{\varphi}$, $\ker \widehat{\varphi}$, is clopen in $F(X)$, and is a proper subset of $G(X)$. Thus, $G(X)$ is disconnected. ∎

Definition 1.2: A real-valued non-negative function N defined on a group G with identity e is called a *semi-norm* if it satisfies the following conditions:

(N1) $N(e) = 0$;
(N2) $N(g) = N(g^{-1})$ for each $g \in G$;
(N3) $N(g \cdot h) \leqslant N(g) + N(h)$ for all $g, h \in G$.

Functions satisfying (N1)–(N3) were called *norms* in [5, 4]. Since there could be elements $g \in G \setminus \{e\}$ with $N(g) = 0$ (and such elements do exist if G does not admit

https://doi.org/10.1017/S0004972700014143 Published online by Cambridge University Press

a coarser metrisable topology), we prefer the term *semi-norm*, analogous to that in the theory of linear spaces.

Let d be a continuous pseudometric on a space X. Denote by $\widehat{d}$ the Graev extention of d to a maximal invariant pseudometric on the open subgroup $G(X)$ of $F(X)$. [Of course, one can extend d to an invariant pseudometric on $F(X)$; however, there is no maximal and there is no "natural" among such extentions. This is the main argument in favor of the consideration of $G(X)$ instead of $F(X)$.] We can define a continuous semi-norm N_d on $G(X)$ by $N_d(g) = \widehat{d}(g, e)$; $g \in G(X)$. By the definition of $\widehat{d}$, we have the following equalities for arbitrary $x, y, z, t \in X$:

(GR) $N_d\left(x \cdot y^{-1}\right) = d(x,y)$ and $N_d\left(x \cdot y \cdot t^{-1} \cdot z^{-1}\right) = d(x,z) + d(y,t)$;
(GL) $N_d\left(x^{-1} \cdot y\right) = d(x,y)$ and $N_d\left(y^{-1} \cdot x^{-1} \cdot z \cdot t\right) = d(x,z) + d(y,t)$.

Suppose we are given another continuous semi-norm N on $G(X)$ that right-induces the same pseudometric d on X, that is, $N\left(x \cdot y^{-1}\right) = d(x,y)$ for all $x, y \in X$. Define a continuous pseudometric d_2 on X^2 by $d_2((x,y),(z,t)) = N\left(x \cdot y \cdot t^{-1} \cdot z^{-1}\right)$ for $x, y, z, t \in X$. The main problem is the following one: can d_2 be any continuous pseudometric on X^2, or must there be some relations between d and d_2?

For example, if $N = N_d$, the pseudometric d completely defines the corresponding pseudometric d_2, for $d_2((x,y),(z,t)) = d(x,y) + d(z,t)$ in this case. In general, we can not hope to generate d_2 by means of d, but at least one relation between them is obvious:

(R1) $d_2((a,x),(b,x)) = N\left(a \cdot x \cdot x^{-1} \cdot b^{-1}\right) = d(a,b)$ for all $a, b, x \in X$.

We can also say that the pseudometric d_2 is invariant with respect to lifting or descent of horizontal intervals in the "plane" X^2. So, $d_2(A,B) = d_2(C,D)$ whenever horizontal intervals $[A,B]$ and $[C,D]$ have the same projections to the first factor X.

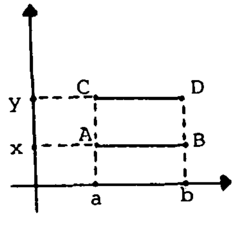

Figure 1

It does not seem surprising that (R1) is not the only relation between the pseudometrics d and d_2 induced by the same semi-norm N. In the fourth part of the paper we give an example explaining this phenomenon. After this preliminary discussion we are ready to present the main notion of the paper. Let X be a set and suppose that d_1 and d_2 are pseudometrics on X and X^2 respectively.

DEFINITION 1.3: Pseudometrics d_1 and d_2 are called right-concordant if they satisfy the following conditions:

(C1) $d_2((a,x),(b,x)) = d_1(a,b)$ for all $a,b,x \in X$;

(C2) $d_2((a_0,x_0),(a_{n+1},x_n)) \leqslant \sum\limits_{i=0}^{n} d_1(a_i,a_{i+1}) + \sum\limits_{i=1}^{n} d_2\big((a_{\pi(i)},x_{i-1}),(a_{\pi(i)},x_i)\big)$ for all $a_0,a_1,\dots,a_{n+1},x_0,x_1,\dots,x_n \in X$ and any permutation π of the set $\{1,\dots,n\}$.

Changing the places of a,b and x in (C1) and a_i,x_j in (C2) gives the definition of left-concordant pseudometrics:

(CL1) $d_2((x,a),(x,b)) = d_1(a,b)$ for all $a,b,x \in X$;

(CL2) $d_2((x_0,a_0),(x_n,a_{n+1})) \leqslant \sum\limits_{i=0}^{n} d_1(a_i,a_{i+1}) + \sum\limits_{i=1}^{n} d_2\big((x_{i-1},a_{\pi(i)}),(x_i,a_{\pi(i)})\big)$ for all $a_0,\dots,a_{n+1},x_0,\dots,x_n \in X$ and any permutation π of the set $\{1,\dots,n\}$.

The condition (C2) of the above definition is more obscure and complicated than (C1). However, if we hope to induce both pseudometrics d_1 and d_2 by means of one semi-norm on $G(X)$ using a "reasonable" construction, it will likely (or inevitably) require some special condition such as (C2) (see Example 4).

The following theorem is our main result on extension of pseudometrics.

THEOREM 1.4. (Right case) *Let d_1 and d_2 be continuous right-concordant pseudometrics on X and X^2 respectively. Then there exists a continuous semi-norm $N = N_r$ on $G(X)$ satisfying the following conditions:*

(R1) $N\big(a\cdot b^{-1}\big) = N\big(a^{-1}\cdot b\big) = d_1(a,b)$ *for all* $a,b \in X$;

(R2) $N\big(a\cdot x\cdot y^{-1}\cdot a^{-1}\big) = d_2((a,x),(a,y))$ *for all* $a,x,y \in X$;

(R3) $N\big(a\cdot x\cdot y^{-1}\cdot b^{-1}\big) \geqslant d_2((a,x),(b,y))$ *for all* $a,b,x,y \in X$.

PROOF: Elements of $G(X)$ having the form $g\cdot x^{\varepsilon}\cdot y^{-\varepsilon}\cdot g^{-1}$, where $g \in F(X)$, $x,y \in X$ and $\varepsilon = \pm 1$, will be called *canonical*. In particular, all elements $x^{\varepsilon}\cdot y^{-\varepsilon}$ with $x,y \in X$ are canonical. For every canonical element $h \in G(X)$, a number $M(h) \geqslant 0$ will be defined as follows. If $h = x^{\varepsilon}\cdot y^{-\varepsilon}$, we put $M(h) = d_1(x,y)$. If $h = a^{\varepsilon}\cdot x^{\varepsilon}\cdot y^{-\varepsilon}\cdot a^{-\varepsilon}$ and $a,x,y \in X$, we put $M(h) = d_2((a,x),(a,y))$; if $h = a^{\varepsilon}\cdot x^{-\varepsilon}\cdot y^{\varepsilon}\cdot a^{-\varepsilon}$, we put $M(h) = d_1(x,y) + d_2((a,x),(a,y))$. Suppose then that $h = g\cdot x^{\varepsilon}\cdot y^{-\varepsilon}\cdot g^{-1}$, where $g = a_1^{\varepsilon_1}\cdot\ldots\cdot a_n^{\varepsilon_n}$ and $a_i \in X$, $\varepsilon_i = \pm 1$ for each $i \leqslant n$; $n \geqslant 2$. In this case we put $M(h) = \widehat{d}_1(h) + \sum\limits_{i=1}^{n} d_2((a_i,x),(a_i,y))$, where $\widehat{d}_1$ is the Graev extension of d_1 to an invariant pseudometric on $G(X)$ [**3, 6, 11**].

https://doi.org/10.1017/S0004972700014143 Published online by Cambridge University Press

Let h be an arbitrary element of $G(X)$. Consider all possible representations of h in the form of a product $h = h_1 \cdot \ldots \cdot h_n$ of canonical elements of $G(X)$. To each of such a representation there corresponds the sum $\sum_{i=1}^{n} M(h_i)$. Denote by $N(h)$ the lower bound of these sums. It is clear that $N(e) = 0$ and $N(h) \geqslant 0$ for each $h \in G(X)$. One easily verifies that the function N satisfies the conditions (N2) and (N3) of Definition 1.2, that is, N is a semi-norm.

Let us show that (R3) holds. We need some preliminary definitions. For a given canonical element $h = a_1^{\varepsilon_1} \cdot \ldots \cdot a_n^{\varepsilon_n} \cdot x^{\varepsilon} \cdot y^{-\varepsilon} \cdot a_n^{-\varepsilon_n} \cdot \ldots \cdot a_1^{-\varepsilon_1}$ of $G(X)$, each of the pairs $\{x^{\varepsilon}, y^{-\varepsilon}\}$, $\{a_1^{\varepsilon_1}, a_1^{-\varepsilon_1}\}, \ldots, \{a_n^{\varepsilon_n}, a_n^{-\varepsilon_n}\}$ will be called h-connected (in h). Furthermore, we shall say that the pair $\{x^{\varepsilon}, y^{-\varepsilon}\}$ h-depends on the pair $\{a_i^{\varepsilon_i}, a_i^{-\varepsilon_i}\}$, $1 \leqslant i \leqslant n$. Let $h_1, \ldots, h_m$ be arbitrary canonical elements of $G(X)$. Consider the word $\overline{h} \equiv h_1 h_2 \ldots h_m$ in the alphabet $X \cup X^{-1}$ generated by writing the words $h_1, h_2, \ldots, h_m$ consecutively. We shall get the element $h = h_1 \cdot h_2 \cdot \ldots \cdot h_m$ of $G(X)$ if we perform all possible consecutive reductions in $\overline{h}$ of coinciding neighbouring letters with opposite exponents. Now fix some order of reductions in $\overline{h}$ that transform $\overline{h}$ to h, and define a partition of letters of the irreducible word h to h-connected pairs. We use an induction on the number of reductions and accompany it with the definition of the notion of connectedness and dependence between pairs.

The partition of the letters of the word $\overline{h}$ to $\overline{h}$-connected pairs is naturally defined by partitions of the letters in each of the canonical elements h_i, $1 \leqslant i \leqslant m$. Suppose that a word h' is obtained by means of k consecutive reductions in $\overline{h}$ and that we have already defined a partition of the letters of h' to h'-connected pairs. Let $h' \equiv p x^{\varepsilon} x^{-\varepsilon} q$, where $x \in X$, $\varepsilon = \pm 1$, and suppose that the $(k+1)$-th reduction in h' is the deletion of the letters x^{ε} and $x^{-\varepsilon}$ from h'. There exist letters $y^{-\varepsilon}$ and z^{ε} in h' such that both pairs $\{y^{-\varepsilon}, x^{\varepsilon}\}$ and $\{x^{-\varepsilon}, z^{\varepsilon}\}$ are h'-connected. In the word $h'' \equiv pq$, h''-connected pairs are exactly all h'-connected pairs of h', without two pairs $\{y^{-\varepsilon}, x^{\varepsilon}\}$ and $\{x^{-\varepsilon}, z^{\varepsilon}\}$, and we add one new h''-connected pair $\{y^{-\varepsilon}, z^{\varepsilon}\}$ that arises instead of these two "old" pairs.

Thus, we have defined the partition of letters of h to h-connected pairs (but the notion of h-dependence has not yet been defined). This partition depends on a representation of h in the form of a product $h = h_1 \cdot \ldots \cdot h_m$ of canonical elements of $G(X)$, and we also used some order of reductions in $\overline{h} \equiv h_1 \ldots h_m$.

The following auxiliary lemmas will be helpfull. We omit the simple proofs of them that use an induction on a number of reductions.

Lemma 1. *A partition of letters in h to h-connected pairs does not depend on the order of reductions in $\overline{h}$.*

Lemma 2. *Suppose that $\{x_1^{\varepsilon}, x_2^{-\varepsilon}\}$ and $\{y_1^{\delta}, y_2^{-\delta}\}$ are h'-connected pairs in h'.*

Then the letters $x_1^\varepsilon, x_2^{-\varepsilon}, y_1^\delta$ and $y_2^{-\delta}$ can occur in the word h' in one of the following ways:

(1) $h' \equiv \ldots x_1^\varepsilon \ldots y_1^\delta \ldots y_2^{-\delta} \ldots x_2^{-\varepsilon} \ldots$; (2) $h' \equiv \ldots x_1^\varepsilon \ldots y_2^{-\delta} \ldots y_1^\delta \ldots x_2^{-\varepsilon} \ldots$;
(3) $h' \equiv \ldots x_2^{-\varepsilon} \ldots y_1^\delta \ldots y_2^{-\delta} \ldots x_1^\varepsilon \ldots$; (4) $h' \equiv \ldots x_2^{-\varepsilon} \ldots y_2^{-\delta} \ldots y_1^\delta \ldots x_1^\varepsilon$;
(5) $h' \equiv \ldots y_1^\delta \ldots x_1^\varepsilon \ldots x_2^{-\varepsilon} \ldots y_2^{-\delta} \ldots$; (6) $h' \equiv \ldots y_1^\delta \ldots x_2^{-\varepsilon} \ldots x_1^\varepsilon \ldots y_2^{-\delta} \ldots$;
(7) $h' \equiv \ldots y_2^{-\delta} \ldots x_1^\varepsilon \ldots x_2^{-\varepsilon} \ldots y_1^\delta \ldots$; (8) $h' \equiv \ldots y_2^{-\delta} \ldots x_2^{-\varepsilon} \ldots x_1^\varepsilon \ldots y_1^\delta \ldots$;
(9) $h' \equiv \ldots x_1^\varepsilon \ldots x_2^{-\varepsilon} \ldots y_1^\delta \ldots y_2^{-\delta} \ldots$; (10) $h' \equiv \ldots x_1^\varepsilon \ldots x_2^{-\varepsilon} \ldots y_2^{-\delta} \ldots y_1^\delta \ldots$;
(11) $h' \equiv \ldots x_2^{-\varepsilon} \ldots x_1^\varepsilon \ldots y_1^\delta \ldots y_2^{-\delta} \ldots$; (12) $h' \equiv \ldots x_2^{-\varepsilon} \ldots x_1^\varepsilon \ldots y_2^{-\delta} \ldots y_1^\delta \ldots$;
(13) $h' \equiv \ldots y_1^\delta \ldots y_2^{-\delta} \ldots x_1^\varepsilon \ldots x_2^{-\varepsilon} \ldots$; (14) $h' \equiv \ldots y_1^\delta \ldots y_2^{-\delta} \ldots x_2^{-\varepsilon} \ldots x_1^\varepsilon \ldots$;
(15) $h' \equiv \ldots y_2^{-\delta} \ldots y_1^\delta \ldots x_1^\varepsilon \ldots x_2^{-\varepsilon} \ldots$; (16) $h' \equiv \ldots y_2^{-\delta} \ldots y_1^\delta \ldots x_2^{-\varepsilon} \ldots x_1^\varepsilon \ldots$.

Applying Lemma 2, we define the notion of h'-dependence between some h'-connected pairs as follows. Let $\{x_1^\varepsilon, x_2^{-\varepsilon}\}$ and $\{y_1^\delta, y_2^{-\delta}\}$ be h'-connected pairs in h'. Then in each of the cases (1)–(4) of occurence of $x_1^\varepsilon, x_2^{-\varepsilon}, y_1^\delta, y_2^{-\delta}$ in h' (see Lemma 2) we say that the pair $\{y_1^\delta, y_2^{-\delta}\}$ h'-depends on the pair $\{x_1^\varepsilon, x_2^{-\varepsilon}\}$. In the cases (5)-(8) we say that the pair $\{x_1^\varepsilon, x_2^{-\varepsilon}\}$ h'-depends on $\{y_1^\delta, y_2^{-\delta}\}$. In cases (9)-(16) there is no relation of h'-dependence between these two pairs.

We proceed to the proof of (R3). Suppose that an element $a \cdot x \cdot y^{-1} \cdot b^{-1}$ (with $a, b, x, y \in X$) is written in the form $h_1 \cdot \ldots \cdot h_m$, where $h_1, \ldots, h_m$ are canonical elements of $G(X)$. By the definition of the semi-norm N it suffices to show that $d_2((a,x),(b,y)) \leqslant \sum_{i=1}^m M(h_i)$. Without loss of generality one can assume that all elements h_i, $1 \leqslant i \leqslant m$, are irreducible. This follows from the next obvious lemma.

LEMMA 3. *Let $\overline{p} \equiv a_1^{\varepsilon_1} \ldots a_n^{\varepsilon_n} x^\varepsilon y^{-\varepsilon} a_n^{-\varepsilon_n} \ldots a_1^{-\varepsilon_1}$ be a reducible word, where $a_1, \ldots, a_n$, $x, y \in X$ and $\varepsilon_1, \ldots, \varepsilon_n, \varepsilon = \pm 1$. Then all possible reductions in $\overline{p}$ transform $\overline{p}$ to a canonical element $p \in G(X)$, and $M(p) \leqslant M(\overline{p})$.*

Denote $g = a \cdot x \cdot y^{-1} \cdot b^{-1}$. It is clear that both pairs $\{a, b^{-1}\}$ and $\{x, y^{-1}\}$ are g-connected and the second pair g-depends on the first one. We need one notion more.

DEFINITION OF A CHAIN: A finite family $\mathcal{P}$ of unordered pairs of letters of the alphabet $X \cup X^{-1}$ is called a *chain* between u and v^{-1} for some $u, v \in X$ if one can enumerate $\mathcal{P} = \{p_i : 1 \leqslant i \leqslant k+1\}$ so that $p_1 = \{u, w_1^{-1}\}$, $p_2 = \{w_1, w_2^{-1}\}$, …, $p_k = \{w_{k-1}, w_k^{-1}\}$ and $p_{k+1} = \{w_k, v^{-1}\}$.

Now we proceed an inductive construction by reductions in $\overline{h} \equiv h_1 \ldots h_m$ (moving in the "inverse" direction, from g to $\overline{h}$). Assume again that an order of reductions in $\overline{h}$ is given.

Let $\mathcal{P}_0$ be the family consisting from one pair $\{a, b^{-1}\}$ and $\mathcal{Q}_0$ the family consisting of the pair $\{x, y^{-1}\}$. We shall say that the pair $\{x, y^{-1}\}$ is g-subordinated to the pair $\{a, b^{-1}\}$.

Suppose that we have a word h' after some number $s \geqslant 0$ of reductions in the word $\overline{h}$, and r is a number of reductions transforming h' to $g \equiv axy^{-1}b^{-1}$. Suppose also that we have defined subfamilies $\mathcal{P}_r$ and $\mathcal{Q}_r$ of the family of h'-connected pairs and a relation of h'-subordination between some pairs of $\mathcal{Q}_r$ and $\mathcal{P}_r$ satisfying the following conditions:

(1_r) the families $\mathcal{P}_r$ and $\mathcal{Q}_r$ are disjoint;

(2_r) the family $\mathcal{P}_r$ is a chain between a and b^{-1}, and $\mathcal{Q}_r$ is a chain betweeen x and y^{-1};

(3_r) every pair of $\mathcal{Q}_r$ is h'-subordinated to some pair of $\mathcal{P}_r$, and if $q \in \mathcal{Q}_r$ is h'-subordinated to some pair $p \in \mathcal{P}_r$ then q h'-depends on p.

Let h'' be the word obtained after $s-1$ reductions in $\overline{h}$. The word h' arises after a reduction of two neighbouring letters t^ε and $t^{-\varepsilon}$ of h''. Consider four possible cases.

(a) $\{t^\varepsilon, t^{-\varepsilon}\}$ is an h''-connected pair. Then we put $\mathcal{P}_{r+1} = \mathcal{P}_r$ and $\mathcal{Q}_{r+1} = \mathcal{Q}_r$. The relation of h''-subordination coincides with the relation of h'-subordination.

(b) $\{t^\varepsilon, u^{-\varepsilon}\}$ and $\{t^{-\varepsilon}, v^\varepsilon\}$ are h''-connected pairs for some letters $u^{-\varepsilon}$, v^ε of h'', and $\{u^{-\varepsilon}, v^\varepsilon\}$ belongs to $\mathcal{P}_r$. Denote

$$\mathcal{P}_{r+1} = \big(\mathcal{P}_r \setminus \{\{u^{-\varepsilon}, v^\varepsilon\}\}\big) \cup \{\{t^\varepsilon, u^{-\varepsilon}\},\ \{t^{-\varepsilon}, v^\varepsilon\}\} \text{ and } \mathcal{Q}_{r+1} = \mathcal{Q}_r.$$

The relation of h'-subordination is considered as a mapping $\varphi_r : \mathcal{Q}_r \to \mathcal{P}_r$; the equality $p = \varphi_r(q)$ with $q \in \mathcal{Q}_r$ and $p \in \mathcal{P}_r$ means that q is h'-subordinated to p. Define a relation $\varphi_{r+1} : \mathcal{Q}_{r+1} \to \mathcal{P}_{r+1}$ as follows. If $q \in \mathcal{Q}_r$, $p = \varphi_r(q)$ and $p \neq \{u^{-\varepsilon}, v^\varepsilon\}$, put $\varphi_{r+1}(q) = p$. If $\varphi_r(q) = \{u^{-\varepsilon}, v^\varepsilon\}$, consider three subcases.

(b_1) Either $h'' \equiv \dots t^\varepsilon t^{-\varepsilon} \dots v^\varepsilon \dots u^{-\varepsilon} \dots$, or $h'' \equiv \dots u^{-\varepsilon} \dots v^\varepsilon \dots t^{-\varepsilon} t^\varepsilon \dots$.
Then put $\varphi_{r+1}(q) = \{t^\varepsilon, u^{-\varepsilon}\}$.

(b_2) Either $h'' \equiv \dots v^\varepsilon \dots u^{-\varepsilon} \dots t^\varepsilon t^{-\varepsilon} \dots$, or $h'' \equiv \dots t^{-\varepsilon} t^\varepsilon \dots u^{-\varepsilon} \dots v^\varepsilon \dots$.
In this case we put $\varphi_{r+1}(q) = \{t^{-\varepsilon}, v^\varepsilon\}$.

(b_3) Either $h'' \equiv \dots u^{-\varepsilon} \dots t^\varepsilon t^{-\varepsilon} \dots v^\varepsilon \dots$, or $h'' \equiv \dots v^\varepsilon \dots t^{-\varepsilon} t^\varepsilon \dots u^{-\varepsilon} \dots$.
Now the definition of $\varphi_{r+1}(q)$ is not straightforward. Let $q = \{z_1^\delta, z_2^{-\delta}\}$. If the letters z_1^δ and $z_2^{-\delta}$ occur in the word h'' between the letters $u^{-\varepsilon}$ and t^ε, we put $\varphi_{r+1}(q) = \{u^{-\varepsilon}, t^\varepsilon\}$. Otherwise z_1^δ and $z_2^{-\delta}$ occur between v^ε and $t^{-\varepsilon}$ (apply Lemma 2), and we put $\varphi_{r+1}(q) = \{t^{-\varepsilon}, v^\varepsilon\}$.

(c) $\{t^\varepsilon, u^{-\varepsilon}\}$ and $\{t^{-\varepsilon}, v^\varepsilon\}$ are h''-connected pairs in h'', and the pair

$\{u^{-\varepsilon}, v^{\varepsilon}\}$ belongs to $\mathcal{Q}_r$. Then put

$$\mathcal{Q}_{r+1} = \big(\mathcal{Q}_r \setminus \{\{u^{-\varepsilon}, v^{\varepsilon}\}\}\big) \cup \{\{t^{\varepsilon}, u^{-\varepsilon}\}, \{t^{-\varepsilon}, v^{\varepsilon}\}\} \text{ and } \mathcal{P}_{r+1} = \mathcal{P}_r.$$

Let $q \in \mathcal{Q}_{r+1}$, but $q \neq \{t^{\varepsilon}, u^{-\varepsilon}\}$ and $q \neq \{t^{-\varepsilon}, v^{\varepsilon}\}$. Then we define $\varphi_{r+1}(q) = \varphi_r(q)$. Put also $\varphi_{r+1}(\{t^{\varepsilon}, u^{-\varepsilon}\}) = \varphi_{r+1}(\{t^{-\varepsilon}, v^{\varepsilon}\}) = \varphi_r(\{u^{-\varepsilon}, v^{\varepsilon}\})$.

(d) $\{t^{\varepsilon}, u^{-\varepsilon}\}$ and $\{t^{-\varepsilon}, v^{\varepsilon}\}$ are h''-connected pairs in h'' and the pair $\{u^{-\varepsilon}, v^{\varepsilon}\}$ does not belong to $\mathcal{P}_r \cup \mathcal{Q}_r$. Then define $\mathcal{P}_{r+1}$, $\mathcal{Q}_{r+1}$ and φ_{r+1} as in the case (a).

Lemma 2 implies that (a)–(d) cover all possible cases. This completes our definition of $\mathcal{P}_{r+1}$, $\mathcal{Q}_{r+1}$ and φ_{r+1}. One can verify that the families $\mathcal{P}_{r+1}$, $\mathcal{Q}_{r+1}$ and the relation of subordination φ_{r+1} satisfy conditions $(1_{r+1}) - (3_{r+1})$ in each of the cases (a)–(d).

Denote by n the number of reductions transforming $\overline{h}$ to $g \equiv axy^{-1}b^{-1}$. Put $\mathcal{P} = \mathcal{P}_n$, $\mathcal{Q} = \mathcal{Q}_n$ and $\varphi = \varphi_n$. Then $\mathcal{P}$ is a chain between a and b^{-1}, $\mathcal{Q}$ is a chain between x and y^{-1} and each pair $q \in \mathcal{Q}$ is g-subordinated to the single pair $\varphi(q) \in \mathcal{P}$; moreover, q $\overline{h}$-depends on p. Let $\mathcal{P} = \{p_i : 1 \leqslant i \leqslant k\}$ and $\mathcal{Q} = \{q_j : 1 \leqslant j \leqslant l\}$ be enumerations of $\mathcal{P}$ and $\mathcal{Q}$ that correspond the definition of a chain between two letters. Thus, we can write $p_1 = \{a, c_1^{-1}\}$, $p_2 = \{c_1, c_2^{-1}\}, \ldots,$ $p_k = \{c_{k-1}, b^{-1}\}$ and $q_1 = \{x, t_1^{-1}\}$, $q_2 = \{t_1, t_2^{-1}\}, \ldots,$ $q_l = \{t_{l-1}, y^{-1}\}$. Without loss of generality one can assume that $\mathcal{Q}$ does not contain pairs of the form $\{t, t^{-1}\}$, because the family $\mathcal{Q}'$ obtained by deletion of such pairs from $\mathcal{Q}$ is again a chain between x and y^{-1}.

The partition of the letters of $\overline{h}$ to $\overline{h}$-connected pairs and the relation of $\overline{h}$-dependence between $\overline{h}$-connected pairs are generated by partitions and relations existing "inside" canonical elements h_i, $1 \leqslant i \leqslant m$. Therefore distinct pairs of $\mathcal{Q}$ lie in different elements h_i and $\varphi(q') \neq \varphi(q'')$ whenever $q' \neq q''$. [$\overline{h}$-connected pairs are considered to be distinct if their letters occupy different places in the word $\overline{h}$. For example, the word $W \equiv aab^{-1}b^{-1}$ consists of two W-connected pairs $\{a, b^{-1}\}$ and $\{a, b^{-1}\}$, the first one contains the left and right letters of W, and the second one contains the middle letters of W. We consider these pairs as different, keeping in mind the places that their letters occupy.] From the definition of subordination it follows that for each $q_j \in \mathcal{Q}$, the pair $p_i = \varphi(q_j)$ has the form $\{c, c^{-1}\}$. Thus, for this $p_i = \{c_{i-1}, c_i\}$, the points c_{i-1} and c_i coincide as elements of X (we put $c_0 = a$ and $c_k = b$), the letters of the pairs p_i and q_j lie in the same canonical element h_s for some $s \leqslant m$, and q_j h_s-depends on p_i. In its turn this implies that $d_2((c_i, t_{j-1}), (c_i, t_j)) \leqslant M(h_s)$; we assume that $t_0 = x$, $t_l = y$. If $1 \leqslant j \leqslant l$, $1 \leqslant i \leqslant k$ and $p_i = \varphi(q_j)$, we put $i = \psi(j)$, thus defining the mapping $\psi : \{1, \ldots, l\} \to \{1, \ldots, k\}$.

Let $\mathcal{P}_1 = \varphi(\mathcal{Q})$. Then $\mathcal{P}_1 = \{p_{i_1}, \ldots, p_{i_l}\}$, where $1 \leqslant i_1 < \ldots < i_l \leqslant k$. Delete from $\mathcal{P}_2 = \mathcal{P} \setminus \mathcal{P}_1$ all pairs of the form $\{c, c^{-1}\}$ and denote the resulting family by $\mathcal{P}_3$.

Obviously, the family $\mathcal{R} = \mathcal{P}_1 \cup \mathcal{P}_3$ is a chain between a and b^{-1}. Since distinct pairs of $\mathcal{R}$ occur in different elements h_i, we have the following inequality

$$\sum_{p\in\mathcal{P}_3} d_1(p) + \sum_{j=1}^{l} d_2\big((c_{\psi(j)}, t_{j-1}), (c_{\psi(j)}, t_j)\big) \leqslant \sum_{s=1}^{m} M(h_s). \tag{1}$$

Here $d_1(p)$ stands for $d_1(u,v)$ if a pair $p \in \mathcal{P}_3$ has the form $p = \{u^\varepsilon, v^{-\varepsilon}\}$, $\varepsilon = \pm 1$. Note that each pair of $\mathcal{P}_1$ is of the form $\{c, c^{-1}\}$, and hence $\mathcal{P}_3$ is a chain between a and b^{-1}. Therefore, the triangle inequality for d_1 implies that

$$\sum_{j=0}^{l} d_1\left(c_{i_j}, c_{i_{j+1}}\right) \leqslant \sum_{p\in\mathcal{P}_3} d_1(p), \tag{2}$$

where $c_{i_0} = a$, $c_{i_{l+1}} = b$. Applying (1) and (2), we get

$$\sum_{j=0}^{l} d_1\left(c_{i_j}, c_{i_{j+1}}\right) + \sum_{j=1}^{l} d_2\big((c_{\psi(j)}, t_{j-1}), (c_{\psi(j)}, t_j)\big) \leqslant \sum_{s=1}^{m} M(h_s). \tag{3}$$

Obviously, we have $\{p_{i_1}, \dots, p_{i_l}\} = \mathcal{P}_1 = \{p_{\psi(1)}, \dots, p_{\psi(l)}\}$. Denote $\overline{a}_j = c_{i_j}$, $1 \leqslant j \leqslant l$, $\overline{a}_0 = a$ and $\overline{a}_{l+1} = b$. Let also u be a mapping from $\{1, \dots, l\}$ to $\{1, \dots, k\}$ defined by $u(j) = i_j$, $1 \leqslant j \leqslant l$. Then $\pi = u^{-1} \circ \psi$ is a bijection of $\{1, \dots, l\}$ onto itself, and we can rewrite (3) as follows:

$$\sum_{j=0}^{l} d_1(\overline{a}_j, \overline{a}_{j+1}) + \sum_{j=1}^{l} d_2\big((\overline{a}_{\pi(j)}, t_{j-1}), (\overline{a}_{\pi(j)}, t_j)\big) \leqslant \sum_{s=1}^{m} M(h_s). \tag{4}$$

By condition (C2) of concordance of d_1 and d_2, the left part of (4) is not less than $d_2((\overline{a}_0, t_0), (\overline{a}_{l+1}, t_l))$. Thus, $d_2((a,x),(b,y)) \leqslant \sum\limits_{s=1}^{m} M(h_s)$. This proves the inequality (R3) of the theorem.

Let $a, x, y \in X$ be arbitrary. From (R3) it follows that $d_2((a,x),(a,y)) \leqslant N\big(a \cdot x \cdot y^{-1} \cdot a^{-1}\big)$. However, the element $a \cdot x \cdot y^{-1} \cdot a^{-1}$ is canonical, and the definition of the semi-norm N implies the inequality $N\big(a \cdot x \cdot y^{-1} \cdot a^{-1}\big) \leqslant d_2((a,x),(a,y))$. Thus, (R2) is proved.

To prove (R1), we need one auxiliary result. Recall that $\widehat{d}_1$ is the Graev extension of d_1 to a maximal invariant pseudometric on $G(X)$.

Lemma 4. *Let $a^\varepsilon \cdot b^{-\varepsilon} = g_0 \cdot a_1 \cdot x_1 \cdot y_1^{-1} \cdot a_1^{-1} \cdot g_1 \cdot a_2 \cdot x_2 \cdot y_2^{-1} \cdot a_2^{-1} \cdot \ldots \cdot a_n \cdot x_n \cdot y_n^{-1} \cdot a_n^{-1} \cdot g_n$, where $a, b, a_i, x_i, y_i \in X$, $\varepsilon = \pm 1$ and $g_i \in G(X)$ for each $i \leqslant n$. Then $d_1(a,b) \leqslant \sum\limits_{i=0}^{n} \widehat{d}_1(g_i, e)$, where e is the identity of $G(X)$.*

Proof of Lemma 4: Suppose the contrary; let $\sum_{i=0}^{n} \widehat{d_1}(g_i, e) < d_1(a, b)$. Then $a \neq b$. Denote by w the word $g_0 a_1 x_1 y_1^{-1} a_1^{-1} g_1 \dots a_n x_n y_n^{-1} a_n^{-1} g_n$ and choose a cancellation order in w that transforms w to $a^\varepsilon \cdot b^{-\varepsilon}$.

Recall the definition of the Graev extension $\widehat{d_1}$ of the pseudometric d_1 (see [3] or [11]). We say that a partition of all letters of the word $g \in G(X)$ to disjoint pairs is a *scheme* for g, if the letters of each pair of the scheme have opposite exponents and every two distinct pairs $\{x^\varepsilon, x^{-\varepsilon}\}$ and $\{y^\delta, y^{-\delta}\}$ of the scheme satisfy the conclusion of Lemma 2. [For example, the word $a_1 b_1^{-1} b_2 a_2^{-1}$ admits two schemes: $\{\{a_1, b_1^{-1}\}, \{a_2^{-1}, b_2\}\}$ and $\{\{a_1, a_2^{-1}\}, \{b_1^{-1}, b_2\}\}$.] For an arbitrary scheme $S = \{\{x_1^{\varepsilon_1}, y_1^{-\varepsilon_1}\}, \dots, \{x_k^{\varepsilon_k}, y_k^{-\varepsilon_k}\}\}$ for g, put $d_1(S) = \sum_{i=1}^{k} d_1(x_i, y_i)$. Then $\widehat{d_1}(g)$ is defined as the minimum of the numbers $d_1(S)$ where S runs through all possible schemes for g. The pairs of a given scheme for g will be called g-*connected.*

For every $i \leqslant n$ choose a scheme S_i for g_i satisfying $d_1(S_i) = \widehat{d_1}(g_i)$. It is also convenient to choose the scheme $T_i = \{\{a_i, a_i^{-1}\}, \{x_i, y_i^{-1}\}\}$ for the element $h_i \equiv a_i x_i y_i^{-1} a_i^{-1}$; the pair $\{x_i, y_i^{-1}\}$ of this scheme will be called an *obstacle*; $1 \leqslant i \leqslant n$. As in the proof of (R3), the cancellation order for w produces a *chain* C between a^ε and $b^{-\varepsilon}$, say $\{a^\varepsilon, c_1^{-\varepsilon}\}$, $\{c_1^\varepsilon, c_2^{-\varepsilon}\}, \dots, \{c_p^\varepsilon, b^{-\varepsilon}\}$. Note that the pairs of C are elements of the schemes S_i and T_j; $0 \leqslant i, j \leqslant n$, $j \neq 0$.

Consider two cases.

I. The chain C does not contain any obstacle pair. Then, by the definition of $\widehat{d_1}$, we have (with $c_0 = a$, $c_{p+1} = b$):

$$d_1(a, b) \leqslant \sum_{i=0}^{p} d_1(c_i, c_{i+1}) \leqslant \sum_{i=0}^{n} d_1(S_i) = \sum_{i=0}^{n} \widehat{d_1}(g_i),$$

a contradiction.

II. The chain C contains an obstacle pair $\{x_j, y_j^{-1}\}$ for some $j \leqslant n$. This is the main case to deal with. The pair $\{x_j, y_j^{-1}\}$ coincides with a pair $\{c_k^\varepsilon, c_{k+1}^{-\varepsilon}\}$ of C for some k, $0 \leqslant k \leqslant p$ (again, we put $c_0 = a$, $c_{p+1} = b$). The idea is to represent the element $a^\varepsilon \cdot b^{-\varepsilon}$ in the form of a product of "new" elements of $G(X)$ as follows. For each $i < k$ both letters c_i^ε, $c_{i+1}^{-\varepsilon}$ occur in exactly one element g_m (or h_m) as a g_m-connected (respectively, h_m-connected) pair; $0 \leqslant m \leqslant n$. We replace c_i^ε by a^ε and $c_{i+1}^{-\varepsilon}$ by $a^{-\varepsilon}$ in the word g_m (respectively, h_m). Analogously, we replace the letters c_i^ε and $c_{i+1}^{-\varepsilon}$ with $i > k$ by b^ε and $b^{-\varepsilon}$ respectively. Finally, we find $r \leqslant n$ such that $\{c_k^\varepsilon, c_{k+1}^{-\varepsilon}\} = \{x_j, y_j^{-1}\}$ is an h_r-connected pair, and then replace c_k^ε by a^ε and $c_{k+1}^{-\varepsilon}$ by $b^{-\varepsilon}$ in the word h_r. If $\{u^\delta, v^{-\delta}\}$ is a g_i-connected (or h_i-connected) pair for some $i \leqslant n$ and it does not belong to C, we replace u^δ by a^δ and $v^{-\delta}$ by $a^{-\delta}$ in the word

https://doi.org/10.1017/S0004972700014143 Published online by Cambridge University Press

g_i (respectively, h_i). After this procedure we obtain another representation of $a^\varepsilon \cdot b^{-\varepsilon}$, say

$$(**) \qquad a^\varepsilon \cdot b^{-\varepsilon} = g_0' \cdot h_0' \cdot g_1' \cdot \ldots \cdot g_{n-1}' \cdot h_n' \cdot g_n',$$

where each of the words g_i', h_i' contains only the letters a, b, a^{-1}, b^{-1}. (To verify the equality $(**)$, use the fact that for each $i \leqslant p$ the letters $c_i^{-\varepsilon}$ and c_{i+1}^ε of the pairs $\{c_{i-1}^\varepsilon, c_i^{-\varepsilon}\}$, $\{c_i^\varepsilon, c_{i+1}^{-\varepsilon}\}$ are deleted from w at some step of the reduction.) To each scheme S_i for g_i and T_i for h_i there naturally corresponds a scheme S_i' for g_i' and T_i' for h_i'. Note that, by the construction, all pairs of the schemes S_i' $(0 \leqslant i \leqslant n)$ and T_i' $(i \neq r)$ have the form $\{a, a^{-1}\}$ or $\{b, b^{-1}\}$. Therefore, $h_i' = e$ for all $i \neq r$ and we can rewrite $(**)$ as $a^\varepsilon \cdot b^{-\varepsilon} = g_1^* \cdot h_r' \cdot g_2^*$, where

$$g_1^* = g_0' \cdot \ldots \cdot g_r' \in G(X),\ g_2^* = g_{r+1}' \cdot \ldots \cdot g_n' \in G(X)$$

and h_r' coincides with one of four irreducible elements $a \cdot a \cdot b^{-1} \cdot a^{-1}$, $a \cdot b \cdot a^{-1} \cdot a^{-1}$, $b \cdot a \cdot b^{-1} \cdot b^{-1}$, $b \cdot b \cdot a^{-1} \cdot b^{-1}$. We claim that $g_1^* = e = g_2^*$, thus obtaining an impossible equality $a^\varepsilon \cdot b^{-\varepsilon} = h_r'$.

Indeed, note that every non-empty scheme S for an arbitrary word $g \in G(X)$ contains a g-connected pair $\{x^\delta, y^{-\delta}\}$ such that the letters x^δ and $y^{-\delta}$ are neighbours in g. Since the schemes S_i, $0 \leqslant i \leqslant n$, contain only the pairs $\{a, a^{-1}\}$ and $\{b, b^{-1}\}$, we conclude that $g_0' = \ldots = g_n' = e$, and hence $g_1^* = e = g_2^*$. This completes the proof of the lemma. $\square$

Let $a, b \in X$ and $\varepsilon \in \{1, -1\}$ be arbitrary. Since $a^\varepsilon \cdot b^{-\varepsilon}$ is a canonical element, the definition of N implies $N(a^\varepsilon \cdot b^{-\varepsilon}) \leqslant d_1(a, b)$. The inverse inequality follows from Lemma 4. This proves (R1).

It remains to show that the semi-norm N is continuous. To this end, it suffices to check that the set $O = \{g \in G(X) : N(g) < 1\}$ contains an open neighbourhood of the identity. (One easily substitutes the number 1 in the definition of O by any real number $\varepsilon > 0$.) This requires some notation. We use a description of a neighbourhood base of $F(X)$ at the identity [**14**].

Denote by X^{-1} a copy of X (with a homeomorphism ${}^{-1}$ of X onto X^{-1}) and put $\overline{X} = X \oplus X^{-1}$, the free topological sum of X and X^{-1}. Let i be the natural embedding of X into $F(X)$. For each integer n denote by i_n the mapping of $\overline{X}^n$ to $F(X)$ defined by $i_n(x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n}) = i(x_1)^{\varepsilon_1} \cdot \ldots \cdot i(x_n)^{\varepsilon_n}$ for all $x_1, \ldots, x_n \in X$ and $\varepsilon_1, \ldots \varepsilon_n = \pm 1$. Let $j_n : \overline{X}^{2n} \to F(X)$ be the mapping defined by $j_n(\overline{x}, \overline{y}) = i_n(\overline{x}) \cdot (i_n(\overline{y}))^{-1}$ for all $\overline{x}, \overline{y} \in \overline{X}^n$. The mappings i_n and j_n are continuous for all integers n.

For every $n \in N^+$ denote by $\mathcal{U}_n$ the finest uniformity of $\overline{X}^n$ compatible with the topology of $\overline{X}^n$, that is, the universal uniformity of $\overline{X}^n$ [**2**, Chapter 8]. For each

https://doi.org/10.1017/S0004972700014143 Published online by Cambridge University Press

sequence $E = \{U_n : n \in N^+\}$ with $U_n \in \mathcal{U}_n$ for all n, we put

$$V(E) = \bigcup_{n \in N^+} \bigcup_{\pi \in S_n} j_{\pi(1)}(U_{\pi(1)}) \cdot \ldots \cdot j_{\pi(n)}(U_{\pi(n)}),$$

where S_n is the group of all permutations of the set $\{1, \ldots, n\}$. By Theorem 1 of [14], the set $V(E)$ is open in $F(X)$ and the family of all sets of this form constitutes a base of $F(X)$ at the identity. Our aim is to define a sequence $E = \{U_n : n \in N^+\}$ satisfying $V(E) \subseteq O$.

For every $\delta > 0$ put

$$U_{1,\delta} = \{(x^\varepsilon, y^{-\varepsilon}) : x, y \in X,\ \varepsilon = \pm 1,\ d_1(x,y) < \delta\}.$$

Then for every $n \geqslant 2$ and every $\delta > 0$ denote by $U_{n,\delta}$ a subset of $\widetilde{X}^{2n}$ that consists of all pairs $(\overline{x}, \overline{y})$ with $\overline{x} = (x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n}) \in \widetilde{X}^n$ and $\overline{y} = (y_1^{\varepsilon_1}, \ldots, y_n^{\varepsilon_n}) \in \widetilde{X}^n$ (with arbitrary $\varepsilon_1, \ldots, \varepsilon_n = \pm 1$) satisfying the conditions $d_1(x_i, y_i) < \delta$ and $d_2((x_i, x_j), (y_i, y_j)) < \delta$ for all i, j, $1 \leqslant i \leqslant j \leqslant n$. One readily verifies that $U_{n,\delta} \in \mathcal{U}_n$ for all $n \geqslant 1$ and $\delta > 0$.

LEMMA 5. *Let $(\overline{x}, \overline{y}) \in U_{n,\delta}$ ($n \geqslant 2$), where $\overline{x} = (x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n})$, $\overline{y} = (y_1^{\varepsilon_1}, \ldots, y_n^{\varepsilon_n})$. Then $d_2((x_i, x_j), (x_i, y_j)) < 2\delta$ and $d_2((y_i, x_j), (y_i, y_j)) < 2\delta$ for all i, j; $1 \leqslant i, j \leqslant n$.*

PROOF: By the definition of $U_{n,\delta}$ and condition (C1) of the right-concordance of d_1 and d_2, we have

$$d_2((x_i, x_j), (y_i, y_j)) < \delta \quad \text{and} \quad d_2((y_i, y_j), (x_i, y_j)) = d_1(y_i, x_i) < \delta.$$

Therefore

$$d_2((x_i, x_j), (x_i, y_j)) \leqslant d_2((x_i, x_j), (y_i, y_j)) + d_2((y_i, y_j), (x_i, y_j)) < 2\delta.$$

An analogous argument shows that $d_2((y_i, x_j), (y_i, y_j)) < 2\delta$. This proves the lemma. □

Put $\delta(n) = 2^{-n}/n(n+1)$ and $U_n = U_{n,\delta(n)}$ for every $n \geqslant 1$. We claim that $V(E) \subseteq O$, where $E = \{U_n : n \in N^+\}$. Indeed, let $n \geqslant 2$ and suppose that an element $g \in G(X)$ has the form $g = x_1^{\varepsilon_1} \cdot \ldots \cdot x_n^{\varepsilon_n} \cdot y_n^{-\varepsilon_n} \cdot \ldots \cdot y_1^{-\varepsilon_1}$ for some $x_i, y_i \in X$ and $\varepsilon_i = \pm 1$, $1 \leqslant i \leqslant n$. We represent g as a product of canonical elements of $G(X)$ as follows. For every $i < n$ put $p_i = x_1^{\varepsilon_1} \cdot \ldots \cdot x_i^{\varepsilon_i}$ and $h_1 = x_1^{\varepsilon_1} \cdot y_1^{-\varepsilon_1}$, $h_i = p_{i-1} \cdot x_i^{\varepsilon_i} \cdot y_i^{-\varepsilon_i} \cdot p_{i-1}^{-1}$, $1 < i \leqslant n$. It is clear that $g = h_n \cdot h_{n-1} \cdot \ldots \cdot h_1$. We shall call this representation of g *standard.*

Let $g \in j_n(U_n)$ be arbitrary, $n \geqslant 2$. Then $g = x_1^{\varepsilon_1} \cdot \ldots \cdot x_n^{\varepsilon_n} \cdot y_n^{-\varepsilon_n} \cdot \ldots \cdot y_1^{-\varepsilon_1}$ for some $x_i, y_i \in X$ and $\varepsilon_i = \pm 1$, $1 \leqslant i \leqslant n$, where the points $\overline{x} = (x_1, \ldots, x_n)$ and $\overline{y} = (y_1, \ldots, y_n)$ satisfy the condition $(\overline{x}, \overline{y}) \in U_n$. Represent g as a standard product,

https://doi.org/10.1017/S0004972700014143 Published online by Cambridge University Press

say $g = h_n \cdot \ldots \cdot h_1$, of canonical factors $h_1, \ldots, h_n$. Lemma 5 implies that the following inequality is valid for every i, $1 < i \leqslant n$:

$$M(h_i) = d_1(x_i, y_i) + \sum_{k=1}^{i-1} d_2((x_k, x_i), (x_k, y_i)) < 2i \cdot \delta(n).$$

Also, we have $M(h_1) = d_1(x_1, y_1) < \delta(n)$. Consequently, the definition of N implies that

$$N(g) \leqslant \sum_{i=1}^{n} M(h_i) < 2\delta(n) \cdot \sum_{i=1}^{n} i = n(n+1) \cdot \delta(n) = 2^{-n}.$$

If $g \in j_1(U_1)$ then $g = x^\varepsilon \cdot y^{-\varepsilon}$ for some $x, y \in X$ and $\varepsilon = \pm 1$, and $N(g) = M(g) = d_1(x, y) < \delta(1) < 2^{-1}$. Thus, we have proved that $N(g) < 2^{-n}$ for each $g \in j_n(U_n)$; $n \in N^+$.

Pick an arbitrary element $g \in V(E)$. By the definition of $V(E)$, there exist $n \in N^+$, a permutation $\pi \in S_n$ and elements $g_1 \in j_1(U_1), \ldots, g_n \in j_n(U_n)$ such that $g = g_{\pi(1)} \cdot \ldots \cdot g_{\pi(n)}$. Consequently, we have

$$N(g) \leqslant \sum_{i=1}^{n} N\big(g_{\pi(i)}\big) = \sum_{i=1}^{n} N(g_i) < \sum_{i=1}^{n} 2^{-i} < 1.$$

The latter means that $N(g) < 1$ for each $g \in V(E)$, that is, $V(E) \subseteq O$. Thus, N is a continuous semi-norm. The theorem is completely proved. $\square$

The next result and Theorem 1.4 are twins.

THEOREM 1.5. (Left case) *Let d_1 and d_2 be left-concordant continuous pseudometrics on X and X^2 respectively. Then there exists a continuous semi-norm N_l on $G(X)$ satisfying the conditions*

(L1) $N_l\big(a^{-1} \cdot b\big) = N\big(a \cdot b^{-1}\big) = d_1(a, b)$ *for all* $a, b \in X$;
(L2) $N_l\big(a^{-1} \cdot x^{-1} \cdot y \cdot a\big) = d_2((x, a), (y, a))$ *for all* $a, x, y \in X$;

(L3) $N_l\big(a^{-1} \cdot x^{-1} \cdot y \cdot b\big) \geqslant d_2((x, a), (y, b))$ *for all* $a, b, x, y \in X$.

Having Theorem 1.4 proved, we can ask whether there exists a non-trivial example of right-concordant pseudometrics, that is, an example of a pair (d_1, d_2) of continuous pseudometrics on X and X^2 respectively, such that d_2 essentially differs from the natural pseudometric d_2^* on X^2 defined by $d_2^*((a, x), (b, y)) = d_1(a, b) + d_1(x, y)$. We present one fairly general method of constructing right-concordant pairs in the next section, thus answering the above question in the affirmative.

2. Constructing concordant pseudometrics

The following theorem is our general tool for investigation of uniformities on X^2 generated by group uniformities of the free topological group $F(X)$. However, all its applications (except one given in the fifth section) will be demonstrated in the forthcoming paper.

Theorem 2.1. *Let ϱ be a continuous pseudometric on X, $\varrho \leqslant 1$, and f a continuous mapping of X to a normed linear space L with a norm $\|\cdot\|$. Then there exist a continuous pseudometric d_2 on X^2 and a continuous semi-norm N on $G(X)$ satisfying the following conditions:*

(RP0) *d_1 and d_2 are right-concordant, where $d_1(a,b) = \|f(a) - f(b)\|$ for all $a, b \in X$;*

(RP1) *$N(a \cdot b^{-1}) = N(a^{-1} \cdot b) = \|f(a) - f(b)\|$ for all $a, b \in X$;*

(RP2) *$N(a \cdot x \cdot y^{-1} \cdot a^{-1}) = d_2((a,x),(a,y)) = \|f(a)\| \cdot \varrho(x,y)$ for all $a, x, y \in X$;*

(RP3) *$N(a \cdot x \cdot y^{-1} \cdot b^{-1}) \geqslant d_2((a,x),(b,y)) \geqslant \max\{\|f(a)\|, \|f(b)\|\} \cdot \varrho(x,y)$ whenever $a, b, x, y \in X$.*

Proof: We define a continuous pseudomertic d_2 on X^2 as follows. Let $A = (x', y')$ and $B = (x'', y'')$ be points of X^2. We shall say that a sequence $\Gamma = \{A_0 = (x_0, y_0),\ A_1 = (x_1, y_1), \dots,\ A_{n+1} = (x_{n+1}, y_{n+1})\}$ of points of X^2 is a *way from A to B* if $A_0 = A$, $A_{n+1} = B$ and for each $i = 0, 1, \dots, n$ either $x_i = x_{i+1}$, or $y_i = y_{i+1}$. We define $P = \{i \leqslant n : x_i = x_{i+1}\}$, $Q = \{j \leqslant n : y_j = y_{j+1}\}$ and put

$$D_\Gamma(A,B) = \sum_{i \in P} \|f(x_i)\| \cdot \varrho(y_i, y_{i+1}) + \sum_{j \in Q} d_1(x_j, x_{j+1}). \tag{1}$$

Then define $d_2(A,B)$ as the lower bound of the numbers $D_\Gamma(A,B)$, where Γ runs through all the ways from A to B. It is clear that d_2 is a pseudometric and we claim that d_2 satisfies the following conditions for all $x_1, x_2, y_1, y_2 \in X$:

$$d_2((x_1, y_1),(x_2, y_1)) = d_1(x_1, x_2); \tag{i}$$

$$d_2((x_1, y_1),(x_1, y_2)) = \|f(x_1)\| \cdot \varrho(y_1, y_2). \tag{ii}$$

Only (ii) requires a proof, because (i) readily follows from the definition of d_2. The inequality $d_2((x_1, y_1),(x_1, y_2)) \leqslant \|f(x_1)\| \cdot \varrho(y_1, y_2)$ is obvious; it suffices to consider the way Γ consisting of two points $A = (x_1, y_1)$ and $B = (x_1, y_2)$. Let us prove the inverse inequality. Consider an arbitrary way Γ from the point A to some point $C = (x_2, y_2)$ whose second coordinate coincides with the second coordinate of B. Suppose that the way Γ consists of points

$$A = A_0 = (a_0, b_0),\ A_1 = (a_1, b_1),\ \dots,\ A_{n+1} = (a_{n+1}, b_{n+1}) = C,$$

where $a_0 = x_1$, $b_0 = y_1$, $a_{n+1} = x_2$, $b_{n+1} = y_2$. (We do not assume that $x_2 = x_1$, that is, that $C = B$.) First, the following inequality has to be proved:

$$\|f(x_1)\| \cdot \varrho(y_1, y_2) \leqslant D_\Gamma(A, C). \tag{2}$$

We have

$$D_\Gamma(A, C) = \sum_{i \in P} \|f(a_i)\| \cdot \varrho(b_i, b_{i+1}) + \sum_{j \in Q} d_1(a_j, a_{j+1}), \tag{3}$$

where P and Q are subsets of $\{0, 1, \ldots, n\}$ defined in the same way as above. We shall not change the number $D_\Gamma(A, C)$ if we delete neighbouring coinciding points from Γ. Therefore the sets P and Q are assumed disjoint. Without loss of generality we can also assume that for each $k \leqslant n - 1$, the numbers k and $k + 1$ belong to different sets P, Q, that is, either $k \in P$ and $k + 1 \in Q$ or $k \in Q$ and $k + 1 \in P$. Indeed, suppose that for some i, $1 \leqslant i \leqslant n$, the first coordinates of the points A_{i-1}, A_i and A_{i+1} coincide, $a_{i-1} = a_i = a_{i+1}$ (that is, $i - 1$ and i belong to P). Denote by Γ_1 the way $A_0, A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n, A_{n+1}$. One easily verifies that

$$D_\Gamma(A, C) - D_{\Gamma_1}(A, C) = \|f(a_i)\| \cdot [\varrho(b_{i-1}, b_i) + \varrho(b_i, b_{i+1}) - \varrho(b_{i-1}, b_{i+1})] \geqslant 0,$$

and hence $D_{\Gamma_1}(A, C) \leqslant D_\Gamma(A, C)$. Thus, we can assume that all even integers $i \leqslant n$ are in P and all odd $i \leqslant n$ are in Q. Assume for convenience that n is odd, $n = 2m+1$ for some integer m. Then $a_{2i} = a_{2i+1}$ and $b_{2i+1} = b_{2i+2}$ for each $i \leqslant m$. Therefore, (3) is equivalent to

$$D_\Gamma(A, C) = \sum_{i=0}^{m} \|f(a_i)\| \cdot \varrho(b_{2i}, b_{2i+2}) + \sum_{i=0}^{m} \|f(a_{2i+2}) - f(a_{2i})\|. \tag{4}$$

Obviously, (4) implies that the following inequality holds for each $p \leqslant m$:

$$D_\Gamma(A, C) \geqslant \|f(a_0) - f(a_{2p})\| + \sum_{i=0}^{m} \|f(a_{2i})\| \cdot \varrho(b_{2i}, b_{2i+2}). \tag{5}$$

Consider two cases.

I. $\sum\limits_{i=0}^{m} \varrho(b_{2i}, b_{2i+2}) \leqslant 1$. We have

$$d_2(A, B) \leqslant \|f(a_0)\| \cdot \varrho(b_0, b_{2m+2}) \leqslant \|f(a_0)\| \cdot \sum_{i=0}^{m} \varrho(b_{2i}, b_{2i+2}).$$

Let $\|f(a_0)-f(a_{2k})\|$ be maximal among the numbers $\|f(a_0)-f(a_{2i})\|$, $0\leqslant i\leqslant m$. In view of (5) (with $p=k$), the inequality (2) will follow from

$$\|f(a_0)\|\cdot\sum_{i=0}^{m}\varrho(b_{2i},b_{2i+2})\leqslant\|f(a_0)-f(a_{2k})\|+\sum_{i=0}^{m}\|f(a_{2i})\|\cdot\varrho(b_{2i},b_{2i+2}),$$

or equivalently, from

$$\sum_{i=0}^{m}\varrho(b_{2i},b_{2i+2})\cdot(\|f(a_0)-f(a_{2i})\|)\leqslant\|f(a_0)-f(a_{2k})\|.$$

The latter, however, follows from our assumption (see I) and the inequalities $\|f(a_0)\|-\|f(a_{2i})\|\leqslant\|f(a_0)-f(a_{2i})\|\leqslant\|f(a_0)-f(a_{2k})\|$, $0\leqslant i\leqslant m$.

II. $\sum_{i=0}^{m}\varrho(b_{2i},b_{2i+2})\geqslant 1$. Let $\|f(a_{2l})\|$ be minimal among the numbers $\|f(a_{2i})\|$, $0\leqslant i\leqslant m$. From (5) (with $p=l$) and the choice of l it follows that

$$\begin{aligned}D_\Gamma(A,C)&\geqslant\|f(a_0)-f(a_{2l})\|+\sum_{i=0}^{m}\|f(a_i)\|\cdot\varrho(b_{2i},b_{2i+2})\\&\geqslant\|f(a_0)-f(a_{2l})\|+\|f(a_{2l})\|\cdot\sum_{i=0}^{m}\varrho(b_{2i},b_{2i+2})\\&\geqslant\|f(a_0)-f(a_{2l})\|+\|f(a_{2l})\|\geqslant\|f(a_0)\|\geqslant\|f(a_0)\|\cdot\varrho(b_0,b_{2m+2}).\end{aligned}$$

So, the inequality $\|f(a_0)\|\cdot\varrho(b_0,b_{2m+1})\geqslant D_\Gamma(A,C)$, that is (2), is proved for an arbitrary point $C=(x_2,y_2)$ whose second coordinate coincides with the second coordinate of $B=(x_1,y_2)$, and for any way Γ from A to C. This implies the inequality

$$\|f(x_1)\|\cdot\varrho(y_1,y_2)\leqslant d_2((x_1,y_1),(x_2,y_2))\text{ for all }x_1,x_2,y_1,y_2\in X.\tag{6}$$

An analogous argument shows that

$$\|f(x_2)\|\cdot\varrho(y_1,y_2)\leqslant d_2((x_1,y_1),(x_2,y_2))\text{ for all }x_1,x_2,y_1,y_2\in X.\tag{6'}$$

To prove (ii) it remains to put $x_2=x_1$ in (6). The continuity of the pseudometric d_2 readily follows from (i) and (ii).

One can easily see that the pseudometrics d_1 and d_2 satisfy the condition (C1) of Definition 1.3. Let us verify (C2).

Suppose we are given points $a_0,a_1,\dots,a_{n+1},x_0,x_1,\dots,x_n\in X$ and a permutation π of $\{1,\dots,n\}$. We prove the following inequality:

$$d_2((a_0,x_0),(a_{n+1},x_n))\leqslant\sum_{i=1}^{n}d_2\big((a_{\pi(i)},x_{i-1}),(a_{\pi(i)},x_i)\big)+\sum_{i=0}^{n}d_1(a_i,a_{i+1}).\tag{7}$$

Put $M = \min\{\|f(a_i)\| : 1 \leqslant i \leqslant n\}$ and choose $k \leqslant n$ with $f(a_k) = M$. By (ii), we have

$$\sum_{i=1}^{n} d_2\big((a_{\pi(i)}, x_{i-1}), (a_{\pi(i)}, x_i)\big) = \sum_{i=1}^{n} \|f(a_{\pi(i)})\| \cdot \varrho(x_{i-1}, x_i)$$
$$\geqslant M \cdot \sum_{i=1}^{n} \varrho(x_{i-1}, x_i) \geqslant \|f(a_k)\| \cdot \varrho(x_0, x_n) = d_2((a_k, x_0), (a_k, x_n)).$$

Thus, we have shown that the right part of (7) is not less than the following expression:

$$d_2((a_k, x_0), (a_k, x_n)) + d_1(a_0, a_k) + d_1(a_k, a_{n+1}). \tag{8}$$

Denote by Γ the way $A = A_0 = (a_0, x_0)$, $A_1 = (a_k, x_0)$, $A_2 = (a_k, x_{n+1})$ and $A_3 = (a_{n+1}, x_{n+1}) = B$. Then $D_\Gamma(A, B)$ is equal to the expression in (8). Therefore, $D_\Gamma(A, B)$ does not exceed the right part of inequality (7). However, we know that $d_2((a_0, x_0), (a_{n+1}, x_n)) \leqslant D_\Gamma(A, B)$ by the definition of d_2. This completes the proof of (7) and (C2), that is, d_1 and d_2 are right-concordant.

Apply Theorem 1.4 to define a continuous semi-norm N on the open subgroup $G(X)$ of $F(X)$ satisfying (R1)–(R3). We need only verify that (RP3) holds. Let points $a, b, x, y \in X$ be arbitrary. By (R3) of Theorem 1.4, $N\big(a \cdot x \cdot y^{-1} \cdot b^{-1}\big) \geqslant d_2((a, x), (b, y))$. Then apply (6) and (6′) to conclude that

$$d_2((a, x), (b, y)) \geqslant \|f(a)\| \cdot \varrho(x, y) \quad \text{and} \quad d_2((a, x), (b, y)) \geqslant \|f(b)\| \cdot \varrho(x, y).$$

The latter proves (RP3) and the theorem. ∎

Remark 2.2. The pseudometric d_2 on X^2 and the semi-norm N defined in the proof of Theorem 2.1 satisfy the condition

$$N\big(a \cdot x \cdot y^{-1} \cdot b^{-1}\big) = d_2((a, x), (b, y)) \text{ for all } a, b, x, y \in X, \tag{9}$$

that is, N right-induces the pseudometric d_2.

Indeed, the inequality $N\big(a \cdot x \cdot y^{-1} \cdot b^{-1}\big) \geqslant d_2((a, x), (b, y))$ follows from Theorem 1.4. The inverse inequality is a consequence of the following observation: for any way Γ from $A = (a, x)$ to $B = (b, y)$ there exists a natural representation of the element $g = a \cdot x \cdot y^{-1} \cdot b^{-1}$ as a product $g = h_1 \cdot \ldots \cdot h_n$ of *canonical* elements of $G(X)$ such that $\sum_{i=1}^{n} M(h_i) \leqslant D_\Gamma(A, B)$. (Here we use the notation and terminology of the proof of Theorem 1.4.) It suffices to illustrate this by an example. Let $A_1 = A$, $A_2 = (a_2, x)$, $A_3 = (a_2, z)$, $A_4 = (a_3, z)$, $A_5 = (a_3, y)$, $A_6 = B$ be a way from A to B. Then we can write $g = \left(a \cdot a_2^{-1}\right) \cdot \left(a_2 \cdot x \cdot z^{-1} \cdot a_2^{-1}\right) \cdot \left(a_2 \cdot a_3^{-1}\right) \cdot \left(a_3 \cdot z \cdot y^{-1} \cdot a_3^{-1}\right) \cdot \left(a_3 \cdot b^{-1}\right)$, where all factors in the right part of the equality are canonical. We have

$$\begin{aligned} M\left(aa_2^{-1}\right) &+ M\left(a_2xz^{-1}a_2^{-1}\right) + M\left(a_2a_3^{-1}\right) + M\left(a_3zy^{-1}a_3^{-1}\right) + M\left(a_3b^{-1}\right) \\ &= d_1(a, a_2) + d_2((a_2, x), (a_2, z)) + d_1(a_2, a_3) + d_2((a_3, z), (a_3, y)) + d_1(a_3, b) \\ &= D_\Gamma(A, B). \end{aligned}$$

It seems to be clear how to write a corresponding representation of g in the case of an arbitrary way Γ from A to B. Since $N(g)$ and $d_2(A, B)$ are defined as lower bounds of the corresponding expressions appeared in the left and right parts of the above equality, the inequality $N\left(a \cdot x \cdot y^{-1} \cdot b^{-1}\right) \leqslant d_2((a, x), (b, y))$ (and hence (9)) is proved.

As in the case of Theorem 1.4, the last result has its twin.

THEOREM 2.3. *Let ϱ be a continuous pseudometric on X, $\varrho \leqslant 1$, and f a continuous mapping of X to a normed linear space L with a norm $\|\cdot\|$. Then there exist a continuous pseudometric d_2 on X^2 and a continuous semi-norm N on $G(X)$ satisfying the following conditions for all $a, b, x, y \in X$:*

(LP0) *d_1 and d_2 are left-concordant, where $d_1(a, b) = \|f(a) - f(b)\|$;*
(LP1) *$N\left(a^{-1} \cdot b\right) = N\left(a \cdot b^{-1}\right) = \|f(a) - f(b)\|$;*
(LP2) *$N\left(a^{-1} \cdot x^{-1} \cdot y \cdot a\right) = d_2((x, a), (y, a)) = \|f(a)\| \cdot \varrho(x, y)$;*
(LP3) *$N\left(a^{-1} \cdot x^{-1} \cdot y \cdot b\right) \geqslant d_2((x, a), (y, b)) \geqslant \max\{\|f(a)\|,\ \|f(b)\|\} \cdot \varrho(x, y)$.*

Theorem 2.1 and Remark 2.2 supply us with a good many continuous semi-norms on $G(X)$. However, under certain circumstances those semi-norms do not vary rapidly enough. The following constructions seem to be more flexible.

3. Two more constructions

Let γ be a locally finite family of non-empty open subsets of X. Suppose that for every $U \in \gamma$ we have defined a continuous pseudometric ϱ_U on X, $\varrho_U \leqslant 1$, and a continuous mapping f_U of X to a normed linear space $(L, \|\cdot\|)$ so that supp $f_U \subseteq clU$, that is, $f_U(x) = 0_L$ for each $x \in X \setminus U$.

Fix an element $U \in \gamma$. Use ϱ_U and f_U to define a pair $(d_{1,U}, d_{2,U})$ of right-concordant continuous pseudometrics respectively on X and X^2 as in Theorem 2.1, where $d_{1,U}(a, b) = \|f_U(a) - f_U(b)\|$ for all $a, b \in X$.

Then we define continuous pseudometrics d_1 on X and d_2 on X^2 by

$$d_1(a, b) = \sum_{U \in \gamma} d_{1,U}(a, b) \text{ and } d_2((a, x), (b, y)) = \sum_{U \in \gamma} d_{2,U}((a, x), (b, y))$$

https://doi.org/10.1017/S0004972700014143 Published online by Cambridge University Press

for all $a, b, x, y \in X$. The continuity of d_1 and d_2 follows from the choice of γ and the functions f_U, $U \in \gamma$, and the right-concordance of d_1 and d_2 is a consequence of that for pairs $(d_{1,U}, d_{2,U})$; $U \in \gamma$. It remains to apply Theorem 1.4 and define a continuous semi-norm N on $G(X)$ that right-induces d_1 and d_2 (apply Remark 2.2).

The following theorem gives us a more tricky way of defining continuous semi-norms on $G(X)$ with the use of some locally finite family of sets in X. The theorem will be applied to consider induced uniformities on the square of a metrisable space.

Theorem 3.1. *Let γ be a locally finite family of sets in a locally compact paracompact space X and suppose that for every $W \in \gamma$ a continuous mapping f_W is given of X to a linear space L with a norm $\|\cdot\|$. Then for any continuous pseudometric ϱ on X there exists a continuous semi-norm N on $G(X)$ satisfying the following condition*

(C) *if $a, b, x, y \in X$, $\varepsilon = \pm 1$, $N(a^\varepsilon \cdot x^\varepsilon \cdot y^{-\varepsilon} \cdot b^{-\varepsilon}) < 1$ and $x, y \in W$ for some $W \in \gamma$, then $N(a^\varepsilon \cdot x^\varepsilon \cdot y^{-\varepsilon} \cdot b^{-\varepsilon}) \geqslant \max\{\|f_W(a)\| \cdot \varrho(x,y),\ \|f_W(b)\| \cdot \varrho(x,y)\}$.*

Proof: Let μ_0 be a locally finite open cover of X such that the closure of every element of μ is compact and intersects only finitely many elements of γ. Denote by μ_1 a locally finite open cover of X which star-refines μ_0. Choose a continuous pseudometric ϱ' on X so that $\{(x,y) \in X^2 : \varrho'(x,y) < 1\} \subseteq \bigcup\{V \times V : V \in \mu_1\}$, and put $\varrho_1 = \max\{\varrho, \varrho'\}$. For every $x \in X$ and $W \in \gamma$ denote by $g_W(x)$ the number $\sup\{\|f_W(y)\| : \varrho_1(x,y) < 1\}$. The number $g_W(x)$ is finite, since f_W is a continuous function and $\{y \in X : \varrho_1(x,y) < 1\} \subseteq St(x, \mu_1) \subseteq V_x$ for some $V_x \in \mu_0$, where clV_x is compact. Here $St(x, \mu_1)$ stands for the set $\bigcup\{U \in \mu_1 : x \in U\}$. For every $x \in X$ we also put $h(x) = 1 + \sum\{g_W(x) : W \in \gamma,\ W \cap St(x, \mu_0) \neq \emptyset\}$. Obviously, $1 \leqslant h(x) < \infty$ by the choice of μ_0 and μ_1.

Let a, x, y be points of X and $\varepsilon = \pm 1$. If $\varrho_1(x,y) < 1$, we put $M(x^\varepsilon y^{-\varepsilon}) = \varrho_1(x,y)$ and $M(a^\varepsilon x^\varepsilon y^{-\varepsilon} a^{-\varepsilon}) = M(a^\varepsilon x^{-\varepsilon} y^\varepsilon a^{-\varepsilon}) = h(a) \cdot \varrho_1(x,y)$. Otherwise put $M(x^\varepsilon y^{-\varepsilon}) = M(a^\varepsilon x^\varepsilon y^{-\varepsilon} a^{-\varepsilon}) = M(a^\varepsilon x^{-\varepsilon} y^\varepsilon a^{-\varepsilon}) = 1$. If $a_1, \ldots, a_n, x, y \in X$ and $\varepsilon_1, \ldots, \varepsilon_n, \varepsilon = \pm 1$ $(n > 1)$, we put

$$M\left(a_1^{\varepsilon_1} \ldots a_n^{\varepsilon_n} x^\varepsilon y^{-\varepsilon} a_n^{-\varepsilon_n} \ldots a_1^{-\varepsilon_1}\right) = \sum_{i=1}^{n} M\left(a_i^{\varepsilon_i} x^\varepsilon y^{-\varepsilon} a_i^{-\varepsilon_i}\right).$$

Thus we have defined the number $M(g)$ for every *canonical* element $g \in G(X)$.

As in the proof of Theorem 1.4, consider all possible representations of an arbitrary element $g \in G(X)$ in the form of a product $g = h_1 \cdot \ldots \cdot h_n$ of canonical elements of $G(X)$. To each such representation there corresponds the sum $\sum\limits_{i=1}^{n} M(h_i)$. Denote by

$N(g)$ the lower bound of these sums. This defines a semi-norm N on $G(X)$. Note that $N(g) \geqslant \widehat{\varrho}_1(g,e)$ for each $g \in G(X)$, where $\widehat{\varrho}_1$ is the Graev extension of ϱ_1 to $G(X)$ and e is the identity of $F(X)$. This follows immediately from the fact that $M(g) \geqslant \widehat{\varrho}_1(g,e)$ for any canonical element $g \in G(X)$.

Let us prove (C). We only consider the case $\varepsilon = 1$; the reasoning for $\varepsilon = -1$ is completely analogous. Suppose that $a,b,x,y \in X$, $N(axy^{-1}b^{-1}) < 1$ and $x,y \in W$ for some $W \in \gamma$. We shall prove that the inequality

$$(1) \qquad \sum_{s=1}^{m} M(h_s) \geqslant \max\{\|f_W(a)\| \cdot \varrho_1(x,y),\ \|f_W(b)\| \cdot \varrho_1(x,y)\}$$

holds for any representation of $g = a\cdot x\cdot y^{-1}\cdot b^{-1}$ in the form of a product $g = h_1\cdot\ldots\cdot h_m$ of canonical elements $h_1,\ldots,h_m \in G(X)$. This will easily imply (C). We first prove that the left part of (1) is not less than $\|f_W(a)\| \cdot \varrho_1(x,y)$. Since $N(g) < 1$, we can assume that $\sum_{s=1}^{m} M(h_s) < 1$. Note that $\varrho_1(a,b) + \varrho_1(x,y) = \widehat{\varrho}_1(g,e) \leqslant N(g) < 1$, so $\varrho_1(a,b) < 1$ and $\varrho_1(x,y) < 1$.

As in the proof of Theorem 1.4, fix an order of cancellations in the word $\overline{g} = h_1\ldots h_m$ that transform $\overline{g}$ to g, and define the relation of $\overline{g}$-dependence. This also gives us a chain $\mathcal{P}$ between a and b^{-1}, a chain $\mathcal{Q}$ between x and y^{-1}, and a relation $\varphi : \mathcal{Q} \to \mathcal{P}$ of $\overline{g}$-subordination. Let $\mathcal{P} = \{p_i : 1 \leqslant i \leqslant k\}$ and $\mathcal{Q} = \{q_j : 1 \leqslant j \leqslant l\}$, where $p_1 = \{a, c_1^{-1}\}$, $p_2 = \{c_1, c_2^{-1}\}, \ldots,$ $p_k = \{c_{k-1}, b^{-1}\}$ and $q_1 = \{x, t_1^{-1}\}$, $q_2 = \{t_1, t_2^{-1}\}, \ldots,$ $q_l = \{t_{l-1}, y^{-1}\}$. Put $c_0 = a$, $c_k = b$, $t_0 = x$, $t_l = y$. For every $j \leqslant l$, denote by $\psi(j)$ the number $i \leqslant k$ such that $p_i = \varphi(q_j)$, thus obtaining the mapping $\psi : \{1,\ldots,l\} \to \{1,\ldots,k\}$. Since both pairs q_j and p_i, $i = \psi(j)$, lie in some element h_s, $1 \leqslant s \leqslant m$, and the pair $p_i = \{c_{i-1}, c_i^{-1}\}$ consists of the same letters, $c_{i-1} = c_i$, we have $h(c_i) \cdot \varrho_1(t_{j-1}, t_j) \leqslant M(h_s)$. It is important to note that different pairs of $\mathcal{Q}$ lie in different elements h_s, $s \leqslant m$, and ψ is a monomorphic mapping.

As in the proof of Theorem 1.4, define families $\mathcal{P}_1 = \varphi(\mathcal{Q})$, $\mathcal{P}_2 = \mathcal{P} \setminus \mathcal{P}_1$ and $\mathcal{P}_3$, a subfamily of $\mathcal{P}_2$ consisting of pairs $p_i = \{c_{i-1}, c_i^{-1}\}$ with $c_{i-1} \neq c_i$. Since different pairs of $\mathcal{P}^* = \mathcal{P}_1 \cup \mathcal{P}_3$ lie in different elements h_s, we have

$$(2) \qquad \sum_{p\in\mathcal{P}_3} \varrho_1(p) + \sum_{j=1}^{l} h\big(c_{\psi(j)}\big) \cdot \varrho_1(t_{j-1}, t_j) \leqslant \sum_{s=1}^{m} M(h_s) < 1,$$

where $\varrho_1(p)$ stands for $\varrho_1(u,v)$ if $p = \{u, v^{-1}\}$. Recall that $\mathcal{P}$ is a chain between a and b^{-1}; hence applying the triangle inequality for ϱ_1 and (2), we have for every $i \leqslant k$:

$$\varrho_1(a, c_i) \leqslant \sum_{r=1}^{i} \varrho_1(p_r) \leqslant \sum_{p\in\mathcal{P}} \varrho_1(p_r) = \sum_{p\in\mathcal{P}_3} \varrho_1(p) < 1.$$

(We also use the fact that every pair $p \in \mathcal{P} \setminus \mathcal{P}_3$ has the form $\{u, u^{-1}\}$.) Thus we have proved that $\varrho_1(a, c_i) < 1$ for each $i \leqslant k$. This and the choice of ϱ_1 together imply that

(3) $$\{c_i : 0 \leqslant i \leqslant k\} \subseteq St(a, \mu_1) \subseteq V \text{ for some } V \in \mu_0.$$

By (3), $c_i \in V$ for each $i \leqslant k$ and $a \in V \cap W \neq \emptyset$. So, the definition of the function h implies that $h(c_i) > g_W(c_i)$; $0 \leqslant i \leqslant k$. Furthermore, since $\varrho_1(a, c_i) < 1$, from the definition of g_W it follows that $g_W(c_i) \geqslant \|f_W(a)\|$, $0 \leqslant i \leqslant k$. This and the inequality (2) gives us the following:

$$\|f_W(a)\| \cdot \varrho(x, y) \leqslant \|f_W(a)\| \cdot \sum_{j=1}^{l} \varrho_1(t_{j-1}, t_j) \leqslant \sum_{j=1}^{l} h(c_{\psi(j)}) \cdot \varrho_1(t_{j-1}, t_j) \leqslant \sum_{s=1}^{m} M(h_s).$$

The same argument shows that $f_W(b) \cdot \varrho_1(x, y) \leqslant \sum_{s=1}^{m} M(h_s)$. This proves (C). The continuity of N can be proved as in Theorem 1.4. □

4. The example

We discuss here whether the condition (C2) of Definition 1.3 of the right-concordance is strictly necessary in Theorem 1.4. The example below shows that it is necessary if we define a continuous semi-norm N_r on $G(X)$ following the construction in the proof of Theorem 1.4 (and very likely, something similar to (C2) is inevitable in general).

Let d_1 be an arbitrary continuous pseudometric on X. Define a continuous pseudometric d_2^* on X^2 by $d_2^*((a, x), (b, y)) = d_1(a, b) + d_1(x, y)$ for all $a, b, x, y \in X$. It is clear that the pseudometrics d_1 and d_2^* are right- and left-concordant. Note also that if (ϱ_1, ϱ_2') and (ϱ_1, ϱ_2'') are two pairs of right-concordant pseudometrics and $\varrho_2 = \max\{\varrho_2', \varrho_2''\}$, then ϱ_1 and ϱ_2 are right-concordant as well. Thus, we can in general assume that a pair (d_1, d_2) of right-concordant continuous pseudometrics satisfies the condition

(C3) $$d_2((a, x), (b, y)) \geqslant d_1(a, b) + d_1(x, y) \text{ for all } a, b, x, y \in X.$$

The use of (C3) simplifies the proof of the equality (R1) of Theorem 1.4. However, our example shows that (C3) can not substitute for any part of (C2) at all.

The idea of the example is based on the following equality:

(1)
$$(a_0 \cdot a_1^{-1}) \cdot (a_1 \cdot z \cdot y^{-1} \cdot a_1^{-1}) \cdot (a_1 \cdot a_2^{-1}) \cdot (a_2 \cdot y \cdot z^{-1} \cdot x \cdot y^{-1} \cdot a_2^{-1}) = a_0 \cdot x \cdot y^{-1} \cdot a_2^{-1},$$

where a_i $(0 \leqslant i \leqslant 2)$ and x, y, z are elements of a space X. Our aim is to define pseudometrics d_1 on X and d_2 on X^2 satisfying (C1) and (C3) such that the semi-norm N on $G(X)$ generated by the pair (d_1, d_2) would not satisfy the conclusion of

Theorem 1.4. In our example the condition (R3) will fail. One can easily do this for a finite space X (consisting of six points a_i, $0 \leqslant i \leqslant 2$, and x, y, z); however, we prefer to give an example constructed on a base of a connected space X, say $X = R$, the reals.

Let d_1 be the usual metric on R, $d_1(x,y) = |x-y|$. Fix an integer $n > 1$ and a real $\varepsilon > 0$. Define elements $a_i, x, y, z \in R$ by $a_0 = x = 0$, $a_1 = z = \varepsilon$ and $a_2 = y = (n+1)\cdot\varepsilon$. It suffices to define a continuous pseudometric d_2 on R^2 (satisfying (C1) and (C3)) so that a corresponding semi-norm N would "evaluate" the left part of (1) less than the right one, that is, so that

$$(2)\quad d_1(a_0,a_1) + d_1(a_1,a_2) + [d_1(x,z) + d_2((a_1,z),(a_1,y)) + d_2((a_2,z),(a_2,x)) + d_2((y,z),(y,x))] < d_2((a_0,x),(a_2,y)),$$

or equivalently,

$$(3)\quad 2d_1(a_0,a_1) + d_1(a_1,a_2) + d_2((a_1,a_2),(a_1,a_1)) + 2d_2((a_2,a_0),(a_2,a_1)) < d_2((a_0,a_0),(a_2,a_2)).$$

Note that the expression in square brackets in the left part of (2) is equal to $M(a_2 \cdot y \cdot z^{-1} \cdot x \cdot y^{-1} \cdot a_2^{-1})$ (see the definition of a semi-norm N in the proof of Theorem 1.4).

The definition of a pseudometric d_2 will be explained with a help of the following figure. Suppose that some point Z can only move horizontally (to the left and to the right) or vertically (up and down) in the plane R^2. Let the speed of a horizontal movement be equal to 1, and the speed of vertical movement depend on a position of the point: if the point does not belong to any of the shaded triangles, its speed is equal to 1; otherwise the speed of the point is equal to $1/(2n+1)$. Both triangles are isosceles and their bases are parallel to the x-axis.

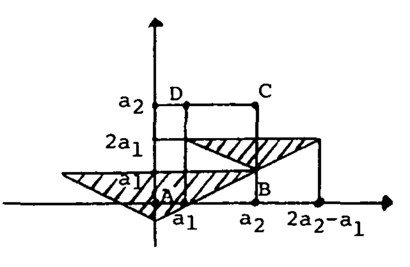

Figure 2.

https://doi.org/10.1017/S0004972700014143 Published online by Cambridge University Press

The distance $d_2(P,Q)$ between arbitrary points P and Q of the plane is defined as the minimal possible time that takes Z to get from P to Q according to the above rules. It is clear that $d_2(P,Q)$ is equal to the lenght of the segment PQ if this segment is horizontal. One can verify that if the segment PQ is vertical, then the minimal time is obtained when Z moves vertically from P to Q (but this is not the only possibility), and this time is not less than $|PQ|$. These two observations imply that d_1 and d_2 satisfy the conditions (C1) and (C3). The continuity of d_2 is obvious.

Let us calculate the distance $d_2(A,C)$ between the points $A=(a_0,a_0)$ and $C=(a_2,a_2)$. We shall say that a way from A to C is *economic* if the point Z always goes from left to the right or upwards. For example, the way

$$A \to B=(a_2,a_0) \to (a_2,a_1) \to (a_1,a_1) \to (a_1,a_2)=D \to C$$

is not economic. (However, the distance $d_2(A,C)$ is attained in this way as we shall see latter.) It is not difficult to show that for an arbitrary way from A to C there exists an economic way with the same end points that requires the same (or less) time. Every economic way from A to C lies in the rectangle $ABCD$. The sum of lengths of horizontal links of such a way is equal to $a_2-a_0=(n+1)\cdot\varepsilon$ and the movement along them requires $(n+1)\varepsilon$ units of time. The sum of lengths of vertical links is equal to $(n+1)\varepsilon$. However, at least ε of that length is in the shaded triangles. Thus, the total time of vertical movement is not less than $(2n+1)\varepsilon+n\varepsilon=(3n+1)\varepsilon$. This gives us the estimate $d_2(A,C)\geqslant(3n+1)\varepsilon+(n+1)\varepsilon=(4n+2)\varepsilon$. On the other hand, this number is attained on the way $A\to D\to C$.

It remains to calculate the left part of (3). We have

$$d_2((a_1,a_2),(a_1,a_1))=n\varepsilon,\ d_2((a_2,a_0),(a_2,a_1))=\varepsilon,\ d_1(a_0,a_1)=\varepsilon,\ d_1(a_1,a_2)=n\varepsilon.$$

Therefore, (3) can be rewritten as $(2n+4)\varepsilon<(4n+2)\varepsilon$, that is true whenever $n>1$. This shows that the conclusion (R3) of Theorem 1.4 fails if we drop the condition (C2) on the right-concordance of pseudometrics or replace it by (C3).

5. Thin subsets of free topological groups

A subset T of a topological group G is said to be *thin* in G (see [**15, 9, 10**]) if for any neighbourhood U of the identity in G there exists a neighbourhood V of the identity such that $g\cdot V\cdot g^{-1}\subseteq U$ for each $g\in T$. Every compact and every pseudocompact subset of a topological group G is thin in G [**15**].

We consider here the following problem: characterise subspaces Y of a given space X which are thin in the free topological group $F(X)$.

This problem was solved by the author in [**15**] with the use of the notion of linearly ordered topological field, considering separately the cases whether X is a P-space or not. Here we present a direct solution of the problem and demonstrate an application of the tecnique developed in the previous sections. The following notion will be useful to describe the subsets in question.

DEFINITION 5.1: (See [15].) Let τ be an infinite cardinal and X a space. We call X a P_τ-space if an intersection of fewer than τ open sets is open in X.

DEFINITION 5.2: A subset Y of X is said pseudo-τ-compact in X if for any discrete (equivalently, locally finite) family γ of open sets in X the cardinality of the family $\{U \cap Y : U \in \gamma\}$ is less than τ.

The following theorem completely solves the above problem.

THEOREM 5.3. *(See [15].) A subset Y of X is thin in $F(X)$ if and only if there exists an infinite cardinal τ such that Y is pseudo-τ-compact in X and X is a P_τ-space.*

PROOF: The existence of a cardinal τ as in the theorem is sufficient to imply thinness of Y in $F(X)$. (Modify the proof of Theorem 1.1 of [1] or consult [15].) So suppose that a subspace Y of X is thin in $F(X)$. Denote by μ the minimal cardinality of a family of open sets in X whose intersection is not open. It is necessary to show that Y is pseudo-μ-compact in X.

Assume the contrary. Then there exists a discrete family $\gamma = \{U_\alpha : \alpha < \mu\}$ of open sets in X such that $U_\alpha \cap Y \neq \emptyset$ for each $\alpha < \mu$. From the definition of μ it follows the existence of a point $x^* \in X$ and a decreasing sequence $\{V_\alpha : \alpha < \mu\}$ of open neighbourhoods of x^* such that x^* does not belong to the interior of the intersection $\bigcap\limits_{\alpha<\mu} V_\alpha$. For every $\alpha < \mu$ pick a point $a_\alpha \in U_\alpha \cap Y$ and define continuous real-valued functions f_α and g_α on X such that $f_\alpha(a_\alpha) = 1$, $g_\alpha(x^*) = 1$, $f_\alpha(x) = 0$ for each $x \in X \setminus U_\alpha$, $g_\alpha(y) = 0$ for each $y \in X \setminus V_\alpha$ and $0 \leqslant f_\alpha, g_\alpha \leqslant 1$. For every $\alpha < \mu$ define continuous pseudometrics $d_{1,\alpha}$ and $\varrho_{1,\alpha}$ on X by $d_{1,\alpha}(x,y) = |f_\alpha(x) - f_\alpha(y)|$ and $\varrho_{1,\alpha}(x,y) = |g_\alpha(x) - g_\alpha(y)|$; $x, y \in X$. Obviously, $\varrho_{1,\alpha} \leqslant 1$. Then apply the reasoning of Section 2 to define right-concordant pairs $(d_{1,\alpha}, d_{2,\alpha})$ of continuous pseudometrics satisfying for all $\alpha < \mu$ and $a, x, y \in X$ the condition

$$\text{(U1)} \qquad d_{2,\alpha}((a,x),(a,y)) = f(a) \cdot \varrho_{1,\alpha}(x,y).$$

Finally, put $d_1 = \sum\limits_{\alpha<\mu} d_{1,\alpha}$, $d_2 = \sum\limits_{\alpha<\mu} d_{2,\alpha}$. The pair (d_1, d_2) is right-concordant, so Theorem 1.4 implies the existence of a continuous semi-norm N on $G(X)$ satisfying the condition

$$\text{(U2)} \qquad d_2((a,x),(a,y)) = N\left(a \cdot x \cdot y^{-1} \cdot a^{-1}\right) \text{ for all } a, x, y \in X.$$

Put $O = \{g \in G(X) : N(g) < 1\}$, an open subset of $G(X)$ and $F(X)$. We claim that the following holds.

(A) For any neighbourhood W of the identity in $F(X)$ there exists $\alpha < \mu$ such that $(a_\alpha \cdot W \cdot a_\alpha^{-1}) \setminus O \neq \emptyset$.

Indeed, for a given neighbourhood W of the identity, put $W^* = X \cap (W \cdot x^*)$. Then W^* is a neighbourhood of x^* in X. Since x^* is not in the interior of $\bigcap_{\alpha<\mu} V_\alpha$, there exists $\alpha < \mu$ such that $W^* \setminus V_\alpha \neq \emptyset$. Pick a point $y \in W^* \setminus V_\alpha$ and put $g = y \cdot (x^*)^{-1}$. Then $g \in W$, and we have

$$\begin{aligned} N(a_\alpha \cdot g \cdot a_\alpha^{-1}) = d_2((a_\alpha, y), (a_\alpha, x^*)) &\geqslant d_{2,\alpha}((a_\alpha, y), (a_\alpha, x^*)) \\ &= f(a_\alpha) \cdot |g_\alpha(y) - g_\alpha(x^*)| = 1, \end{aligned}$$

for $f(a_\alpha) = g_\alpha(x^*) = 1$ and $g_\alpha(y) = 0$. Thus, $N(a_\alpha \cdot g \cdot a_\alpha^{-1}) \geqslant 1$ and, a fortiori, $a_\alpha \cdot g \cdot a_\alpha^{-1} \notin O$. This proves (A). However, (A) implies that Y is not thin in $F(X)$, a contradiction. □

References

[1] W.W. Comfort and K.A. Ross, 'Pseudocompactness and uniform continuity in topological groups', *Pacific J. Math.* **16** (1966), 483–496.

[2] R. Engelking, *General topology* (PWN, Warsaw, 1977).

[3] M.I. Graev, 'Free topological groups', (in Russian), *Izv. Akad. Nauk SSSR Ser. Mat.* **12** (1948), 279–324. English translation: *Transl. Math. Monographs* **8** (1962), 305–364.

[4] M.I. Graev, 'Theory of topological groups', (in Russian), *Uspekhi Mat. Nauk* **5** (1950), 3–56.

[5] A.A. Markov, 'On free topological groups', (in Russian), *Izv. Akad. Nauk SSSR Ser. Mat.* **9** (1945), 3–64. English translation: *Transl. Math. Monographs* **8** (1962), 195–272.

[6] S.A. Morris and P. Nickolas, 'Locally invariant topologies on free groups', *Pacific J. Math.* **103** (1982), 523–537.

[7] E.C. Nummela, 'Uniform free topological groups and Samuel compactifications', *Topology Appl.* **13** (1982), 77–83.

[8] V.G. Pestov, 'Some properties of free topological groups', (in Russian), *Vestnik Moscov. Univ. Ser. Matem.* (1982), 35–37. English translation: *Moscow Univ. Math. Bull.* **37**, 46–49.

[9] V.G. Pestov V.G., 'Thin sets in topological groups and a new compactness theorem', (in Russian), *Izv. Vyssh. Uchebn. Zaved. Ser. Matem.* **11** (1987), 64–66. English translation: *Soviet Math.* **31**, 81–84.

[10] O.V. Sipacheva and M.G. Tkačenko, 'Thin and bounded subsets of free topological groups', *Topology Appl.* **36** (1990), 143-156.

[11] O.V. Sipacheva and V.V. Uspenskiĭ, 'Free topological groups with no small subgroups and Graev metrics', (in Russian), *Vestnik Moscov. Univ. Ser. Mat.* **4** (1987), 21–24. English translation: *Moscow Univ. Math. Bull.* **42**, 24–29.

[12] S. Świerczkowski, 'Topologies in free algebras', *Proc. London Math. Soc.* **14** (1964), 566–576.

[13] M.G. Tkačenko, 'On completeness of topological groups', (in Russian), *Sibirsk. Mat. J.* **25** (1984), 146–158. English translation: *Siberian Math. J.* **25**, 122–131.

[14] M.G. Tkačenko, 'On topologies of free groups', *Czechoslovak. Math. J.* **34** (1984), 541–551.

[15] M.G. Tkačenko, 'On some properties of free topological groups', (in Russian), *Mat. Zametki* **37** (1985), 110–118. English translation: *Math. Notes* **37**.

[16] M.G. Tkačenko, 'Concordant extension of pseudometrics from subspaces of free topological groups', in *Algebraic and logic constructions (Collection)*, (in Russian) (Kalinin State Univ., 1989), pp. 94–103.

Departamento de Matemáticas
Universidad Autónoma, Metropolitana
Unidad Iztapalapa
Mexico 13
e-mail: mich@xanum.uam.mx