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RESEARCH ARTICLE

The stable cohomology of self-equivalences of connected sums of products of spheres

Robin Stoll

Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden; E-mail: robin.stoll@math.su.se.

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Abstract

We identify the cohomology of the stable classifying space of homotopy automorphisms (relative to an embedded disk) of connected sums of $S^k \times S^l$, where $3 \leq k < l \leq 2k-2$. The result is expressed in terms of Lie graph complex homology.

Contents

1. Introduction

The starting point of this article is the following manifold with boundary

$$M_{g,1}^{k,l} := \#_g(S^k \times S^l) \setminus \mathring{D}^{k+l}$$

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(i.e., the g-fold connected sum of $S^k \times S^l$ with an open disk removed). Given this, a natural object to consider is the classifying space

$$\mathrm{Baut}_\partial\left(M_{g,1}^{k,l}\right)$$

of the topological monoid of homotopy automorphisms of $M_{g,1}^{k,l}$ that fix the boundary pointwise. It classifies fibrations with fiber $M_{g,1}^{k,l}$ under the trivial fibration with fiber its boundary (see, for example, [HL15, Appendix B]). In particular, the cohomology of $\mathrm{Baut}_\partial(M_{g,1}^{k,l})$ consists of the characteristic classes for such fibrations.

This cohomology appears to be hard to understand completely. However, at least rationally, it is possible to say something about the behavior of the *stabilization maps*

$$\mathrm{H}^*\left(\mathrm{Baut}_\partial(M_{g+1,1}^{k,l});\mathbb{Q}\right) \longrightarrow \mathrm{H}^*\left(\mathrm{Baut}_\partial(M_{g,1}^{k,l});\mathbb{Q}\right) \tag{1.1}$$

obtained by gluing on, along one of its boundary components, a copy of $S^k \times S^l$ with two disjoint open disks removed. Namely, in the case $2 \leq k = l$, Berglund–Madsen [BM20] have shown this map to be an isomorphism in a certain stable range of degrees,[1] and this was extended by Grey [Gre19] to the cases $3 \leq k \leq l \leq 2k-2$.

Moreover, [BM20] contains, in the case $3 \leq k = l$, a combinatorial description (in terms of graph complex homology) of the *stable cohomology* (i.e., the limit of the maps (1.1) as g goes to infinity). In this paper, we provide a similar description when $k \neq l$. The following is our main result.

Theorem 1.1 (see Corollary 5.9). *Let $3 \leq k < l \leq 2k-2$ and $2 \leq g$ be integers. Then there is, in cohomological degrees $\leq g-2$, an isomorphism of graded algebras*

$$\mathrm{H}^*\left(\mathrm{Baut}_\partial(M_{g,1}^{k,l});\mathbb{Q}\right) \cong \mathrm{H}^*(\mathrm{GL}(\mathbb{Z});\mathbb{Q}) \otimes \mathrm{H}^*\left(\mathfrak{UG}^{k+l-2}(\mathscr{L}\mathrm{ie})^\vee\right)$$

compatible with the stabilization maps on the left-hand side. Here we set $\mathrm{GL}(\mathbb{Z}) \coloneqq \mathrm{colim}_{g\in\mathbb{N}} \mathrm{GL}_g(\mathbb{Z})$, denote by $\mathfrak{UG}^m(\mathscr{L}\mathrm{ie})$ the m-twisted graph complex associated to the cyclic Lie operad (with its canonical coalgebra structure), and write $(-)^\vee$ for linear dualization.

In particular, after stabilizing, we obtain an isomorphism

$$\lim_{g\in\mathbb{N}} \mathrm{H}^*\left(\mathrm{Baut}_\partial(M_{g,1}^{k,l});\mathbb{Q}\right) \cong \mathrm{H}^*(\mathrm{GL}(\mathbb{Z});\mathbb{Q}) \otimes \mathrm{H}^*\left(\mathfrak{UG}^{k+l-2}(\mathscr{L}\mathrm{ie})^\vee\right)$$

of graded algebras.

Remark. We also prove a version of Theorem 1.1 for cohomology with certain local coefficients; see Theorem 5.7. In this generality, the description we give is in terms of a *directed* graph complex (see below). It should be possible to relate this to an undirected graph complex, just as in the case of trivial coefficients above, but we do not carry this out in this paper.

Remark. The proof of Theorem 1.1 does not rely on the stability result of Grey [Gre19] (though we do reuse some of its ingredients). In particular, we obtain a new proof of his result (for the manifolds $M_{g,1}^{k,l}$), while also roughly doubling the stable range. This is made possible by a result of Li–Sun [LS19], which yields a slope 1 vanishing of the rational cohomology of $\mathrm{GL}_n(\mathbb{Z})$ with certain coefficients. That a result of this form would lead to an improvement of the stable range had been suggested by Krannich [Kra20, p. 1070].

Surprisingly, the graph complex appearing in Theorem 1.1 only depends on the dimension $k+l$ and not on k and l themselves. Moreover, when $k+l$ is even, it is the same graph complex that appears in

[1] This stable range was extended significantly by Krannich [Kra20].

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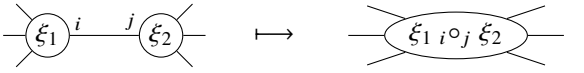
[BM20] – that is, in the case $k = l$ (when $k + l$ is odd, it is defined similarly to the even case but appears to have quite different homology; see further below). For both of these observations, there seems to be no a priori reason to expect them to be true. The second one is particularly interesting when considering our proof, where first a different graph complex appears, of which we then show that it actually has the same homology as $\mathfrak{UG}^{k+l-2}(\mathscr{L}\mathrm{ie})$.

We will now describe the terms appearing on the right-hand sides of the isomorphisms of Theorem 1.1 in more detail. First, note that

$$\mathrm{H}^*(\mathrm{GL}(\mathbb{Z});\mathbb{Q}) \cong \Lambda\mathbb{Q}\langle x_i \mid i \in \mathbb{N}_{\geq 1}\rangle \qquad \text{where} \quad |x_i| = 4i+1$$

by a classical result of Borel [Bor74]. The graph complex $\mathfrak{UG}^m(\mathscr{C})$, for $\mathscr{C}$ a cyclic operad, is also a well-known object. Versions of it were first described by Kontsevich [Kon93, Kon94], but they have, since then, appeared in many different places; see, for example, [GK98, CV03, LV08, BM20].

We now sketch the definition of this graph complex. It is generated by graphs (potentially with loops and parallel edges) equipped with an orientation of each edge and a labeling of each vertex with an element of $\mathscr{C}$ of cyclic arity the valence of the vertex (which we assume to be at least three). We consider a vertex to have homological degree $1 - m$ and an edge to have homological degree m. The resulting graded vector space is quotiented by the actions of the automorphism groups of the graphs, taking into account the homological degrees of the vertices and edges and yielding an extra sign $(-1)^{m+1}$ whenever the orientation of an edge is flipped. The differential is defined to be the sum over all possible contractions of non-loop edges. When an edge is contracted, the label of the new vertex is obtained via the cyclic operad composition of the labels of the two contracted vertices, as in the following picture.



The graph complex has the structure of a cocommutative differential graded coalgebra with comultiplication given by the sum over all possible ways to partition the connected components of a graph into two sets.

Note that, up to regrading, the graph complex and, in particular, its homology only depend on the parity of m. Even though this homology could theoretically be computed directly from its definition, this has only be feasible in low degrees so far. Its structure in higher degrees remains largely mysterious. By a result of Kontsevich [Kon93] (for m even; see also Conant–Vogtmann [CV03]) and Lazarev–Voronov [LV08] (for m odd), Lie graph complex homology is closely related to the rational homology (with twisted coefficients if m is odd) of the group $\mathrm{Out}(F_g)$ of outer automorphisms of the free group on g generators. See Remark 4.30 for a summary of what is known about this. In particular, computations of Brun–Willwacher [BW23] imply that $\mathrm{H}_p(\mathfrak{UG}^m(\mathscr{L}\mathrm{ie}))$ is trivial for $0 < p \leq 3m+2$ when $m \geq 3$ is odd (see Remark 4.30). Using this, we obtain the following consequence of our main theorem.

Corollary 1.2. *Let $3 \leq k < l \leq 2k-2$ and $2 \leq g$ be integers such that $k + l$ is odd. Then there is, in cohomological degrees $\leq \min(g-2, 3(k+l)-4)$, an isomorphism of graded algebras*

$$\mathrm{H}^*\left(\mathrm{Baut}_\partial(M_{g,1}^{k,l});\mathbb{Q}\right) \cong \mathrm{H}^*(\mathrm{GL}(\mathbb{Z});\mathbb{Q})$$

compatible with the stabilization maps on the left-hand side.

Sketch of the proof of Theorem 1.1

From now on, we fix some $3 \leq k < l \leq 2k-2$ and drop them from the notation. The starting point of our proof are methods recently developed by Berglund–Zeman [BZ22] which in particular yield an

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isomorphism

$$\mathrm{H}^*\big(\mathrm{Baut}_\partial(M_{g,1}); P\big) \cong \mathrm{H}^*\Big(\Gamma_g^{\mathbb{Z}}; \mathrm{H}^*_{\mathrm{CE}}(\mathfrak{g}_g) \otimes P\Big) \tag{1.2}$$

for any $\Gamma_g^{\mathbb{Z}}$-representation P (the details of applying their methods to relative self-equivalences are worked out in the upcoming paper [BS] joint with Berglund). Here, the differential graded Lie algebra $\mathfrak{g}_g$ is the Quillen model of the 1-connected cover of $\mathrm{Baut}_\partial(M_{g,1})$. We write $\mathrm{H}^*_{\mathrm{CE}}(\mathfrak{g}_g)$ for its Chevalley–Eilenberg cohomology, and we set

$$\Gamma_g^R := \mathrm{Aut}\Big(\widetilde{\mathrm{H}}_*(M_{g,1}; R), \langle -, - \rangle_\cap\Big) \cong \mathrm{GL}_g(R),$$

where R is a commutative ring and $\langle -, - \rangle_\cap$ is the intersection pairing. The Lie algebra $\mathfrak{g}_g$ has an explicit description which was shown by Berglund–Madsen [BM20] to be isomorphic to

$$(\mathrm{s}^{-(k+l-2)}\, \mathscr{L}\mathrm{ie})\big[\mathrm{s}^{-1}\widetilde{\mathrm{H}}_*(M_{g,1}; \mathbb{Q})\big]^+$$

equipped with the trivial differential. Here, $\mathscr{L}\mathrm{ie}$ is (the symmetric sequence underlying) the cyclic Lie operad, s denotes a degree shift, $\mathscr{A}[-]$ is the Schur functor associated to $\mathscr{A}$, and $(-)^+$ denotes positive truncation.

The next step is to simplify the right-hand side of (1.2). To this end, we prove, using the previously mentioned work of Li–Sun [LS19], that the canonical map

$$\mathrm{H}^p\Big(\Gamma_g^{\mathbb{Z}}; \mathbb{Q}\Big) \otimes \mathrm{H}^q\Big(\big(\mathrm{C}^*_{\mathrm{CE}}(\mathfrak{g}_g) \otimes P\big)^{\Gamma_g^{\mathbb{Q}}}\Big) \longrightarrow \mathrm{H}^p\Big(\Gamma_g^{\mathbb{Z}}; \mathrm{H}^q_{\mathrm{CE}}(\mathfrak{g}_g) \otimes P\Big)$$

is an isomorphism in a stable range when P is an algebraic representation of $\Gamma_g^{\mathbb{Q}}$. (See also work of Krannich [Kra20], where an analogous argument is made to obtain a similar stable range in the case $k = l$.)

This shows that it is enough to identify the (stable) cohomology of the invariants $\big(\mathrm{C}^*_{\mathrm{CE}}(\mathfrak{g}_g) \otimes P\big)^{\Gamma_g^{\mathbb{Q}}}$. For sake of exposition, we will now focus on the case $P = \mathbb{Q}$. In the paper, we do carry out the first part of the argument for a certain class of nontrivial representations as well, though. Similar situations, arising from Sp_{2g} or $O_{g,g}$ instead of GL_g, have been considered by Kontsevich [Kon93, Kon94] (see also Conant–Vogtmann [CV03]) in the symplectic case and by Berglund–Madsen [BM20] in both cases. In our situation, we use coinvariant theory methods similar to those of [BM20] to obtain a description of the stable cohomology in terms of a directed version of the Lie graph complex, which we denote by $\mathfrak{DG}^m(\mathscr{L}\mathrm{ie})$. It is defined in the same way as $\mathfrak{UG}^m(\mathscr{L}\mathrm{ie})$, except that every edge is equipped with a direction, which (unlike an orientation) must be respected by automorphisms of the graph.

Theorem 1.3 (see Corollary 3.33). *There is an isomorphism*

$$\operatorname*{colim}_{g \in \mathbb{N}} \mathrm{C}_*^{\mathrm{CE}}(\mathfrak{g}_g)_{\Gamma_g^{\mathbb{Q}}} \cong \mathfrak{DG}^{k+l-2}_{\mathrm{trunc}}(\mathscr{L}\mathrm{ie})$$

of differential graded coalgebras, compatible with the stabilization maps on the left-hand side. Here, $\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{L}\mathrm{ie})$ *is the subcomplex of* $\mathfrak{DG}^m(\mathscr{L}\mathrm{ie})$ *spanned by truncated directed graphs, which are those that do not have vertices of valence zero or one and whose vertices of valence two have only incoming edges.*

Remark. At the heart of this theorem actually lies an unstable identification of a certain subcomplex of $\mathrm{C}_*^{\mathrm{CE}}(\mathfrak{g}_g)_{\Gamma_g}$. Moreover, we prove a more general statement, which also applies to a certain class of nontrivial representations P, as well as to all cyclic operads and not just $\mathscr{L}\mathrm{ie}$.

Although less ubiquitous than undirected graph complexes, directed versions have appeared in the literature before; see, for example, [Wil14, Živ20, DR19]. Surprisingly, it turns out that the homology

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of the truncated directed graph complex $\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{L}\mathrm{ie})$ is actually isomorphic to the one of $\mathfrak{UG}^m(\mathscr{L}\mathrm{ie})$. We will now explain the idea of this argument. Restricting the differential of $\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{L}\mathrm{ie})$ to the two edges incident to a vertex of valence two (which must be labeled by the identity operation) yields the following picture.

Thus, intuitively, taking homology should kill the difference between the two orientations of an edge. This can be formalized using a spectral sequence argument to prove the following theorem. Together with the preceding results, this implies Theorem 1.1.

Theorem 1.4 (see Theorem 4.28). *Let $\mathscr{C}$ be a cyclic operad such that $\mathscr{C}(\!(2)\!) \cong \mathbb{Q}\langle \mathrm{id}\rangle$. Then there is, for any $m \in \mathbb{N}_0$, an isomorphism*

$$\mathrm{H}_*\left(\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})\right) \cong \mathrm{H}_*(\mathfrak{UG}^m(\mathscr{C}))$$

of graded coalgebras.

Remark. We only carry out this argument in the situation arising from trivial coefficients $P = \mathbb{Q}$. There should exist a more general statement of this form, though.

Remark. For the nontruncated directed graph complex, an argument similar to this has been sketched by Willwacher [Wil14, Appendix K].

Related work and further research

Building on their work for homotopy automorphisms, Berglund–Madsen [BM20] also computed, for $k \geq 3$, the stable cohomology of the classifying spaces $\mathrm{B}\widetilde{\mathrm{Diff}}_\partial\left(M^{k,k}_{g,1}\right)$ of block diffeomorphisms. Combining their methods with the methods of our paper (as well as an upgrade of the methods of [BZ22] to automorphisms of bundles, which is planned to be contained in the upcoming paper [BS] joint with Berglund or a follow-up) should yield a computation of the stable cohomology of $\mathrm{B}\widetilde{\mathrm{Diff}}_\partial\left(M^{k,l}_{g,1}\right)$ for $3 \leq k < l \leq 2k-2$. This is currently work in progress. It should significantly simplify and extend a result of Ebert–Reinhold [ER22], who compute, in a hands-on way, the stable cohomology of $\mathrm{B}\widetilde{\mathrm{Diff}}_\partial\left(M^{k,k+1}_{g,1}\right)$ up to degree roughly k. From this, they deduce, using classical methods as well as a recent result of Krannich [Kra22], the stable cohomology of $\mathrm{BDiff}_\partial\left(M^{k,k+1}_{g,1}\right)$ in the same range. A complete calculation of the stable cohomology of block diffeomorphisms would, in the same way, allow to extend this result to degree roughly $2k$. Moreover, one might hope to extend this calculation even further, for example by using methods recently developed by Krannich–Randal-Williams [KR21]. This is the goal of current work in progress joint with Krannich.

One question one might hope to attack with this approach is to determine the images of the cohomology classes coming from the graph complex under the morphism

$$\mathrm{H}^*\left(\mathrm{Baut}_\partial\left(M^{k,l}_{g,1}\right);\mathbb{Q}\right) \longrightarrow \mathrm{H}^*\left(\mathrm{BDiff}_\partial\left(M^{k,l}_{g,1}\right);\mathbb{Q}\right)$$

induced by the forgetful map. When $k = l$, the stable cohomology of the right-hand side has been computed completely in celebrated work of Galatius–Randal-Williams [GR14], but even in that case, it is mostly unknown what the graph cohomology classes map to. Results in this direction have the potential to yield very interesting information about both the cohomology of the diffeomorphism groups as well as the homology of the graph complex and thus the homology of $\mathrm{Out}(F_g)$.

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Structure of this paper

In Section 2, we fix some conventions, recall various definitions and prove basic lemmas needed throughout the rest of the paper. In Section 3, we use coinvariant theory to prove Theorem 1.3. In Section 4, we prove Theorem 1.4 using a spectral sequence argument. Finally, Section 5 contains the topological part, in which we combine everything to obtain Theorem 1.1.

2. Preliminaries

In this section, we collect a number of basic conventions, notations, definitions and lemmas which we will use throughout the rest of this paper.

2.1. Graded vector spaces

Convention 2.1. The base field is $\mathbb{Q}$.

Notation 2.2. We denote by **Vect** the category of vector spaces and linear maps.

Convention 2.3. All gradings are by $\mathbb{Z}$.

Notation 2.4. We denote by **GrVect** the category of graded vector spaces and grading preserving linear maps. Sometimes we will implicitly consider an ungraded vector space (such as $\mathbb{Q}$) as a graded vector space concentrated in degree 0.

Notation 2.5. For $k \in \mathbb{Z}$, we denote by $\mathrm{s}^k(-)$ a k-fold degree shift. For V a graded vector space, the graded vector space $\mathrm{s}^k V$ is given by $(\mathrm{s}^k V)_n := V_{n-k}$. When $k = 1$, we just write $\mathrm{s} := \mathrm{s}^1$.

Notation 2.6. Let $m \in \mathbb{Z}$. A *graded symplectic form* of degree $-m$ on a graded vector spaces V is a nondegenerate bilinear form of degree $-m$

$$\langle -, - \rangle \colon V \otimes V \longrightarrow \mathbb{Q}$$

such that $\langle v, w \rangle = (-1)^{|v||w|+1} \langle w, v \rangle$ (i.e. it is graded anti-symmetric).

We denote by $\mathbf{Sp}^m$ the category of finite-dimensional graded vector spaces equipped with a graded symplectic form of degree $-m$ with morphisms those linear maps of degree 0 that preserve the bilinear form.

Remark 2.7. Note that a morphism of $\mathbf{Sp}^m$ is automatically injective.

2.2. Double complexes

Convention 2.8. A *double complex* is a bigraded vector space $C_{*,*}$ equipped with a differential d^1 of bidegree $(-1, 0)$ and a differential d^2 of bidegree $(0, -1)$ such that $d^1 \circ d^2 = -d^2 \circ d^1$.

Notation 2.9. Let $(C_{*,*}, d^1, d^2)$ be a double complex. We denote by $\operatorname{Tot} C$ its *total complex* – that is, the chain complex with underlying graded vector space

$$(\operatorname{Tot} C)_p := \bigoplus_{k+l=p} C_{k,l}$$

and differential $d = d^1 + d^2$.

Lemma 2.10. *Let $(C_{*,*}, d^1, d^2)$ be a double complex concentrated in nonnegative degrees such that $\mathrm{H}_*(C_{p,*}, d^2)$ is concentrated in degree 0 for all p. Denote by*

$$f \colon \operatorname{Tot} C \to \left(\mathrm{H}_0(C_{\bullet,*}, d^2), d^1\right)$$

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the map of chain complexes given by the canonical projections $C_{p,0} \to \mathrm{H}_0(C_{p,*}, d^2)$ and the trivial map on $C_{p,q}$ for $q > 0$. Then f is a quasi-isomorphism.

Proof. The map f canonically lifts to a map $g\colon C_{*,*} \to \big(\mathrm{H}_0(C_{\bullet,*}, d^2), d_1\big)$ of double complexes, where the target is equipped with the double complex structure concentrated in bidegrees $(p, 0)$. The map g induces an isomorphism between the E^1-pages of the spectral sequences associated to these double complexes whose zeroth differential is given by d^2 (see, for example, Weibel [Wei95, §5.6]). By the Comparison Theorem for spectral sequences, this implies that f is a quasi-isomorphism (see, for example, [Wei95, Theorem 5.2.12]). □

2.3. Group actions

Definition 2.11. Let $\mathcal{G}$ be a groupoid and $\mathcal{C}$ a category. A *left $\mathcal{G}$-module in $\mathcal{C}$* is a functor $M\colon \mathcal{G} \to \mathcal{C}$, and a *right $\mathcal{G}$-module* is a functor $M\colon \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$.

Remark 2.12. Identifying a group G with its associated one-object groupoid, a left G-module in $\mathcal{C}$ is an object of $\mathcal{C}$ together with a left G-action, and a right G-module in $\mathcal{C}$ is an object of $\mathcal{C}$ together with a right G-action.

Convention 2.13. Unless otherwise stated, 'module in $\mathcal{C}$' will mean 'left module in $\mathcal{C}$'. In particular, group actions will be from the left. If we omit the category $\mathcal{C}$, we mean a (left or right) module in the category **GrVect**.

Definition 2.14. For $\mathcal{G}$ a groupoid, $\mathcal{C}$ a category and M a (left) $\mathcal{G}$-module in $\mathcal{C}$, we will denote by M^{op} the right $\mathcal{G}$-module in $\mathcal{C}$ given by the composition

$$\mathcal{G}^{\mathrm{op}} \xrightarrow{\cong} \mathcal{G} \xrightarrow{M} \mathcal{C},$$

where the first map is the isomorphism given by the identity on objects and by taking the inverses of morphisms.

Remark 2.15. If $\mathcal{G} = G$ is a group, this means that M^{op} is the right G-module with the opposite action (i.e., $m^{\mathrm{op}} \,.\, g := (g^{-1} \,.\, m)^{\mathrm{op}}$).

2.4. Kan extensions

The following lemmas will be useful later.

Lemma 2.16. *Let G be a group and S a G-set (i.e., a functor $S\colon G \to$ **Set**). Then a functor X from the Grothendieck construction $\int_G S$ to some category $\mathcal{C}$ can be equivalently described as a family $(X_s)_{s\in S}$ of objects of $\mathcal{C}$ equipped with morphisms $g_s\colon X_s \to X_{g.s}$ for every $g \in G$ and $s \in S$ such that they are functorial in the sense that $h_{g.s} \circ g_s = (hg)_s$.*

Moreover, let $H \trianglelefteq G$ be a normal subgroup. If $\mathcal{C}$ is cocomplete, then there is an isomorphism

$$\left(\coprod_{s\in S} X_s\right)_H \cong \mathrm{Lan}_{\int_G S \to G/H} X$$

of (G/H)-modules in $\mathcal{C}$. On the left-hand side $g \in G$, acts on $\coprod_{s\in S} X_s$ by mapping X_s to $X_{g.s}$ via g_s; the action of G/H is the induced action on the coinvariants. On the right-hand side, we have the left Kan extension of X along the composite $\int_G S \to G \to G/H$ of the canonical projection and the quotient map.

Proof. The first claim follows directly from the definition of the Grothendieck construction. For the second claim, note that the G-module $\coprod_{s\in S} X_s$ is isomorphic to the left Kan extension of X along the canonical projection $\int_G S \to G$. Taking H-coinvariants of this corresponds to left Kan extending further

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along the quotient map $G \to G/H$. Since Kan extensions compose, the result is isomorphic to the left Kan extension of X along the composite $\int_G S \to G/H$. □

We will now provide an explicit identification of the left Kan extension in the lemma above. This takes the form of the following (more general) lemmas.

Lemma 2.17. *Let $\mathcal{G}$ be a groupoid, H a group and $F\colon \mathcal{G} \to H$ a functor. Denote by $\mathcal{G}^F \subseteq \mathcal{G}$ the wide subcategory of those morphisms that F maps to the neutral element $e_H \in H$. Also denote by $*$ the unique object of H considered as a groupoid, and by $F \downarrow *$ the corresponding comma category. Then there is a fully faithful functor*

$$\Phi_F\colon \mathcal{G}^F \longrightarrow F \downarrow *$$

given by sending an object $G \in \mathcal{G}^F$ to $(G, e_H\colon F(G) = * \to *)$ and a morphism $g\colon G \to G'$ to itself.

Moreover, assume that, for all $G \in \mathcal{G}$ and $h \in H$, there exists a morphism $g_{G,h}\colon G \to K_{G,h}$ in $\mathcal{G}$ such that $F(g_{G,h}) = h$. Then Φ_F is an equivalence of categories with an inverse (up to natural isomorphism) given by the functor

$$\Psi_F\colon F \downarrow * \longrightarrow \mathcal{G}^F$$

given on objects by $(G, h) \mapsto K_{G,h}$ and on morphisms by $(g\colon (G, h) \to (G', h')) \mapsto g_{G',h'} g g_{G,h}^{-1}$. (Note that Ψ_F depends on the choice of $g_{G,h}$ for all $G \in \mathcal{G}$ and $h \in H$.) The natural isomorphism $\mathrm{id}_{F\downarrow *} \to \Phi_F \circ \Psi_F$ is, at an object (G, h), given by the map $g_{G,h}\colon G \to K_{G,h}$.

Proof. This follows easily from the definitions. □

Lemma 2.18. *In the situation of the second part of Lemma 2.17, let $\mathcal{C}$ be a cocomplete category and $X\colon \mathcal{G} \to \mathcal{C}$ a functor. Furthermore, for an element $h \in H$, let A_h denote the functor $\mathcal{G}^F \to \mathcal{G}^F$ that is given by $G \mapsto K_{G,h}$ on objects and that sends a morphism $g\colon G \to G'$ to $g_{G',h} g g_{G,h}^{-1}$. Then there is an isomorphism*

$$\mathrm{Lan}_{F\colon \mathcal{G} \to H} X \cong \mathrm{colim}_{\mathcal{G}^F} (X \circ \mathrm{inc}_{\mathcal{G}^F})$$

of H-modules in $\mathcal{C}$. Here an element $h \in H$ acts on the right-hand side via the functor A_h and the natural transformation $\mathrm{inc}_{\mathcal{G}^F} \to \mathrm{inc}_{\mathcal{G}^F} \circ A_h$ given, at an object $G \in \mathcal{G}^F$, by $g_{G,h}\colon G \to K_{G,h}$.

Proof. The left Kan extension in the statement is, at the unique object $*$ of H, isomorphic to $\mathrm{colim}_{F\downarrow *} (X \circ \mathrm{pr})$, where $\mathrm{pr}\colon F \downarrow * \to \mathcal{G}$ denotes the projection. The action of an element $h \in H$ on this object is induced by the functor $F \downarrow * \to F \downarrow *$ given by $(G, h') \mapsto (G, hh')$ (note that this functor commutes with the projection). That this is isomorphic, as an H-module, to the description given in the statement follows from Lemma 2.17. □

2.5. Symmetric sequences

Definition 2.19. We denote by **LinSet** the category with objects finite sets equipped with a linear order and morphisms the maps of sets (not necessarily respecting the order), by $\mathbf{\Sigma} := \mathrm{Core}(\mathbf{LinSet})$ its maximal subgroupoid, and by $\mathbf{P}$ the skeleton of $\mathbf{\Sigma}$ spanned by the objects $\underline{n} := \{1 < \cdots < n\}$ for $n \in \mathbb{N}_0$.

Remark 2.20. There exist canonical equivalences of categories $\mathbf{LinSet} \to \mathbf{FinSet}$, $\mathbf{\Sigma} \to \mathrm{Core}(\mathbf{FinSet})$, and $\mathbf{P} \to \mathrm{Core}(\mathbf{FinSet})$, though neither of them admit a canonical inverse equivalence. The inclusion $\mathbf{P} \to \mathbf{\Sigma}$, however, is an equivalence that does admit a canonical inverse equivalence – namely, by identifying an object $S \in \mathbf{\Sigma}$ with an object in $\mathbf{P}$ via the unique map that respects the linear order.

Definition 2.21. Let $\mathcal{V}$ be a symmetric monoidal category and $V \in \mathcal{V}$. Then there is a functor $V^{\otimes -}\colon \mathbf{P} \to \mathcal{V}$ given on objects by $\underline{n} \mapsto V^{\otimes n}$ and defined on morphisms via the symmetrizer isomorphism. Pulling

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this back via the canonical equivalence $\mathbf{\Sigma} \to \mathbf{P}$, we obtain a functor $\mathbf{\Sigma} \to \mathcal{V}$ which we also denote by $V^{\otimes -}$.

Similarly, for $S \in \mathbf{\Sigma}$ and $(V_s)_{s\in S}$ a family of objects of $\mathcal{V}$, we write $\bigotimes_{s\in S} V_s$ for $V_{s(1)} \otimes \ldots \otimes V_{s(|S|)}$, where $s\colon \underline{|S|} \to S$ is the unique order-preserving bijection. We will also use the notation $\bigotimes_{s\in S} v_s$ to denote elementary tensors of this tensor product.

Definition 2.22. For $\mathscr{M}$ a $\mathbf{\Sigma}$-module and $a\colon S \to T$ a morphism of **LinSet**, we write

$$\mathscr{M}(a) \coloneqq \bigotimes_{t\in T} \mathscr{M}\Big(a^{-1}(t)\Big),$$

where $a^{-1}(t) \subseteq S$ has the linear order induced from the one of S. This construction is functorial in $a \in \mathbf{\Sigma} \downarrow \mathbf{\Sigma}$, with the comma category being taken over **LinSet**.

Definition 2.23. Let $\mathscr{P}$ and $\mathscr{Q}$ be two $\mathbf{\Sigma}$-modules. Then we can form their *composition product* – that is, the $\mathbf{\Sigma}$-module $\mathscr{P} \circ \mathscr{Q}$ given on objects by

$$S \longmapsto \operatorname*{colim}_{(a\colon S\to T)\in S\downarrow\mathbf{\Sigma}} (\mathscr{P}(T) \otimes \mathscr{Q}(a)),$$

where the slice category $S \downarrow \mathbf{\Sigma}$ is taken over **LinSet**. A morphism $f\colon S \to S'$ of $\mathbf{\Sigma}$ acts via the map

$$\operatorname*{colim}_{(a\colon S\to T)\in S\downarrow\mathbf{\Sigma}} (\mathscr{P}(T) \otimes \mathscr{Q}(a)) \longrightarrow \operatorname*{colim}_{(a\colon S'\to T)\in S'\downarrow\mathbf{\Sigma}} (\mathscr{P}(T) \otimes \mathscr{Q}(a))$$

induced by the functor $f^{-1} \downarrow \mathbf{\Sigma}\colon S \downarrow \mathbf{\Sigma} \to S' \downarrow \mathbf{\Sigma}$ as well as the natural transformation $\mathscr{P}(T) \otimes \mathscr{Q}(a) \to \mathscr{P}(T) \otimes \mathscr{Q}(a \circ f^{-1})$ given by the identity of $\mathscr{P}(T)$ and the map $\mathscr{Q}(a) \to \mathscr{Q}(a \circ f^{-1})$ induced by the morphism $(f, \mathrm{id}_T)\colon a \to a \circ f^{-1}$ of $\mathbf{\Sigma} \downarrow \mathbf{\Sigma}$.

2.6. Schur functors

Notation 2.24. Let $\mathscr{M}$ a $\mathbf{\Sigma}$-module. We define a functor $\mathscr{M}[-]\colon \mathbf{GrVect} \to \mathbf{GrVect}$ by

$$V \longmapsto \bigoplus_{k\in\mathbb{N}_0} \Big(\mathscr{M}(\underline{k})^{\mathrm{op}} \otimes_{\Sigma_{\underline{k}}} V^{\otimes \underline{k}}\Big)$$

and call it the *Schur functor* associated to $\mathscr{M}$.

Remark 2.25. There is a canonical, and natural in V, isomorphism

$$\mathscr{M}[V] \cong \operatorname*{colim}_{\underline{k}\in\mathbf{P}} \Big(\mathscr{M}(\underline{k}) \otimes V^{\otimes \underline{k}}\Big),$$

where $\mathbf{P}$ acts diagonally on the tensor product. We can also replace $\mathbf{P}$ by $\mathbf{\Sigma}$ since they are canonically equivalent.

Lemma 2.26. *Let $\mathscr{P}$ and $\mathscr{Q}$ be $\mathbf{\Sigma}$-modules. Then there is a natural isomorphism*

$$(\mathscr{P} \circ \mathscr{Q})[-] \xrightarrow{\cong} \mathscr{P}[\mathscr{Q}[-]]$$

with an explicit description as given in the proof.

Proof. We define the isomorphism by sending the element of

$$(\mathscr{P} \circ \mathscr{Q})[V] = \operatorname*{colim}_{S\in\mathbf{\Sigma}} \Big(\operatorname*{colim}_{(a\colon S\to T)\in S\downarrow\mathbf{\Sigma}} \mathscr{P}(T) \otimes \mathscr{Q}(a)\Big) \otimes V^{\otimes S}$$

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represented by $S \in \mathbf{\Sigma}$ and $(a\colon S \to T) \in S \downarrow \mathbf{\Sigma}$ as well as the elements

$$p \otimes \bigotimes_{t\in T} q_t \in \mathscr{P}(T) \otimes \mathscr{Q}(a) \qquad \text{and} \qquad \bigotimes_{s\in S} v_s \in V^{\otimes S}$$

to $\varepsilon \in \{\pm 1\}$ times the element of

$$\mathscr{P}[\mathscr{Q}[-]] = \operatorname*{colim}_{T\in\mathbf{\Sigma}} \mathscr{P}(T) \otimes \left(\operatorname*{colim}_{F\in\mathbf{\Sigma}} \mathscr{Q}(F) \otimes V^{\otimes F}\right)^{\otimes T}$$

represented by $T \in \mathbf{\Sigma}$ and $(F_t \coloneqq a^{-1}(t) \in \mathbf{\Sigma})_{t\in T}$ as well as the elements

$$p \in \mathscr{P}(T) \qquad \text{and} \qquad q_t \otimes \bigotimes_{s\in F_t} v_s \in \mathscr{Q}(F_t) \otimes V^{\otimes F_t}$$

where ε is the sign incurred by permuting the two expressions

$$p \otimes \bigotimes_{t\in T} q_t \otimes \bigotimes_{s\in S} v_s \quad \text{and} \quad p \otimes \bigotimes_{t\in T} \left(q_t \otimes \bigotimes_{s\in F_t} v_s\right)$$

into each other. □

2.7. *Differential graded Lie algebras*

Definition 2.27. Let L be a differential graded Lie algebra. Its *positive truncation* is the differential graded Lie subalgebra $L^+ \subseteq L$ given by

$$(L^+)_i \coloneqq \begin{cases} L_i, & \text{for } i \geq 2 \\ \ker(\delta\colon L_1 \to L_0), & \text{for } i = 1\,, \\ 0, & \text{for } i \leq 0 \end{cases}$$

where δ is the differential of L.

Definition 2.28. Let L be a differential graded Lie algebra. We denote by $\mathrm{C}_*^{\mathrm{CE}}(L)$ its *Chevalley–Eilenberg complex* (i.e., the underlying graded vector space of the free graded commutative algebra $\Lambda(\mathrm{s}L)$), equipped with the differential $d = d_0 + d_1$, where

$$\begin{aligned} d_0(\mathrm{s}x_1 \wedge \cdots \wedge \mathrm{s}x_k) &\coloneqq \sum_{i=1}^{k} (-1)^{b_i} \mathrm{s}x_1 \wedge \cdots \wedge \mathrm{s}\delta(x_i) \wedge \cdots \wedge \mathrm{s}x_k \\ b_i &\coloneqq 1 + \sum_{j=1}^{i-1} |\mathrm{s}x_j| \\ d_1(\mathrm{s}x_1 \wedge \cdots \wedge \mathrm{s}x_k) &\coloneqq \sum_{i<j} (-1)^{b_{i,j}} \mathrm{s}[x_i, x_j] \wedge \mathrm{s}x_1 \wedge \cdots \wedge \widehat{\mathrm{s}x_i} \wedge \cdots \wedge \widehat{\mathrm{s}x_j} \wedge \cdots \wedge \mathrm{s}x_k \\ b_{i,j} &\coloneqq |\mathrm{s}x_i|(1 + |\mathrm{s}x_j|) + |\mathrm{s}x_i| \sum_{l=1}^{i-1} |\mathrm{s}x_l| + |\mathrm{s}x_j| \sum_{l=1}^{j-1} |\mathrm{s}x_l|, \end{aligned}$$

where δ is the differential of L.

The Chevalley–Eilenberg complex comes equipped with the structure of a cocommutative differential graded coalgebra. Its counit $\varepsilon\colon \mathrm{C}_*^{\mathrm{CE}}(L) \to \mathbb{Q}$ is given by 1 on the empty wedge and 0 on all higher wedges. The comultiplication $\Delta\colon \mathrm{C}_*^{\mathrm{CE}}(L) \to \mathrm{C}_*^{\mathrm{CE}}(L) \otimes \mathrm{C}_*^{\mathrm{CE}}(L)$ is given by

$$\Delta(\mathrm{s}x_1 \wedge \cdots \wedge \mathrm{s}x_k) \coloneqq \sum_{A\subseteq \underline{k}} \varepsilon(A)(\textstyle\bigwedge_{a\in A} \mathrm{s}x_a) \otimes (\bigwedge_{b\in \underline{k}\setminus A} \mathrm{s}x_b),$$

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where the sum runs over all subsets A of the set $\underline{k} = \{1, \dots, k\}$. Here, $\varepsilon(A) \in \{\pm 1\}$ is the sign incurred by permuting $\mathrm{s}x_1 \otimes \dots \otimes \mathrm{s}x_k$ into $\bigotimes_{a\in A} \mathrm{s}x_a \otimes \bigotimes_{b\in \underline{k}\setminus A} \mathrm{s}x_b$.

Moreover, we write $\mathrm{H}^*_{\mathrm{CE}}(L)$ for the *Chevalley–Eilenberg cohomology* of L – that is, the cohomology of the linear dual of $\mathrm{C}^{\mathrm{CE}}_*(L)$, equipped with its graded commutative algebra structure.

Lemma 2.29. *Let* $\Lambda\mathrm{s}$ *denote the* $\mathbf{\Sigma}$*-module given by* $\Lambda\mathrm{s}(S) = (\mathrm{s}\mathbb{Q})^{\otimes S}$. *Then there is a canonical isomorphism of graded vector spaces, natural in* $V \in \mathbf{GrVect}$,

$$\begin{aligned}(\Lambda\mathrm{s})[V] &\xrightarrow{\cong} \Lambda(\mathrm{s}V)\\ [(\mathrm{s}1)^{\otimes k} \otimes (v_1 \otimes \dots \otimes v_k)] &\longmapsto (-1)^c \mathrm{s}v_1 \wedge \dots \wedge \mathrm{s}v_k,\end{aligned}$$

where $c \coloneqq \sum_{i=1}^k |v_i|(k-i)$.

Proof. The map lifts to an isomorphism $(\mathrm{s}\mathbb{Q})^{\otimes k} \otimes V^{\otimes k} \to (\mathrm{s}V)^{\otimes k}$ that is compatible with the Σ_k-actions. Passing to quotients again, we obtain the desired statement. □

2.8. The convolution Lie algebra

Recall that a *cyclic operad* is an operad $\mathscr{C}$ such that the right action of Σ_n on $\mathscr{C}(n)$ extends to a right action of Σ_{n+1} in a way compatible with the composition operations (see Getzler–Kapranov [GK95, §2]). Unless explicitly stated otherwise, we will always work in the category of graded vector spaces.

Notation 2.30. For a cyclic operad $\mathscr{C}$, we write $\mathscr{C}(\!(n)\!)$ for $\mathscr{C}(n-1)$ equipped with its right Σ_n-action. We consider a cyclic operad as a $\mathbf{P}$-module via (the opposite of) this extended right action. (Note that this is different from the $\mathbf{P}$-module associated to the underlying operad.) In particular, the Schur functor associated to $\mathscr{C}$ is $\mathscr{C}[V] = \bigoplus_{n\geq 1} \mathscr{C}(\!(n)\!) \otimes_{\Sigma_n} V^{\otimes n}$.

Definition 2.31. For a cyclic operad $\mathscr{C}$, natural numbers $k, l \in \mathbb{N}_{\geq 1}$, operations $p \in \mathscr{C}(\!(k)\!)$ and $q \in \mathscr{C}(\!(l)\!)$, and $1 \leq i \leq k$ and $1 \leq j \leq l$, we set

$$p \;{}_i\circ_j\; q \coloneqq p \circ_i (q \cdot \mathrm{t}_l^j) \in \mathscr{C}(\!(k+l-2)\!),$$

where $\mathrm{t}_n \coloneqq (12\dots n) \in \Sigma_n$ is the cyclic permutation and $\circ_i$ denotes the i-th partial composition of the underlying operad of $\mathscr{C}$. In the case that $i = k$, we set $p \circ_k q \coloneqq ((p \cdot \mathrm{t}_k) \circ_{k-1} q) \cdot \mathrm{t}_{k+l-2}^{-1}$.

Remark 2.32. Via the canonical equivalence $\mathbf{\Sigma} \to \mathbf{P}$, the operations ${}_i\circ_j$ of Definition 2.31 generalize to operations

$${}_s\circ_t \colon \mathscr{C}(\!(S)\!) \otimes \mathscr{C}(\!(T)\!) \longrightarrow \mathscr{C}(\!(S \;{}_s\circ_t\; T)\!),$$

where $S, T \in \mathbf{\Sigma}$, and $s \in S$ and $t \in T$. Here,

$$S \;{}_s\circ_t\; T \coloneqq (S \setminus \{s\}) \amalg (T \setminus \{t\})$$

equipped with the linear order such that

$$S_{<s} < T_{>t} < T_{<t} < S_{>s}$$

with the linear order on each block inherited from S respectively T.

Definition 2.33. Let $V \in \mathbf{Sp}^m$, $k, l \in \mathbb{N}_0$, $1 \leq i \leq k$ and $1 \leq j \leq l$. We denote by

$${}_i\circ_j \colon V^{\otimes k} \otimes V^{\otimes l} \to V^{\otimes k+l-2}$$

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the map given by

$$(v_1 \otimes \ldots \otimes v_k)\ {}_i\circ_j\ (w_1 \otimes \ldots \otimes w_l) \coloneqq (-1)^c \cdot v_{1,i-1} \otimes w_{j+1,l} \otimes w_{1,j-1} \otimes v_{i+1,k} \langle w_j, v_i \rangle$$
$$c \coloneqq (|v_{i,k}| - m)(|w_{1,l}| - m) + |w_{1,j}||w_{j+1,l}| + 1,$$

where, for $a \leq b$, we set $v_{a,b} \coloneqq v_a \otimes v_{a+1} \otimes \ldots \otimes v_b$ and analogously for $w_{a,b}$.

Lemma 2.34. *Let $\mathscr{C}$ be a cyclic operad in graded vector spaces that is concentrated in even degrees and let $V \in \mathbf{Sp}^m$. Then $(\mathrm{s}^{-m}\mathscr{C})[V]$ becomes a graded Lie algebra by setting*

$$[\mathrm{s}^{-m}\xi \otimes v, \mathrm{s}^{-m}\zeta \otimes w] \coloneqq \sum_{i \in \underline{k}, j \in \underline{l}} \mathrm{s}^{-m}(\xi\ {}_i\circ_j\ \zeta) \otimes (v\ {}_i\circ_j\ w)$$

for $\xi \in \mathscr{C}(\!(k)\!)$, $v \in V^{\otimes k}$, $\zeta \in \mathscr{C}(\!(l)\!)$, and $w \in V^{\otimes l}$. This is functorial – that is, the restricted functor $(\mathrm{s}^{-m}\mathscr{C})[-]|_{\mathbf{Sp}^m} \colon \mathbf{Sp}^m \to \mathbf{GrVect}$ lifts to a functor to graded Lie algebras, which we also denote by $(\mathrm{s}^{-m}\mathscr{C})[-]$.

Proof. Checking antisymmetry and the Jacobi identity are straightforward, though tedious, computations. The only trick one has to use is that $v\ {}_i\circ_j\ w$ being nonzero implies that $|v_i| + |w_j| = m$. We omit the details. The claim about the functoriality is clear from the definitions.

In the case $\mathscr{C} = \mathscr{L}\mathrm{ie}$, a different proof of this fact can be found in [BM20, equation (6.7)]. □

Remark 2.35. The condition that $\mathscr{C}$ is concentrated in even degrees is purely for convenience, as it simplifies the signs occurring.

3. The directed graph complex

In this section, our goal is to prove a version of Theorem 1.3 with coefficients. That is, we want to identify (in a stable range)

$$\left(\mathrm{C}_*^{\mathrm{CE}}\left((\mathrm{s}^{-(k+l-2)}\ \mathscr{L}\mathrm{ie})[\mathrm{s}^{-1}\widetilde{\mathrm{H}}_*(M_{g,1}^{k,l};\mathbb{Q})]^+\right) \otimes Q\right)_{\Gamma_g^{\mathbb{Q}}},$$

where $\Gamma_g^{\mathbb{Q}}$ is the group of automorphisms of the graded vector space $\widetilde{\mathrm{H}}_*(M_{g,1};\mathbb{Q})$ that respect the intersection pairing, Q is some $\Gamma_g^{\mathbb{Q}}$-module, and the graded Lie algebra structure on the Schur functor is obtained from Lemma 2.34.

To this end, we begin, in the first two subsections, with studying $(\mathscr{P}[H] \otimes Q)_{\mathrm{Aut}(H)}$, where $\mathscr{P}$ is a $\mathbf{\Sigma}$-module, H is an object of $\mathbf{Sp}^m$ that is concentrated in two degrees $\neq \frac{m}{2}$, and Q is an $\mathrm{Aut}(H)$-representation. From the results we obtain for this situation, we will, via Lemmas 2.26 and 2.29, be able to deduce a description as desired. The arguments we will use are similar to those of [BM20, §9] for the case where H is concentrated in degree $\frac{m}{2}$. We will restrict ourselves to the case that $Q = V^{\otimes I} \otimes W^{\otimes J}$ for some finite linearly ordered sets I and J, where V and W are the two nontrivial homogeneous pieces of H. Identifying the result in this case with its induced $(\Sigma_I \times \Sigma_J)$-action allows one to deduce the result for more general representations via Schur–Weyl duality (see e.g. [FH04, §6.1] or [Eti+11, §5.19]).

Throughout this section, we fix two integers n and m such that $m \neq 2n$. (They correspond to $k-1$ and $k+l-2$ above.)

3.1. Coinvariants of monomial functors

In this subsection, we let $H = V \oplus W$ be a finite-dimensional graded vector space such that V is concentrated in degree n and W is concentrated in degree $m - n \neq n$. Moreover, we assume that H is equipped with a graded symplectic form $\langle -, - \rangle$ of degree $-m$ (see Notation 2.6). We write $\Gamma \coloneqq \mathrm{Aut}_{\mathbf{Sp}^m}(H, \langle -, - \rangle)$ for the group of degree 0 symplectic automorphisms of H.

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Let S, I and J be finite linearly ordered sets and P a Σ_S-module. Our goal for this subsection is to identify the coinvariants

$$\left((P^{\mathrm{op}} \otimes_{\Sigma_S} H^{\otimes S}) \otimes V^{\otimes I} \otimes W^{\otimes J}\right)_\Gamma$$

as a $(\Sigma_I \times \Sigma_J)$-module only in terms of P, without reference to H. We begin with some basic observations about the structures of H and Γ.

Lemma 3.1. *The map $W \to \mathrm{s}^m V^\vee$ given by $w \mapsto \mathrm{s}^m\langle w, -\rangle$ is a degree 0 isomorphism. After identifying W with $\mathrm{s}^m V^\vee$ via this isomorphism, the inner product on $H = V \oplus \mathrm{s}^m V^\vee$ is given by $\langle (v, \mathrm{s}^m\phi), (v', \mathrm{s}^m\phi')\rangle = \phi(v') - (-1)^{n(m-n)}\phi'(v)$.*

Proof. The composition

$$\begin{array}{ccccc} V \oplus W & \xrightarrow{\cong} & (V \oplus W)^\vee & \xrightarrow{\cong} & V^\vee \oplus W^\vee \\ h & \longmapsto & \langle h, - \rangle & & \\ & & & \phi & \longmapsto (\phi|_V, \phi|_W) \end{array}$$

is a degree $-m$ isomorphism of the form

$$\begin{pmatrix} 0 & f \\ f' & 0 \end{pmatrix}$$

since $\langle v, v'\rangle = 0 = \langle w, w'\rangle$ for all $v, v' \in V$ and $w, w' \in W$ by our assumption that $m \neq 2n$ (which also implies $m \neq 2(m-n)$). In particular, f and f' are degree $-m$ isomorphisms. Noting that $f(w) = \langle w, -\rangle$, this implies the first claim. The second follows from $f(w)(v) = \langle w, v\rangle = -(-1)^{n(m-n)}\langle v, w\rangle$. □

Using the preceding lemma we will, in the following, implicitly identify H with $V \oplus \mathrm{s}^m V^\vee$. We now give a description of Γ in these terms.

Lemma 3.2. *The map $\mathrm{GL}(V) \to \mathrm{Aut}(V \oplus \mathrm{s}^m V^\vee, \langle -, -\rangle) = \Gamma$ given by $f \mapsto f \oplus (f^{-1})^\vee$ is an isomorphism of groups. (Here, we implicitly identify automorphisms of $\mathrm{s}^m V^\vee$ with automorphisms of $V^\vee$.)*

Proof. Let $f \in \mathrm{GL}(V)$. Then the computation

$$\begin{aligned} &\langle \Big(f(v), (f^{-1})^\vee(\mathrm{s}^m\phi)\Big), \Big(f(v'), (f^{-1})^\vee(\mathrm{s}^m\phi')\Big)\rangle \\ &= (\phi \circ f^{-1})(f(v')) - (-1)^{n(m-n)}(\phi' \circ f^{-1})(f(v)) \\ &= \phi(v') - (-1)^{n(m-n)}\phi'(v) \\ &= \langle (v, \mathrm{s}^m\phi), (v', \mathrm{s}^m\phi')\rangle \end{aligned}$$

shows that the map in the statement is actually well defined. It is clearly injective. We will now show that it is also surjective. For this purpose, let $f \oplus g \in \Gamma$, where $f \in \mathrm{GL}(V)$ and $g \in \mathrm{GL}(\mathrm{s}^m V^\vee)$. By using that $f^{-1} \oplus g^{-1} \in \Gamma$ as well, we obtain

$$g(\phi)(v) = \langle g(\mathrm{s}^m\phi), v\rangle = \langle \mathrm{s}^m\phi, f^{-1}(v)\rangle = \phi(f^{-1}(v)) = (f^{-1})^\vee(\phi)(v)$$

and hence $g = (f^{-1})^\vee$. This shows surjectivity. □

By basic linear algebra, there is a canonical isomorphism of graded vector spaces $V \otimes \mathrm{s}^m V^\vee \cong \mathrm{s}^m \mathrm{End}(V)$ (here, $\mathrm{End}(V)$ is concentrated in degree 0 since V is concentrated in a single degree). We will now use this to identify $(H^{\otimes S} \otimes V^{\otimes I} \otimes W^{\otimes J})_\Gamma$ in terms of $\mathrm{End}(V)$. To make this precise, we need to introduce some notation.

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Definition 3.3. Let S be a finite linearly ordered set, A and B two disjoint subsets of S such that $A \cup B = S$ (i.e., an ordered partition of S into two subsets), and V_1 and V_2 two graded vector spaces. Then we denote by $(V_1, V_2)^{\otimes(A,B)}$ the tensor product $\bigotimes_{s\in S} V_s$, where $V_s := V_1$ if $s \in A$ and $V_s := V_2$ if $s \in B$.

A morphism $f\colon S \to S'$ of $\mathbf{\Sigma}$ induces a map

$$f_*\colon (V_1, V_2)^{\otimes(A,B)} \to (V_1, V_2)^{\otimes(f(A),f(B))}$$

by permuting the factors. When V_1 is concentrated in degree n and V_2 in degree $m - n$, we denote by $\mathrm{sgn}_{A,B}(f) \in \{\pm 1\}$ the sign incurred by this permutation – that is, it is chosen such that

$$f_*(\textstyle\bigotimes_{s\in S} v_s) = \mathrm{sgn}_{A,B}(f) \cdot \bigotimes_{s'\in S'} v_{f^{-1}(s')}$$

holds.

Definition 3.4. Let S be a finite linearly ordered set. We denote by $\mathrm{OM}(S)$ the set of *ordered matchings* of S (i.e., (ordered) pairs (A, B) such that A and B are disjoint subsets of S with $|A| = |B|$ and $A \cup B = S$). (In particular, $\mathrm{OM}(S)$ is empty if $|S|$ is odd.) This assembles into a functor $\mathrm{OM}\colon \mathbf{\Sigma} \to \mathbf{Set}$ by letting a morphism $f\colon S \to S'$ act via $(A, B) \mapsto (f(A), f(B))$.

Also note that the linear order of S restricts to a linear order on A and a linear order on B. In particular, there is a unique order-preserving bijection $A \to B$. We will denote this bijection by $\mu_{A,B}$.

Definition 3.5. Let S, I and J be finite linearly ordered sets. We denote by $\mathrm{OM}(S, I, J) \subseteq \mathrm{OM}(S \amalg I \amalg J)$ the subset of those ordered matchings (A, B) such that $I \subseteq A$ and $J \subseteq B$. This assembles into a functor $\mathrm{OM}\colon \mathbf{\Sigma} \times \mathbf{\Sigma} \times \mathbf{\Sigma} \to \mathbf{Set}$ by letting a morphism (f, g, h) act via $f \amalg g \amalg h$.

Remark 3.6. Note that there is a canonical bijection $\mathrm{OM}(S, I, J) \cong \mathrm{OM}(S)$, which is natural in $\mathbf{\Sigma}\times\mathbf{\Sigma}\times\mathbf{\Sigma}$.

Lemma 3.7. *Let S, I and J be finite linearly ordered sets. Then there is an isomorphism of $(\Sigma_S \times \Sigma_I \times \Sigma_J)$-modules*

$$\Theta_1\colon \bigoplus_{(A,B)\in \mathrm{OM}(S,I,J)} \mathrm{s}^{m|A|}\left(\mathrm{End}(V)^{\otimes A}\right)_\Gamma \xrightarrow{\cong} \left(H^{\otimes S} \otimes V^{\otimes I} \otimes W^{\otimes J}\right)_\Gamma,$$

where $g \in \mathrm{GL}(V) \cong \Gamma$ acts on $f \in \mathrm{End}(V)$ via the conjugation $g \,.\, f = gfg^{-1}$, and diagonally on the tensor products. The $(\Sigma_S \times \Sigma_I \times \Sigma_J)$-module structure on the left-hand side is given as follows: a morphism (f, g, h) of $\Sigma_S \times \Sigma_I \times \Sigma_J$ maps the factor corresponding to $(A, B) \in \mathrm{OM}(S, I, J)$ into the factor corresponding to $(c(A), c(B))$ via the map given by

$$\mathrm{s}^{m|A|}[\textstyle\bigotimes_{a\in A} \mathrm{E}_{i_a,j_a}] \longmapsto \mathrm{s}^{m|A|}[\mathrm{sgn}_{A,B}(c) \cdot \bigotimes_{a'\in c(A)} \mathrm{E}_{i_{\sigma^{-1}(a')},j_{\tau^{-1}(a')}}],$$

where $c := f \amalg g \amalg h$, $\sigma := c|_A$, and $\tau := (\mu_{c(A),c(B)})^{-1} \circ c|_B \circ \mu_{A,B}$, and the $\mathrm{E}_{i,j}$ are single-entry matrices with respect to some fixed basis of V (the resulting map is independent of this choice).

Moreover, Θ_1 can be chosen such that, for any basis e of V, it fulfills, on the factor corresponding to $(A, B) \in \mathrm{OM}(S, I, J)$,

$$\Theta_1(\mathrm{s}^{m|A|}[\textstyle\bigotimes_{a\in A} \mathrm{E}_{i_a,j_a}]) = [\bigotimes_{s\in S\amalg I\amalg J} h_s]$$
$$h_s := \begin{cases} e_{i_s} \in V, & \text{if } s \in A \\ \mathrm{s}^m e^\vee_{j_{\mu_{A,B}^{-1}(s)}} \in \mathrm{s}^m V^\vee, & \text{if } s \in B \end{cases},$$

where the $E_{i,j}$ are with respect to e.

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Proof. There is a canonical isomorphism

$$H^{\otimes S} \otimes V^{\otimes I} \otimes W^{\otimes J} \cong \bigoplus_{(A,B)} (V, \mathrm{s}^m V^\vee)^{\otimes (A,B)},$$

where (A, B) runs over all (ordered) partitions of $S \amalg I \amalg J$ into two subsets such that $I \subseteq A$ and $J \subseteq B$. This is an isomorphism of $(\Sigma_S \times \Sigma_I \times \Sigma_J \times \Gamma)$-modules where a morphism (f, g, h) of $\Sigma_S \times \Sigma_I \times \Sigma_J$ acts on both sides by permuting the factors (on the right-hand side this maps the (A, B) summand into the $(c(A), c(B))$ summand, where $c := f \amalg g \amalg h$) and an element $g \in \mathrm{GL}(V) \cong \Gamma$ acts diagonally (acting on $V^\vee$ by $(g^{-1})^\vee$).

Now note that the Γ-coinvariants of $(V, \mathrm{s}^m V^\vee)^{\otimes (A,B)}$ are trivial when $|A| \neq |B|$ since $2 \in \mathrm{GL}(V)$ acts via multiplication by $2^{|A|}/2^{|B|}$. Hence, the inclusion

$$\bigoplus_{(A,B) \in \mathrm{OM}(S,I,J)} (V, \mathrm{s}^m V^\vee)^{\otimes (A,B)} \hookrightarrow H^{\otimes S} \otimes V^{\otimes I} \otimes W^{\otimes J}$$

becomes an isomorphism after taking Γ-coinvariants.

Now let T be a finite linearly ordered set. Then there is, for any $(A, B) \in \mathrm{OM}(T)$, a canonical isomorphism of Γ-modules

$$\Psi_{A,B} \colon (V, \mathrm{s}^m V^\vee)^{\otimes (A,B)} \xrightarrow{\cong} (\mathrm{s}^m V^\vee \otimes V)^{\otimes A}$$

given by the following permutation: we consider the tensor product on the right-hand side to be indexed over the set $A \times \{0, 1\}$ equipped with the lexicographical order and permute according to the bijection $\psi_{A,B} \colon T \to A \times \{0, 1\}$ that sends $a \in A$ to $(a, 1)$ and $b \in B$ to $((\mu_{A,B})^{-1}(b), 0)$.

The isomorphisms $\mathrm{sgn}_{A,B}(\psi_{A,B}) \cdot \Psi_{A,B}$ assemble into an isomorphism of $(\Sigma_T \times \Gamma)$-modules

$$\Psi \colon \bigoplus_{(A,B) \in \mathrm{OM}(T)} (V, \mathrm{s}^m V^\vee)^{\otimes (A,B)} \xrightarrow{\cong} \bigoplus_{(A,B) \in \mathrm{OM}(T)} (\mathrm{s}^m V^\vee \otimes V)^{\otimes A},$$

where an element $c \in \Sigma_T$ acts on the right-hand side by sending the factor corresponding to (A, B) into the factor corresponding to $(c(A), c(B))$ via

$$c_*^{A,B}\Big(\bigotimes_{a \in A} (\mathrm{s}^m \phi_a \otimes v_a)\Big) = \mathrm{sgn}_{A,B}(c) \cdot \Big(\bigotimes_{a \in A} (\mathrm{s}^m \phi_{\tau^{-1}(a)} \otimes v_{\sigma^{-1}(a)})\Big),$$

where σ and τ are as in the statement of the lemma. Said differently, we have

$$c_*^{A,B} = \mathrm{sgn}_{A,B}(\psi_{A,B}) \, \mathrm{sgn}_{c(A),c(B)}(\psi_{c(A),c(B)}) \cdot (\Psi_{c(A),c(B)} \circ c_* \circ \Psi_{A,B}^{-1}),$$

which maybe makes it clearer why Ψ is compatible with the Σ_T-action.

Now consider the case that $T = S \amalg I \amalg J$. Then it is clear that the isomorphism Ψ restricts to an isomorphism

$$\bigoplus_{(A,B) \in \mathrm{OM}(S,I,J)} (V, \mathrm{s}^m V^\vee)^{\otimes (A,B)} \xrightarrow{\cong} \bigoplus_{(A,B) \in \mathrm{OM}(S,I,J)} (\mathrm{s}^m V^\vee \otimes V)^{\otimes A}$$

of $(\Sigma_S \times \Sigma_I \times \Sigma_J \times \Gamma)$-modules.

Lastly, note that there is a Γ-module isomorphism

$$\begin{aligned} \Phi \colon \mathrm{s}^m V^\vee \otimes V &\longrightarrow \mathrm{s}^m \mathrm{End}(V) \\ \mathrm{s}^m \phi \otimes v &\longmapsto \mathrm{s}^m (v' \mapsto \phi(v') v), \end{aligned}$$

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where $\Gamma \cong \mathrm{GL}(V)$ acts on $\mathrm{End}(V)$ by conjugation. Together with the canonical isomorphism $(\mathrm{s}^m\mathrm{End}(V))^{\otimes A} \cong \mathrm{s}^{m|A|}(\mathrm{End}(V))^{\otimes A}$ these isomorphisms combine into an isomorphism

$$\Theta_1 \colon \bigoplus_{(A,B)\in \mathrm{OM}(S,I,J)} \mathrm{s}^{m|A|}\Big(\mathrm{End}(V)^{\otimes A}\Big)_\Gamma \xrightarrow{\cong} \Big(H^{\otimes S}\otimes V^{\otimes I}\otimes W^{\otimes J}\Big)_\Gamma$$

as desired. For the claims made about the induced $(\Sigma_S \times \Sigma_I \times \Sigma_J)$-module structure on the left-hand side and the explicit description of Θ_1, note that, when $e := (e_1, \dots, e_k)$ is a basis of V, then $\Phi(\mathrm{s}^m e_j^\vee \otimes e_i) = \mathrm{s}^m \mathrm{E}_{i,j}$, where $\mathrm{E}_{i,j}$ is with respect to e. □

We have now reduced finding a description of $\big(H^{\otimes S}\otimes V^{\otimes I}\otimes W^{\otimes J}\big)_\Gamma$ independent of H to finding a description of $\Big(\mathrm{End}(V)^{\otimes A}\Big)_\Gamma$ that is independent of V. Such a description is provided by the classical coinvariant theory for the general linear group.

Proposition 3.8 (Fundamental theorem of coinvariant theory). *Let $k \in \mathbb{N}_0$ such that $k \le \dim V$. Then there is an isomorphism of vector spaces*

$$\Big(\mathrm{End}(V)^{\otimes k}\Big)_\Gamma \xrightarrow{\cong} \mathbb{Q}[\Sigma_k]$$

such that $[\mathrm{E}_{\omega(1),\sigma(1)}\otimes\ldots\otimes\mathrm{E}_{\omega(k),\sigma(k)}] \mapsto \omega^{-1}\sigma$ for all $\omega, \sigma \in \Sigma_k$ and any basis of V.

Proof. This can, for example, be found in [Lod98, 9.2.5, 9.2.8, and 9.2.10].[2] □

Applying this to our situation, we obtain the following corollary.

Corollary 3.9. *Let S, I and J be finite linearly ordered sets such that $|S| + |I| + |J| \le 2(\dim V)$. Then there is an isomorphism of $(\Sigma_S \times \Sigma_I \times \Sigma_J)$-modules*

$$\Theta_2 \colon \bigoplus_{(A,B)\in \mathrm{OM}(S,I,J)} \mathrm{s}^{m|A|}\Big(\mathrm{End}(V)^{\otimes A}\Big)_\Gamma \xrightarrow{\cong} \bigoplus_{(A,B)\in \mathrm{OM}(S,I,J)} \mathrm{s}^{m|A|}\mathbb{Q}[\Sigma_A],$$

where $\Sigma_S \times \Sigma_I \times \Sigma_J$ acts on the left-hand side as in Lemma 3.7 and on the right-hand side as follows: an element (f, g, h) maps the factor corresponding to $(A, B) \in \mathrm{OM}(S, I, J)$ into the factor corresponding to $(c(A), c(B))$ via the map

$$\begin{aligned} \mathrm{s}^{m|A|}\mathbb{Q}[\Sigma_A] &\longrightarrow \mathrm{s}^{m|A|}\mathbb{Q}[\Sigma_{c(A)}] \\ \mathrm{s}^{m|A|}\omega &\longmapsto \mathrm{s}^{m|A|}(\mathrm{sgn}_{A,B}(c)\cdot\sigma\omega\tau^{-1}), \end{aligned}$$

where $c := f \amalg g \amalg h$, and σ and τ are as in Lemma 3.7.

Moreover, Θ_2 can be chosen such that, for any basis of V and $\omega \in \Sigma_A$, it fulfills, on the factor corresponding to $(A, B) \in \mathrm{OM}(S, I, J)$, that

$$\Theta_2\Big(\mathrm{s}^{m|A|}[\mathrm{E}_{1,\omega(1)}\otimes\ldots\otimes\mathrm{E}_{a,\omega(a)}]\Big) = \mathrm{s}^{m|A|}\omega,$$

where we identify A with $\{1, \dots, a := |A|\}$ via the unique order-preserving bijection.

Proof. On each summand, Θ_2 is given by a shift of the isomorphism in Proposition 3.8. Now let $(A, B) \in \mathrm{OM}(S, I, J)$ and $\omega \in \Sigma_A$, and fix a basis of V. We will also use the notation from Lemma 3.7. First note that by Proposition 3.8, the elements $[\mathrm{E}_{1,\omega(1)}\otimes\ldots\otimes\mathrm{E}_{a,\omega(a)}]$ of $\big(\mathrm{End}(V)^{\otimes A}\big)_\Gamma$, with ω running over

[2] Note that in the proof of 9.2.10, there is a small mistake, leading to the result being stated as $\sigma\omega^{-1}$ instead of the correct $\omega^{-1}\sigma$.

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Σ_A, form a basis. Hence, it is enough to check the compatibility of Θ_2 with the $(\Sigma_S \times \Sigma_I \times \Sigma_J)$-action on those elements. To this end, we note that $\Theta_2\big(\mathrm{s}^{m|A|}[\mathrm{E}_{1,\omega(1)} \otimes \ldots \otimes \mathrm{E}_{a,\omega(a)}]\big) = \mathrm{s}^{m|A|}\omega$ and

$$\begin{aligned}\Theta_2\Big(&\mathrm{sgn}_{A,B}(c) \cdot \mathrm{s}^{m|A|}[\mathrm{E}_{\sigma^{-1}(1),\omega(\tau^{-1}(1))} \otimes \ldots \otimes \mathrm{E}_{\sigma^{-1}(a),\omega(\tau^{-1}(a))}]\Big)\\ &= \mathrm{s}^{m|A|}(\mathrm{sgn}_{A,B}(c) \cdot \sigma\omega\tau^{-1}),\end{aligned}$$

which is what we wanted to show. □

Remark 3.10. Setting $T = S \amalg I \amalg J$, we note that Θ_2 restricts to an isomorphism of $(\Sigma_S \times \Sigma_I \times \Sigma_J)$-modules on the summands indexed by $\mathrm{OM}(S, I, J) \subseteq \mathrm{OM}(T)$.

We have now almost achieved what we set out do to in this subsection, as we have obtained, when $|S| + |I| + |J| \leq 2(\dim V)$, an isomorphism

$$\Big(H^{\otimes S} \otimes V^{\otimes I} \otimes W^{\otimes J}\Big)_\Gamma \cong \bigoplus_{(A,B)\in \mathrm{OM}(S,I,J)} \mathrm{s}^{m|A|}\mathbb{Q}[\Sigma_A] \tag{3.1}$$

with the right-hand side independent of H as desired. To generalize this to expressions of the form $((P^{\mathrm{op}} \otimes_{\Sigma_S} H^{\otimes S}) \otimes V^{\otimes I} \otimes W^{\otimes J})_\Gamma$, we will use the abstract lemmas of Section 2.4. To state the result in its simplest form, we need to introduce a bit of notation.

Definition 3.11. Let I and J be finite linearly ordered sets. We denote by $\mathbf{Match}_{I,J}$ the following groupoid:

- Objects are tuples (S, A, B, μ) with S a finite linearly ordered set, $(A, B) \in \mathrm{OM}(S, I, J)$ an ordered matching, and $\mu\colon A \to B$ a bijection.
- A morphism $(S, A, B, \mu) \to (S', A', B', \mu')$ is a tuple (f, g, h) of bijections $f\colon S \to S'$, $g \in \Sigma_I$, and $h \in \Sigma_J$, such that the map $c \coloneqq f \amalg g \amalg h$ fulfills $c(A) = A'$, $c(B) = B'$, and $c|_B \circ \mu = \mu' \circ c|_A$.

Moreover, for $S \in \mathbf{\Sigma}$, we denote by $\mathbf{Match}^S_{I,J} \subseteq \mathbf{Match}_{I,J}$ the full subgroupoid spanned by objects of the form (S', A', B', μ') with $S' = S$. Also note that there is a canonical functor $\mathrm{pr}_{I,J}\colon \mathbf{Match}_{I,J} \to \Sigma_I \times \Sigma_J$ given on morphisms by sending (f, g, h) to (g, h). We denote by $\mathbf{Match}_{\mathrm{id}_I,\mathrm{id}_J} \subseteq \mathbf{Match}_{I,J}$ the wide subgroupoid of those morphisms that are sent to $(\mathrm{id}_I, \mathrm{id}_J)$ by $\mathrm{pr}_{I,J}$, and analogously for $\mathbf{Match}^S_{\mathrm{id}_I,\mathrm{id}_J} \subseteq \mathbf{Match}^S_{I,J}$.

Definition 3.12. We write $\mathrm{sgn}_{n,m}$ for the $\mathbf{Match}_{I,J}$-module given by

$$(S, A, B, \mu) \longmapsto (\mathrm{s}^n\mathbb{Q}, \mathrm{s}^{m-n}\mathbb{Q})^{\otimes(A,B)}$$

(see Definition 3.3). Said differently, it is $\mathrm{s}^{m|A|}\mathbb{Q}$ on which a morphism (f, g, h) acts by $\mathrm{sgn}_{A,B}(c)$, where $c \coloneqq f \amalg g \amalg h$.

With only a little bit more work, the lemmas of Section 2.4 now specialize to the following. (See [BM20, Corollary 9.8] for a similar statement in the case where H is concentrated in a single degree and $I = J = \varnothing$.)

Proposition 3.13. *Let S, I and J be finite linearly ordered sets such that $|S|+|I|+|J| \leq 2(\dim V) = \dim H$, and let P be a Σ_S-module. Then there is an isomorphism of $(\Sigma_I \times \Sigma_J)$-modules*

$$\operatorname*{colim}_{\mathbf{Match}^S_{\mathrm{id}_I,\mathrm{id}_J}} (P \otimes \mathrm{sgn}_{n,m}) \xrightarrow{\cong} \Big((P^{\mathrm{op}} \otimes_{\Sigma_S} H^{\otimes S}) \otimes V^{\otimes I} \otimes W^{\otimes J}\Big)_\Gamma,$$

where a morphism $(f, \mathrm{id}_I, \mathrm{id}_J)$ of $\mathbf{Match}^S_{\mathrm{id}_I,\mathrm{id}_J}$ acts on P by $f \in \Sigma_S$, and Γ acts trivially on P and diagonally on the tensor product. The action of an element $(g, h) \in \Sigma_I \times \Sigma_J$ on the left-hand side is

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given by sending an element represented by $(A, B, \mu) \in \mathbf{Match}^S_{\mathrm{id}_I, \mathrm{id}_J}$ *and* $x \in P \otimes \mathrm{sgn}_{n,m}$ *to the element represented by* $(A, B, c|_B \circ \mu \circ (c|_A)^{-1})$ *and* $\mathrm{sgn}_{A,B}(c)x$, *where* $c \coloneqq \mathrm{id}_S \amalg g \amalg h$.

Moreover, the isomorphism above can be chosen such that, for any basis e of V, it maps the element of the colimit represented by $(A, B, \mu) \in \mathbf{Match}^S_{\mathrm{id}_I, \mathrm{id}_J}$ *and* $p \otimes \mathrm{s}^{m|A|}1 \in P \otimes \mathrm{sgn}_{n,m}$ *to* $[p \otimes \bigotimes_{t \in S \amalg I \amalg J} h_t]$ *where*

$$h_t \coloneqq \begin{cases} e_{a^{-1}(t)} \in V, & \text{if } t \in A \\ \mathrm{s}^m e^\vee_{a^{-1}(\mu^{-1}(t))} \in \mathrm{s}^m V^\vee, & \text{if } t \in B \end{cases}$$

(here $a\colon \underline{|A|} \to A$ *is the unique order-preserving bijection).*

Proof. We start by rewriting the expression yielded by Corollary 3.9 and Remark 3.10. We have isomorphisms of $(\Sigma_S \times \Sigma_I \times \Sigma_J)$-modules

$$\begin{aligned} \bigoplus_{(A,B) \in \mathrm{OM}(S,I,J)} \mathrm{s}^{m|A|}\mathbb{Q}[\Sigma_A] &\cong \bigoplus_{(A,B) \in \mathrm{OM}(S,I,J)} \mathrm{s}^{m|A|}\mathbb{Q}\langle Bij(A,B)\rangle \\ &\cong \bigoplus_{(A,B,\mu) \in \mathrm{OM}_\mu(S,I,J)} \mathrm{s}^{m|A|}\mathbb{Q}\langle\mu\rangle, \end{aligned}$$

where, for the first isomorphism, we identify Σ_A with the set of (not necessarily order preserving) bijections $Bij(A, B)$ via the map $\omega \mapsto \mu_{A,B} \circ \omega^{-1}$. For the second isomorphism, we set $\mathrm{OM}_\mu(S, I, J) \coloneqq \mathrm{Ob}(\mathbf{Match}^S_{I,J})$. An element $(f, g, h) \in \Sigma_S \times \Sigma_I \times \Sigma_J$ acts on $\mathrm{OM}_\mu(S, I, J)$ via

$$(A, B, \mu) \longmapsto \Big(c(A), c(B), c \,.\, \mu \coloneqq c|_B \circ \mu \circ (c|_A)^{-1}\Big),$$

where $c \coloneqq f \amalg g \amalg h$, and on $\mathbb{Q}\langle\mu\rangle$ by sending μ to $\mathrm{sgn}_{A,B}(c) \cdot (c \,.\, \mu)$.

Now we have isomorphisms of $(\Sigma_I \times \Sigma_J)$-modules

$$\begin{aligned} \Big((P^{\mathrm{op}} \otimes_{\Sigma_S} H^{\otimes S}) \otimes V^{\otimes I} \otimes W^{\otimes J}\Big)_\Gamma &= \Big((P \otimes H^{\otimes S})_{\Sigma_S} \otimes V^{\otimes I} \otimes W^{\otimes J}\Big)_\Gamma \\ &\cong \Big((P \otimes H^{\otimes S} \otimes V^{\otimes I} \otimes W^{\otimes J})_\Gamma\Big)_{\Sigma_S} \\ &\cong \Big(P \otimes (H^{\otimes S} \otimes V^{\otimes I} \otimes W^{\otimes J})_\Gamma\Big)_{\Sigma_S} \end{aligned}$$

since the actions of $\Sigma_S \times \Sigma_I \times \Sigma_J$ and Γ commute and since Γ acts trivially on P. By Lemma 3.7, Corollary 3.9 and Remark 3.10, as well as the preceding discussion, the rightmost expression above is isomorphic to

$$\Big(P \otimes \bigoplus_{(A,B) \in \mathrm{OM}(S,I,J)} \mathrm{s}^{m|A|}\mathbb{Q}[\Sigma_A]\Big)_{\Sigma_S} \cong \Big(\bigoplus_{(A,B,\mu) \in \mathrm{OM}_\mu(S,I,J)} \Big(P \otimes \mathrm{s}^{m|A|}\mathbb{Q}\langle\mu\rangle\Big)\Big)_{\Sigma_S} \tag{3.2}$$

as $(\Sigma_I \times \Sigma_J)$-modules. To describe the action on the sum on the right-hand side, let $(f, g, h) \in \Sigma_S \times \Sigma_I \times \Sigma_J$ and set $c \coloneqq f \amalg g \amalg h$. Then it sends the summand corresponding to (A, B, μ) to the summand corresponding to $(c(A), c(B), c \,.\, \mu)$ by acting on P via f and on $\mathbb{Q}\langle\mu\rangle$ as described above. The quotient is taken with respect to the action of $\Sigma_S \cong \Sigma_S \times \{\mathrm{id}_I\} \times \{\mathrm{id}_J\}$.

Now we can apply Lemma 2.16. Noting that there is a canonical isomorphism

$$\int_{\Sigma_S \times \Sigma_I \times \Sigma_J} \mathrm{OM}_\mu(S, I, J) \cong \mathbf{Match}^S_{I,J}$$

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of categories over $\Sigma_I \times \Sigma_J$, we obtain an isomorphism of the right-hand side of (3.2) with

$$\mathrm{Lan}_{\mathbf{Match}^S_{I,J} \to \Sigma_I \times \Sigma_J}(P \otimes \mathrm{sgn}_{n,m})$$

as $(\Sigma_I \times \Sigma_J)$-modules, where a morphism (f, g, h) of $\mathbf{Match}^S_{I,J}$ acts on P by $f \in \Sigma_S$. To this we can, in turn, apply Lemma 2.18 to obtain the desired description in terms of a colimit: given an object $(A, B, \mu) \in \mathbf{Match}^S_{I,J}$ and an element $(g, h) \in \Sigma_I \times \Sigma_J$, we pick the lift $(\mathrm{id}_S, g, h)\colon (A, B, \mu) \to (A, B, c \,.\, \mu)$, where $c \coloneqq \mathrm{id}_S \amalg g \amalg h$.

The explicit description of the isomorphism is obtained by chasing through the explicit descriptions of its constituent parts. □

3.2. Coinvariants of Schur functors

In this subsection, we want to compute

$$\left(\mathscr{P}[H] \otimes V^{\otimes I} \otimes W^{\otimes J}\right)_\Gamma$$

as a $(\Sigma_I \times \Sigma_J)$-module, where $\mathscr{P}$ is some $\mathbf{\Sigma}$-module, and H, V, W and Γ are as in the preceding subsection. Noting that $\mathscr{P}[H]$ is a sum of monomial functors in H, Proposition 3.13 yields an identification as desired in a range depending on the dimension of H. To obtain a description of the whole thing, we thus need to let the dimension of H go to infinity.

More precisely, we fix the following notation for the rest of this section:

$$H_1 \to H_2 \to H_3 \to \dots$$

is a sequence of maps of $\mathbf{Sp}^m$ (see Notation 2.6) such that H_i is concentrated in dimensions n and $m - n$ and such that the sequence $\dim H_i$ diverges. Moreover, we write $\Gamma_i \coloneqq \mathrm{Aut}_{\mathbf{Sp}^m}(H_i)$ and denote by V_i and W_i the degree n and $m - n$ parts of H_i, respectively.

Note that there are maps $\Gamma_i \to \Gamma_{i+1}$ given by extending an automorphism of H_i by the identity on the symplectic complement of H_i in H_{i+1}. Moreover, the maps $H_i \to H_{i+1}$ become Γ_i-equivariant when H_{i+1} is equipped with the restricted action. In particular, for any functor F from $\mathbf{Sp}^m$ to some cocomplete category $\mathcal{C}$, we obtain a sequence $F(H_1)_{\Gamma_1} \to F(H_2)_{\Gamma_2} \to \dots$ and thus can make sense of the expression $\mathrm{colim}_{i \in \mathbb{N}} F(H_i)_{\Gamma_i}$.

Definition 3.14. A *compatible basis* of the V_i is a sequence $e_1, e_2, \dots$ of elements of $\mathrm{colim}_i V_i$ such that, for each $i \geq 1$, each of the elements $e_1, \dots, e_{\dim V_i}$ is represented by an (automatically unique) element of V_i, and the set of these representatives forms a basis of V_i.

Proposition 3.15. *Let I and J be finite linearly ordered sets and $\mathscr{P}$ a $\mathbf{\Sigma}$-module. Then there is an isomorphism of $(\Sigma_I \times \Sigma_J)$-modules*

$$\operatorname*{colim}_{(S,A,B,\mu) \in \mathbf{Match}_{\mathrm{id}_I, \mathrm{id}_J}} \left(\mathscr{P}(S) \otimes \mathrm{sgn}_{n,m}\right) \xrightarrow{\cong} \operatorname*{colim}_{i \in \mathbb{N}} \left(\mathscr{P}[H_i] \otimes V_i^{\otimes I} \otimes W_i^{\otimes J}\right)_{\Gamma_i},$$

where a morphism $(f, \mathrm{id}_I, \mathrm{id}_J)$ of $\mathbf{Match}_{\mathrm{id}_I, \mathrm{id}_J}$ acts on P by $f \in \Sigma_S$. The action of an element $(g, h) \in \Sigma_I \times \Sigma_J$ on the left-hand side is given by sending an element represented by $(S, A, B, \mu) \in \mathbf{Match}_{\mathrm{id}_I, \mathrm{id}_J}$ and $x \in \mathscr{P}(S) \otimes \mathrm{sgn}_{n,m}$ to the element represented by $(S, A, B, c|_B \circ \mu \circ (c|_A)^{-1})$ and $\mathrm{sgn}_{A,B}(c)x$, where $c \coloneqq \mathrm{id}_S \amalg g \amalg h$.

Moreover, this isomorphism can be chosen such that, for any compatible basis $(e_j)_{j \in \mathbb{N}_{\geq 1}}$ of the V_i, it maps an element represented by $(S, A, B, \mu) \in \mathbf{Match}_{\mathrm{id}_I, \mathrm{id}_J}$ and $p \otimes \mathrm{s}^{m|A|}1 \in \mathscr{P}(S) \otimes \mathrm{sgn}_{n,m}$ to the

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element $[p \otimes \bigotimes_{t \in S \amalg I \amalg J} h_t]$ *of*

$$\operatorname*{colim}_{i \in \mathbb{N}} \left((\mathscr{P}(S)^{\mathrm{op}} \otimes_{\Sigma_S} H_i^{\otimes S}) \otimes V_i^{\otimes I} \otimes W_i^{\otimes J} \right)_{\Gamma_i} \subseteq \operatorname*{colim}_{i \in \mathbb{N}} \left(\mathscr{P}[H_i] \otimes V_i^{\otimes I} \otimes W_i^{\otimes J} \right)_{\Gamma_i},$$

where h_t *is as in Proposition 3.13 (and we use the definition of* $\mathscr{P}[H_i]$ *of Remark 2.25).*

Proof. We have

$$\left(\mathscr{P}[H_i] \otimes V_i^{\otimes I} \otimes W_i^{\otimes J} \right)_{\Gamma_i} \cong \bigoplus_{k \in \mathbb{N}_0} \left((\mathscr{P}(\underline{k})^{\mathrm{op}} \otimes_{\Sigma_{\underline{k}}} H_i^{\otimes \underline{k}}) \otimes V_i^{\otimes I} \otimes W_i^{\otimes J} \right)_{\Gamma_i}$$

by definition of $\mathscr{P}[-]$. When $k \leq \dim H_i - |I| - |J|$, there is an isomorphism

$$\operatorname*{colim}_{\mathbf{Match}^{\underline{k}}_{\mathrm{id}_I, \mathrm{id}_J}} \left(\mathscr{P}(\underline{k}) \otimes \mathrm{sgn}_{n,m} \right) \xrightarrow{\cong} \left((\mathscr{P}(\underline{k})^{\mathrm{op}} \otimes_{\Sigma_{\underline{k}}} H_i^{\otimes \underline{k}}) \otimes V_i^{\otimes I} \otimes W_i^{\otimes J} \right)_{\Gamma_i} \tag{3.3}$$

provided by Proposition 3.13. It is compatible with the maps $H_i \to H_{i+1}$ (i.e., it is a natural isomorphism of sequential diagrams, where the left-hand side is considered as a constant functor). To see this, we chose a basis of V_i (the degree n part of H_i), extend it to a basis of V_{i+1}, and use the explicit description of the isomorphism (3.3). Hence we have, for each $k \in \mathbb{N}_0$, an isomorphism

$$\operatorname*{colim}_{\mathbf{Match}^{\underline{k}}_{\mathrm{id}_I, \mathrm{id}_J}} \left(\mathscr{P}(\underline{k}) \otimes \mathrm{sgn}_{n,m} \right) \xrightarrow{\cong} \operatorname*{colim}_{i \in \mathbb{N}} \left((\mathscr{P}(\underline{k})^{\mathrm{op}} \otimes_{\Sigma_{\underline{k}}} H_i^{\otimes \underline{k}}) \otimes V_i^{\otimes I} \otimes W_i^{\otimes J} \right)_{\Gamma_i},$$

which, by setting $\mathbf{Match}^{\mathbf{P}}_{\mathrm{id}_I, \mathrm{id}_J} \coloneqq \coprod_{k \in \mathbb{N}_0} \mathbf{Match}^{\underline{k}}_{\mathrm{id}_I, \mathrm{id}_J} \subset \mathbf{Match}_{\mathrm{id}_I, \mathrm{id}_J}$, assemble into an isomorphism

$$\begin{aligned} \operatorname*{colim}_{\mathbf{Match}^{\mathbf{P}}_{\mathrm{id}_I, \mathrm{id}_J}} \left(\mathscr{P} \otimes \mathrm{sgn}_{n,m} \right) &\cong \bigoplus_{k \in \mathbb{N}_0} \operatorname*{colim}_{\mathbf{Match}^{\underline{k}}_{\mathrm{id}_I, \mathrm{id}_J}} \left(\mathscr{P}(\underline{k}) \otimes \mathrm{sgn}_{n,m} \right) \\ &\xrightarrow{\cong} \operatorname*{colim}_{i \in \mathbb{N}} \left(\mathscr{P}[H_i] \otimes V_i^{\otimes I} \otimes W_i^{\otimes J} \right)_{\Gamma_i} \end{aligned}$$

since colimits commute with direct sums. Since the inclusion $\mathbf{Match}^{\mathbf{P}}_{\mathrm{id}_I, \mathrm{id}_J} \to \mathbf{Match}_{\mathrm{id}_I, \mathrm{id}_J}$ is an equivalence of categories, it induces a canonical isomorphism

$$\operatorname*{colim}_{\mathbf{Match}^{\mathbf{P}}_{\mathrm{id}_I, \mathrm{id}_J}} \left(\mathscr{P} \otimes \mathrm{sgn}_{n,m} \right) \xrightarrow{\cong} \operatorname*{colim}_{\mathbf{Match}_{\mathrm{id}_I, \mathrm{id}_J}} \left(\mathscr{P} \otimes \mathrm{sgn}_{n,m} \right),$$

which finishes the construction.

We will now prove the claimed explicit description of the composed isomorphism. To this end, consider the element represented by $(S, A, B, \mu) \in \mathbf{Match}_{\mathrm{id}_I, \mathrm{id}_J}$ and $p \otimes \mathrm{s}^{m|A|} 1 \in \mathscr{P}(S) \otimes \mathrm{sgn}_{n,m}$. It is equal, for any choice of bijection $f \colon S \to \underline{|S|}$, to the element represented by $\left(\underline{|S|}, c(A), c(B), c|_B \circ \mu \circ (c|_A)^{-1} \right)$ and $\mathrm{sgn}_{A,B}(c) \cdot \left(f_*(p) \otimes \mathrm{s}^{m|A|} 1 \right)$, where $c \coloneqq f \amalg \mathrm{id}_I \amalg \mathrm{id}_J$. On such an element, the isomorphism is, by construction, given by the isomorphism of Proposition 3.13. The representative one obtains using the explicit description of that isomorphism represents the same element as the representative we provided in the statement of this proposition (by translating it back to S via f^{-1}). In the same way, one proves that the action of $\Sigma_I \times \Sigma_J$ is as claimed. □

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3.3. Coinvariants of double Schur functors

In this subsection, our goal is to compute

$$\operatorname*{colim}_{i\in\mathbb{N}} \left(\mathscr{P}[\mathscr{Q}[H_i]] \otimes V_i^{\otimes I} \otimes W_i^{\otimes J}\right)_{\Gamma_i}$$

as a $(\Sigma_I \times \Sigma_J)$-module, where $\mathscr{P}$ and $\mathscr{Q}$ are $\mathbf{\Sigma}$-modules, and H_i, V_i, W_i and Γ_i are as in the preceding subsection. By Lemma 2.26, we have an isomorphism $\mathscr{P}[\mathscr{Q}[H_i]] \cong (\mathscr{P} \circ \mathscr{Q})[H_i]$. Hence, we can use Proposition 3.15 to obtain a description as desired. We now introduce some notation to state the result in a nice way.

Definition 3.16. Let I and J be finite linearly ordered sets. We denote by $\mathbf{DirGraph}_{\mathrm{id}_I,\mathrm{id}_J}$ the following groupoid:

- Objects are *directed* (I,J)*-graphs* – that is, tuples (F,N,S,T,μ,a) with F and N finite linearly ordered sets, $(S,T) \in \mathrm{OM}(F,I,J)$ an ordered matching, $\mu\colon S \to T$ a bijection, and $a\colon F \to N$ a map of sets.
- A morphism from (F,N,S,T,μ,a) to (F',N',S',T',μ',a') is a tuple of bijections $(f\colon F \to F',\ k\colon N \to N')$ such that $c(S) = S'$, $c(T) = T'$, $c|_T \circ \mu = \mu' \circ c|_S$, and $k \circ a = a' \circ f$, where we set $c \coloneqq f \amalg \mathrm{id}_I \amalg \mathrm{id}_J$.

When $I = J = \varnothing$, we just write **DirGraph** and call its objects *directed graphs*.

Remark 3.17. We think of a directed (I,J)-graph as a directed graph with I inputs and J outputs. Under this viewpoint, we have the following: N is the set of *vertices* (or nodes), F is the set of *internal flags* (or half-edges), a describes to which vertex a flag is attached, S is the set of those flags which are *sources* (i.e., outgoing) which includes the inputs, similarly T is the set of those flags which are *targets* (i.e., incoming) which includes the outputs, and μ describes the (directed) edges (i.e., $s \in S$ and $\mu(s) \in T$ are connected and form an edge).

Lemma 3.18. *There is an isomorphism from* $\mathrm{sgn}_{n,m}$ *to the* $\mathbf{Match}_{I,J}$*-module given by*

$$(S,A,B,\mu) \longmapsto (\mathrm{s}^m\mathbb{Q})^{\otimes A},$$

where a morphism $(f,g,h)\colon (S,A,B,\mu) \to (S',A',B',\mu')$ *of* $\mathbf{Match}_{I,J}$ *acts by permuting the factors according to* $c|_A\colon A \to A'$*, where* $c \coloneqq f \amalg g \amalg h$*. This isomorphism maps* $\mathrm{s}^{m|A|}1 \in \mathrm{sgn}_{n,m}$ *to* $\varepsilon \cdot \mathrm{s}^{m|A|}1 \in (\mathrm{s}^m\mathbb{Q})^{\otimes A}$*, where* ε *is the sign incurred by permuting*

$$\bigotimes_{t\in S\amalg I\amalg J} h_t \quad \text{into} \quad \bigotimes_{a\in A}(h_a \otimes h_{\mu(a)})$$

where $h_t \coloneqq \mathrm{s}^n 1$ *if* $t \in A$ *and* $h_t \coloneqq \mathrm{s}^{m-n}1$ *if* $t \in B$.

Proof. Let $p,q\colon \mathbf{Match}_{I,J} \to \mathbf{Match}_{\varnothing,\varnothing}$ be the canonical functors given, on objects, by

$$p(S,A,B,\mu) = (S \amalg I \amalg J, A, B, \mu)$$
$$q(S,A,B,\mu) = (T, A, B, \mu),$$

where T is equal to $S \amalg I \amalg J$ as a set but is equipped with the linear order uniquely determined by the following two properties:

- The element $a \in A$ is the predecessor of $\mu(a) \in B$.
- Its restriction to A agrees with the linear order on A obtained by restricting the one of $S \amalg I \amalg J$.

The identity of the underlying sets determines a natural isomorphism $p \to q$. The map in the statement is obtained by applying $\mathrm{sgn}_{n,m}$ to this natural transformation. □

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The following proposition contains the desired description of the $(\Sigma_I \times \Sigma_J)$-module $\operatorname{colim}_{i\in\mathbb{N}} (\mathscr{P}[\mathscr{Q}[H_i]] \otimes V_i^{\otimes I} \otimes W_i^{\otimes J})_{\Gamma_i}$. (See [BM20, Theorem 9.12] for a similar statement in the case where H is concentrated in a single degree and $I = J = \varnothing$.)

Proposition 3.19. *Let I and J be finite linearly ordered sets and let $\mathscr{P}$ and $\mathscr{Q}$ be $\mathbf{\Sigma}$-modules. Then there is an isomorphism of $(\Sigma_I \times \Sigma_J)$-modules*

$$\operatorname*{colim}_{(F,N,S,T,\mu,a)\in \mathbf{DirGraph}_{\mathrm{id}_I,\mathrm{id}_J}} \left(\mathscr{P}(N) \otimes \mathscr{Q}(a) \otimes (\mathrm{s}^m\mathbb{Q})^{\otimes S}\right) \xrightarrow{\cong} \operatorname*{colim}_{i\in\mathbb{N}} \left(\mathscr{P}[\mathscr{Q}[H_i]] \otimes V_i^{\otimes I} \otimes W_i^{\otimes J}\right)_{\Gamma_i},$$

where a morphism (f,k) of $\mathbf{DirGraph}_{\mathrm{id}_I,\mathrm{id}_J}$ acts on $\mathscr{P}(N)$ by k, on $\mathscr{Q}(a)$ by (f,k) considered as a morphism of $\mathbf{\Sigma}\downarrow\mathbf{\Sigma}$, and on $(\mathrm{s}^m\mathbb{Q})^{\otimes S}$ by permuting the factors according to $c|_S$ where $c \coloneqq f \amalg \mathrm{id}_I \amalg \mathrm{id}_J$. The action of an element $(g,h) \in \Sigma_I \times \Sigma_J$ on the left-hand side is given by sending an element represented by $(F,N,S,T,\mu,a) \in \mathbf{DirGraph}_{\mathrm{id}_I,\mathrm{id}_J}$ and $x \in \mathscr{P}(N) \otimes \mathscr{Q}(a) \otimes (\mathrm{s}^m\mathbb{Q})^{\otimes S}$ to the element represented by $(F,N,S,T,c|_T \circ \mu \circ (c|_S)^{-1}, a)$ and $\operatorname{sgn}(g)^m x$, where $c \coloneqq \mathrm{id}_F \amalg g \amalg h$.

Moreover, this isomorphism can be chosen such that, for any compatible basis $(e_j)_{j\in\mathbb{N}_{\geq 1}}$ of the V_i, it maps an element represented by $(F,N,S,T,\mu,a) \in \mathbf{DirGraph}_{\mathrm{id}_I,\mathrm{id}_J}$ and

$$p \otimes \textstyle\bigotimes_{v\in N} q_v \otimes \mathrm{s}^{m|S|}1 \in \mathscr{P}(N) \otimes \mathscr{Q}(a) \otimes (\mathrm{s}^m\mathbb{Q})^{\otimes S}$$

to the element

$$\varepsilon \cdot [p \otimes \textstyle\bigotimes_{v\in N}\left(q_v \otimes \bigotimes_{f\in a^{-1}(v)} h_f\right) \otimes \bigotimes_{f\in I\amalg J} h_f] \in \operatorname*{colim}_{i\in\mathbb{N}} \left(\mathscr{P}[\mathscr{Q}[H_i]] \otimes V_i^{\otimes I} \otimes W_i^{\otimes J}\right)_{\Gamma_i},$$

where

$$h_f \coloneqq \begin{cases} e_{s^{-1}(f)} \in V, & \text{if } f \in S \\ \mathrm{s}^m e^\vee_{s^{-1}(\mu^{-1}(f))} \in \mathrm{s}^m V^\vee, & \text{if } f \in T \end{cases}$$

and $s\colon \underline{|S|} \to S$ is the unique order-preserving bijection. The sign ε is the sign incurred by permuting

$$\textstyle\bigotimes_{v\in N} q_v \otimes \bigotimes_{s\in S}(h_s \otimes h_{\mu(s)}) \quad \text{into} \quad \bigotimes_{v\in N}\left(q_v \otimes \bigotimes_{f\in a^{-1}(v)} h_f\right) \otimes \bigotimes_{f\in I\amalg J} h_f$$

according to the Koszul sign rule.

Proof. We have isomorphisms

$$\begin{aligned}\operatorname*{colim}_{(F,S,T,\mu)\in\mathbf{Match}_{\mathrm{id}_I,\mathrm{id}_J}} \left((\mathscr{P}\circ\mathscr{Q})(F) \otimes (\mathrm{s}^m\mathbb{Q})^{\otimes S}\right) &\cong \operatorname*{colim}_{i\in\mathbb{N}} \left((\mathscr{P}\circ\mathscr{Q})[H_i] \otimes V_i^{\otimes I} \otimes W_i^{\otimes J}\right)_{\Gamma_i} \\ &\cong \operatorname*{colim}_{i\in\mathbb{N}} \left(\mathscr{P}[\mathscr{Q}[H_i]] \otimes V_i^{\otimes I} \otimes W_i^{\otimes J}\right)_{\Gamma_i}\end{aligned}$$

by Proposition 3.15 and Lemmas 3.18 and 2.26. Moreover, by definition,

$$(\mathscr{P}\circ\mathscr{Q})(F) = \operatorname*{colim}_{(a\colon F\to N)\in F\downarrow\mathbf{\Sigma}} (\mathscr{P}(N) \otimes \mathscr{Q}(a))$$

so that the leftmost colimit above is isomorphic to

$$\operatorname*{colim}_{(F,S,T,\mu)\in\mathbf{Match}_{\mathrm{id}_I,\mathrm{id}_J}} \operatorname*{colim}_{(a\colon F\to N)\in F\downarrow\mathbf{\Sigma}} \left(\mathscr{P}(N) \otimes \mathscr{Q}(a) \otimes (\mathrm{s}^m\mathbb{Q})^{\otimes S}\right),$$

and hence, it is enough to show that the Grothendieck construction of the covariant functor $U\colon \mathbf{Match}_{\mathrm{id}_I,\mathrm{id}_J} \to \mathbf{Cat}$ given by $(F,S,T,\mu) \mapsto F \downarrow \mathbf{\Sigma}$ (where a map acts by precomposition with its inverse) is canonically isomorphic to $\mathbf{DirGraph}_{\mathrm{id}_I,\mathrm{id}_J}$. For this, we define a functor

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$\int U \to \mathbf{DirGraph}_{\mathrm{id}_I, \mathrm{id}_J}$, given, on objects respectively morphisms, by

$$((F, S, T, \mu), a\colon F \to N) \longmapsto (F, N, S, T, \mu, a)$$
$$\Big((f\colon F \to F', \mathrm{id}_I, \mathrm{id}_J), k\colon a \circ f^{-1} \to a'\Big) \longmapsto (f, \mathrm{pr}(k)),$$

where $\mathrm{pr}\colon F \downarrow \mathbf{\Sigma} \to \mathbf{\Sigma}$ is the projection. This is clearly an isomorphism.

The explicit description of the $(\Sigma_I \times \Sigma_J)$-action and the explicit description of the isomorphism follow from the explicit descriptions in Proposition 3.15 and Lemmas 3.18 and 2.26. □

3.4. Coinvariants of CE chains of convolution Lie algebras

In this subsection, our goal is to compute the $(\Sigma_I \times \Sigma_J)$-module

$$\operatorname*{colim}_{i \in \mathbb{N}} \left(\mathrm{C}^{\mathrm{CE}}_*((\mathrm{s}^{-m}\mathscr{C})[H_i]) \otimes V_i^{\otimes I} \otimes W_i^{\otimes J}\right)_{\Gamma_i}, \tag{3.4}$$

where $\mathscr{C}$ is a cyclic operad, and H_i, V_i, W_i and Γ_i are as in the preceding subsections. (For ease of dealing with signs, we will only consider the case where $\mathscr{C}$ is concentrated in even degrees.) The cocommutative coalgebra structure on the Chevalley–Eilenberg chains induces a cocommutative coalgebra structure on (3.4) when $I = J = \varnothing$, and the structure of a comodule over this coalgebra for general I and J. We will also identify this coalgebra (resp. comodule) structure explicitly.

By Lemma 2.29, there is an isomorphism

$$\mathrm{C}^{\mathrm{CE}}_*((\mathrm{s}^{-m}\mathscr{C})[H_i]) \cong (\Lambda \mathrm{s})[(\mathrm{s}^{-m}\mathscr{C})[H_i]]$$

of graded vector spaces. Hence, we can apply Proposition 3.19 to obtain a description of the underlying graded vector space of (3.4). The rest of this subsection is concerned with also identifying its differential in these terms. We begin with some useful auxiliary definitions.

Definition 3.20. Let I and J be finite linearly ordered sets and (F, N, S, T, μ, a) a directed (I, J)-graph. For a vertex $v \in N$, we set

- $|v| \coloneqq |a^{-1}(v)|$ its *degree*.
- $|v|^{\mathrm{out}} \coloneqq |a^{-1}(v) \cap |$ its *out-degree*.
- $|v|^{\mathrm{in}} \coloneqq |a^{-1}(v) \cap T|$ its *in-degree*.
- $\mathrm{w}(v) \coloneqq n|v|^{\mathrm{out}} + (m-n)|v|^{\mathrm{in}}$ its *weight*.

Definition 3.21. Let I and J be finite linearly ordered sets, $G = (F, N, S, T, \mu, a)$ a directed (I, J)-graph, and $s \in F \cap S$ an internal outgoing flag. Moreover, we set $v \coloneqq a(s)$ and $v' \coloneqq a(\mu(s))$ and assume that $v \neq v'$. In this situation, we denote by $\mathrm{ctr}_s(G)$ the directed (I, J)-graph $(F', N', S', T', \mu', a')$ with

- $F' \coloneqq F \setminus \{s, \mu(s)\}$ and $N' \coloneqq N/(v \sim v')$,
- $S' \coloneqq S \setminus \{s\}$ and $T' \coloneqq T \setminus \{\mu(s)\}$,
- $\mu' \coloneqq \mu|_{S'}$ and $a' \coloneqq \mathrm{pr} \circ (a|_{F'})$, where $\mathrm{pr}\colon N \to N'$ is the quotient map.

Here, N' has the linear order such that $\mathrm{pr}(v) = \mathrm{pr}(v')$ is the smallest element and such that pr is order-preserving on all other elements. The set F' is equipped with the linear order determined as follows: the elements of $a^{-1}(\{v, v'\})$ are smaller than all other elements; these other elements have the order restricted from F; on $a^{-1}(\{v, v'\})$, the order is given by

$$a^{-1}(v)_{<s} < a^{-1}(v')_{>\mu(s)} < a^{-1}(v')_{<\mu(s)} < a^{-1}(v)_{>s},$$

where on each block the order is the one restricted from F. (This convention is chosen such that it interacts nicely with the composition operations of a cyclic operad of Remark 2.32.)

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Definition 3.22. Let I and J be finite linearly ordered sets and $G = (F, N, S, T, \mu, a)$ a directed (I, J)-graph. A *neighbor-closed vertex set* of G is a subset $N' \subseteq N$ such that $\mu(S \cap a^{-1}(N')) = T \cap a^{-1}(N')$. A *closed connected component* of G is a neighbor-closed vertex set that is inclusion minimal among nonempty neighbor-closed vertex sets.

A neighbor-closed vertex set N' of G determines a directed $(\varnothing, \varnothing)$-graph and a directed (I, J)-graph

$$\begin{aligned} \mathrm{ind}_G(N') &\coloneqq \left(F', N', S', T', \mu|_{S'}, a|_{F'}\right) \in \mathbf{DirGraph} \\ \mathrm{coind}_G(N') &\coloneqq \left(F \setminus F', N \setminus N', S \setminus S', T \setminus T', \mu|_{S\setminus S'}, a|_{F\setminus F'}\right) \in \mathbf{DirGraph}_{\mathrm{id}_I, \mathrm{id}_J}, \end{aligned}$$

where $F' \coloneqq a^{-1}(N')$, $S' \coloneqq S \cap a^{-1}(N')$, and $T' \coloneqq T \cap a^{-1}(N')$.

Remark 3.23. Any neighbor-closed vertex set of G is a disjoint union of a unique set of closed connected components of G. Moreover, when $I = J = \varnothing$ and $N' \subseteq N$ is a neighbor-closed vertex set of a directed graph $G = (F, N, S, T, \mu, a)$, then $N \setminus N'$ is one as well, and we have $\mathrm{ind}_G(N \setminus N') = \mathrm{coind}_G(N')$.

Definition 3.24. Let I and J be linearly ordered sets. For a $\mathbf{\Sigma}$-module $\mathscr{C}$, we define the $(\Sigma_I \times \Sigma_J)$-module of *$\mathscr{C}$-decorated directed (I, J)-graphs* to be

$$\mathfrak{DG}^m(\mathscr{C})_{I,J} \coloneqq \operatorname*{colim}_{\mathbf{DirGraph}_{\mathrm{id}_I, \mathrm{id}_J}} \mathrm{Dec}^m_{\mathscr{C};I,J}$$

(when $I = J = \varnothing$, then we omit them from the notation). Here, $\mathrm{Dec}^m_{\mathscr{C};I,J}$ is the functor $\mathbf{DirGraph}_{\mathrm{id}_I, \mathrm{id}_J} \to \mathbf{GrVect}$ given, on a directed (I, J)-graph $G \coloneqq (F, N, S, T, \mu, a)$, by

$$\mathrm{Dec}^m_{\mathscr{C};I,J}(G) \coloneqq (\mathrm{s}\mathbb{Q})^{\otimes N} \otimes (\mathrm{s}^{-m}\mathscr{C})(a) \otimes (\mathrm{s}^m\mathbb{Q})^{\otimes S},$$

where a morphism (f, k) of $\mathbf{DirGraph}_{\mathrm{id}_I, \mathrm{id}_J}$ acts on $(\mathrm{s}^{-m}\mathscr{C})(a)$ by (f, k) considered as a morphism of $\mathbf{\Sigma} \downarrow \mathbf{\Sigma}$, on $(\mathrm{s}\mathbb{Q})^{\otimes N}$ by permuting the factors according to k, and on $(\mathrm{s}^m\mathbb{Q})^{\otimes S}$ by permuting the factors according to $(f \amalg \mathrm{id}_I \amalg \mathrm{id}_J)|_S$. The action of an element $(g, h) \in \Sigma_I \times \Sigma_J$ on $\mathfrak{DG}^m(\mathscr{C})_{I,J}$ is given by sending an element represented by $(F, N, S, T, \mu, a) \in \mathbf{DirGraph}_{\mathrm{id}_I, \mathrm{id}_J}$ and $x \in \mathrm{Dec}^m_{\mathscr{C};I,J}$ to the element represented by $(F, N, S, T, c|_T \circ \mu \circ (c|_S)^{-1}, a)$ and $\mathrm{sgn}(g)^m x$, where $c \coloneqq \mathrm{id}_F \amalg g \amalg h$.

Given a family $\left(\xi_v \in \mathscr{C}(a^{-1}(v))\right)_{v \in N}$ of elements of $\mathscr{C}$, we call

$$\mathrm{s}^{|N|}1 \otimes \textstyle\bigotimes_{v \in N} \mathrm{s}^{-m}\xi_v \otimes \mathrm{s}^{m|S|}1 \in \mathrm{Dec}^m_{\mathscr{C}}(G)$$

the element *represented* by (G and) the decorations ξ_v.

If $\mathscr{C}$ is concentrated in even degrees and equipped with the structure of a cyclic operad, then $\mathfrak{DG}^m(\mathscr{C})_{I,J}$ becomes a chain complex, which we will call the *directed (I, J)-graph complex* associated to $\mathscr{C}$. We now describe its differential d. For every directed (I, J)-graph $G \coloneqq (F, N, S, T, \mu, a)$, every $y \in \mathrm{Dec}^m_{\mathscr{C};I,J}(G)$, and every internal outgoing flag $s \in S \cap F$ (or, equivalently, every internal edge $e \coloneqq (s, \mu(s))$), we define an element $d^G_s(y) \in \mathfrak{DG}^m(\mathscr{C})_{I,J}$ by specifying the following properties:

- For every isomorphism $\chi\colon G \to G'$ of directed (I, J)-graphs holds that $d^G_s(y) = d^{G'}_{\chi(s)}(\chi \,.\, y)$.
- If $a(s) = a(\mu(s))$ (i.e., if e is a loop), then $d^G_s(y) = 0$.
- Assume that $v \coloneqq a(s)$ and $v' \coloneqq a(\mu(s))$ are, in this order, the first two elements of N. Moreover, assume that a is order-preserving and that s is the last element of $a^{-1}(v)$ and that $\mu(s)$ is the first element of $a^{-1}(v')$. Then d^G_s applied to the element represented by some decorations ξ_v is $(-1)^{m(|v|^{\mathrm{out}}-1)}$ times the element represented by the directed (I, J)-graph $\mathrm{ctr}_s(G)$ decorated by $\xi_v \;{}_s\!\circ_{\mu(s)}\; \xi_{v'}$ at the collapsed vertex $\mathrm{pr}(v) = \mathrm{pr}(v')$ and by ξ_w at all other vertices w.

For $x \in \mathfrak{DG}^m(\mathscr{C})_{I,J}$ the element represented by G and y, we set $d(x) \coloneqq \sum_{s \in S \cap F} d^G_s(y)$.

When $I = J = \varnothing$, we equip $\mathfrak{DG}^m(\mathscr{C})$ with the structure of a cocommutative differential graded coalgebra. The counit $\varepsilon\colon \mathfrak{DG}^m(\mathscr{C}) \to \mathbb{Q}$ sends the element represented by the empty graph $\varnothing$ and

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$1 \in \mathbb{Q} = \mathrm{Dec}^m_{\mathscr{C}}(\varnothing)$ to 1 and an element represented by any other graph to 0. The comultiplication $\Delta\colon \mathfrak{DG}^m(\mathscr{C}) \to \mathfrak{DG}^m(\mathscr{C}) \otimes \mathfrak{DG}^m(\mathscr{C})$ is given as follows: for an element $x \in \mathfrak{DG}^m(\mathscr{C})$ represented by G and $(\xi_v)_{v\in N}$, we set

$$\Delta(x) \coloneqq \sum_{N'} (-1)^{m|N'||N\setminus N'|} \varepsilon(N') x_{N'} \otimes x_{N\setminus N'},$$

where the sum runs over all neighbor-closed vertex subsets N' of G and x_A is the element of $\mathfrak{DG}^m(\mathscr{C})$ represented by $\mathrm{ind}_G(A)$ and $(\xi_v)_{v\in A}$. The sign $\varepsilon(N') \in \{\pm 1\}$ is the sign incurred by permuting

$$(\mathrm{s}1)^{\otimes N} \otimes \bigotimes_{v\in N} \mathrm{s}^{-m}\xi_v \otimes (\mathrm{s}^m 1)^{\otimes S}$$

into

$$(\mathrm{s}1)^{\otimes N'} \otimes \bigotimes_{v\in N'} \mathrm{s}^{-m}\xi_v \otimes (\mathrm{s}^m 1)^{\otimes S'} \otimes (\mathrm{s}1)^{\otimes N\setminus N'} \otimes \bigotimes_{v\in N\setminus N'} \mathrm{s}^{-m}\xi_v \otimes (\mathrm{s}^m 1)^{\otimes S\setminus S'},$$

where $S' \coloneqq S \cap a^{-1}(N')$.

For general I and J, we equip $\mathfrak{DG}^m(\mathscr{C})_{I,J}$ with the structure of a differential graded left comodule over $\mathfrak{DG}^m(\mathscr{C})$. Its structure map $\rho\colon \mathfrak{DG}^m(\mathscr{C})_{I,J} \to \mathfrak{DG}^m(\mathscr{C}) \otimes \mathfrak{DG}^m(\mathscr{C})_{I,J}$ is given as follows: for an element $x \in \mathfrak{DG}^m(\mathscr{C})_{I,J}$ represented by G and $(\xi_v)_{v\in N}$, we set

$$\rho(x) \coloneqq \sum_{N'} (-1)^{m|N'||N\setminus N'|} \varepsilon(N') x_{N'} \otimes x'_{N'},$$

where the sum runs over all neighbor-closed vertex subsets N' of G, $x_{N'}$ is the element of $\mathfrak{DG}^m(\mathscr{C})$ represented by $\mathrm{ind}_G(N')$ and $(\xi_v)_{v\in N'}$, and $x'_{N'}$ is the element of $\mathfrak{DG}^m(\mathscr{C})_{I,J}$ represented by $\mathrm{coind}_G(N')$ and $(\xi_v)_{v\in N\setminus N'}$. The sign $\varepsilon(N') \in \{\pm 1\}$ is the same as above.

Remark 3.25. It will follow from Theorem 3.29 that the differential d of $\mathfrak{DG}^m(\mathscr{C})_{I,J}$ is well defined and actually a differential, that, in the case $I = J = \varnothing$, the maps ε and Δ actually equip it with the structure of a cocommutative differential graded coalgebra, and, for general I and J, that ρ actually equips it with the structure of a differential graded comodule.

Note that, for defining the counit and comultiplication (respectively, the comodule structure map), it is not necessary for $\mathscr{C}$ to be a cyclic operad. However, our proof below that they yield the structure of a cocommutative graded coalgebra (respectively comodule) only applies in that case; it is certainly also true without this assumption though. The condition that $\mathscr{C}$ is concentrated in even degrees is purely for convenience; it avoids having to deal with even more signs.

Remark 3.26. We think of elements of $\mathfrak{DG}^m(\mathscr{C})_{I,J}$ as being represented by a directed (I,J)-graph $(F,N,S,T,\mu,a) \in \mathbf{DirGraph}_{\mathrm{id}_I,\mathrm{id}_J}$ together with a decoration $\xi_v \in \mathscr{C}(a^{-1}(v))$ for every vertex $v \in N$. To understand the action of an (I,J)-graph isomorphism, we should additionally think of such an element as being equipped with an orientation on the set of vertices if m is even or an orientation on the set of edges if m is odd; when an (I,J)-graph isomorphism flips the orientation of the respective set, it induces an extra sign.

Remark 3.27. The sign $(-1)^{m(|v|^{\mathrm{out}}-1)}$ in the definition of the differential above corresponds to moving the to-be-contracted outgoing flag s to the beginning of S.

Remark 3.28. In the case of the cyclic commutative operad $\mathscr{C} = \mathscr{C}\mathrm{om}$, the directed graph complex $\mathfrak{DG}^m(\mathscr{C})$ (or versions of it) has appeared in the work of, for example, Willwacher [Wil14, Appendix K], Živković [Živ20] and Dolgushev–Rogers [DR19].

We are now ready to state and prove the main results of this subsection. (See [BM20, Theorem 9.14] for a similar statement in the case where H is concentrated in a single degree.)

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Theorem 3.29. *Let I and J be finite linearly ordered sets and let $\mathscr{C}$ be a cyclic operad concentrated in even degrees. Then there is an isomorphism of differential graded $(\Sigma_I \times \Sigma_J)$-modules*

$$\Phi_{I,J} : \mathfrak{DG}^m(\mathscr{C})_{I,J} \xrightarrow{\cong} \operatorname*{colim}_{i\in\mathbb{N}} \left(\mathrm{C}_*^{\mathrm{CE}}((\mathrm{s}^{-m}\mathscr{C})[H_i]) \otimes V_i^{\otimes I} \otimes W_i^{\otimes J}\right)_{\Gamma_i},$$

where $(\mathrm{s}^{-m}\mathscr{C})[H_i]$ is equipped with the graded Lie algebra structure of Lemma 2.34. When $I = J = \varnothing$, this is an isomorphism of differential graded coalgebras, and, for general I and J, it is compatible with the comodule structures over these algebras.

Remark 3.30. Tracing the dimension restriction of Proposition 3.13 through our arguments, the preceding theorem can be strengthened to an unstable isomorphism in a certain stable range that is compatible with the stabilization maps. More precisely, we obtain an isomorphism from the differential graded subcoalgebra (resp. subcomodule) of $\mathfrak{DG}^m(\mathscr{C})_{I,J}$ spanned by the decorated graphs with at most $\dim H_i - |I| - |J|$ internal flags to the differential graded subcoalgebra (resp. subcomodule) of $\mathrm{C}_*^{\mathrm{CE}}\big((\mathrm{s}^{-m}\mathscr{C})[H_i] \otimes V_i^{\otimes I} \otimes W_i^{\otimes J}\big)_{\Gamma_i}$ spanned by the elementary tensors involving at most $\dim H_i$ elements of H_i (including V_i and W_i).

Proof of Theorem 3.29. Lemma 2.29 and Proposition 3.19 provide us with an isomorphism $\Phi'_{I,J}$ on the underlying $(\Sigma_I \times \Sigma_J)$-modules (in graded vector spaces). Let $G = (F, N, S, T, \mu, a) \in \mathbf{DirGraph}_{\mathrm{id}_I,\mathrm{id}_J}$ be a directed (I, J)-graph and

$$x \coloneqq (\mathrm{s}1)^{\otimes N} \otimes \bigotimes\nolimits_{v\in N} \mathrm{s}^{-m}\xi_v \otimes \mathrm{s}^{m|S|}1 \quad \in \quad (\mathrm{s}\mathbb{Q})^{\otimes N} \otimes (\mathrm{s}^{-m}\mathscr{C})(a) \otimes (\mathrm{s}^m\mathbb{Q})^{\otimes S} \tag{3.5}$$

an element. Denoting by $[x]$ the element represented by G and x, we set

$$\begin{aligned} \Phi_{I,J}([x]) &\coloneqq \gamma(G)\Phi'_{I,J}([x]) \\ \gamma(G) &\coloneqq (-1)^{m\frac{1}{2}|N|(|N|+1)+n|N|} \end{aligned}$$

(the sign $\gamma(G)$ serves to cancel out a sign encountered below). We have $\Phi_{I,J}([x]) = [\Phi^G_{I,J}(x)]$, where we set $\Phi^G_{I,J}(x) \coloneqq \gamma(G)\varepsilon(x)y$ with y given by

$$\begin{aligned} y &\coloneqq \tilde{y} \otimes \bigotimes\nolimits_{f\in I\amalg J} h_f \\ \tilde{y} &\coloneqq \bigwedge\nolimits_{v\in N} \mathrm{s}y_v \\ y_v &\coloneqq \mathrm{s}^{-m}\xi_v \otimes \bigotimes\nolimits_{f\in a^{-1}(v)} h_f \\ h_f &\coloneqq \begin{cases} e_{s^{-1}(f)} \in V, & \text{if } f \in S \\ \mathrm{s}^m e^\vee_{s^{-1}(\mu^{-1}(f))} \in \mathrm{s}^m V^\vee, & \text{if } f \in T \end{cases} \end{aligned} \tag{3.6}$$

(where $s\colon \underline{|S|} \to S$ is the unique order-preserving bijection) and the sign $\varepsilon(x) \in \{\pm 1\}$ is the sign incurred by permuting

$$(\mathrm{s}1)^{\otimes N} \otimes \bigotimes_{v\in N} \mathrm{s}^{-m}\xi_v \otimes \bigotimes_{s\in S} (h_s \otimes h_{\mu(s)}) \quad \text{into} \quad \bigotimes_{v\in N} \mathrm{s}\left(\mathrm{s}^{-m}\xi_v \otimes \bigotimes_{f\in a^{-1}(v)} h_f\right) \otimes \bigotimes_{f\in I\amalg J} h_f$$

according to the Koszul sign rule.

We will now identify the result of pulling the Chevalley–Eilenberg differential $d = d_0 + d_1$ (cf. Definition 2.28) back to $\mathfrak{DG}^m(\mathscr{C})_{I,J}$ along the isomorphism $\Phi_{I,J}$. To this end, we first evaluate it on

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$\tilde{y}$. We have $d_0 = 0$ (since the differential of $(\mathrm{s}^{-m}\mathscr{C})[H_i]$ is trivial) and

$$\begin{aligned} d_1(\tilde{y}) &= \sum_{\substack{v,v' \in N \\ v<v'}} (-1)^{b_{v,v'}} \mathrm{s}[y_v, y_{v'}] \wedge \textstyle\bigwedge_{w \in V \setminus \{v,v'\}} \mathrm{s} y_w \\ [y_v, y_{v'}] &= \sum_{\substack{i \in a^{-1}(v) \\ j \in a^{-1}(v')}} \mathrm{s}^{-m}(\xi_{v\ i} \circ_j \xi_{v'}) \otimes (h_{v\ i} \circ_j h_{v'}) \qquad (3.7) \\ b_{v,v'} &\coloneqq |\mathrm{s} y_v|(1 + |\mathrm{s} y_{v'}|) + |\mathrm{s} y_v| \sum_{\substack{w \in N \\ w<v}} |\mathrm{s} y_w| + |\mathrm{s} y_{v'}| \sum_{\substack{w \in N \\ w<v'}} |\mathrm{s} y_w|, \end{aligned}$$

where $h_v \coloneqq \bigotimes_{f \in a^{-1}(v)} h_f$. In the following, we will, for ease of notation, identify each $a^{-1}(v)$ with $\{1, \dots, |v|\}$ via the unique order-preserving bijection and set $h_{v;i} \coloneqq h_{f(v,i)}$, where $f(v,i)$ is the i-th element of $a^{-1}(v)$. We recall

$$\begin{aligned} h_{v\ i} \circ_j h_{v'} &\coloneqq (-1)^{a^{i,j}_{v,v'}} h_{v;1,i-1} \otimes h_{v';j+1,|v'|} \otimes h_{v';1,j-1} \otimes h_{v;i+1,|v|} \langle h_{v';j}, h_{v;i} \rangle \\ a^{i,j}_{v,v'} &\coloneqq (|h_{v;i,\deg v}| - m)(|h_{v'}| - m) + |h_{v';1,j}||h_{v';j+1,\deg v'}| + 1 \\ h_{w;k,l} &\coloneqq \textstyle\bigotimes_{i=k}^{l} h_{w;i} \end{aligned}$$

and note that, by definition of h_f, the expression $\langle h_{v';j}, h_{v;i} \rangle$ is zero unless $\{i,j\} = \{s, \mu(s)\}$ for some $s \in S \cap F$. Hence, the same is true for the summand in (3.7) corresponding to (i,j). We thus obtain the following description:

$$\begin{aligned} d(y) &= \sum_{\substack{s \in S \cap F \\ a(s) \neq a(\mu(s))}} d^G_s(y) \\ d^G_s(y) &\coloneqq \delta_s \cdot \Big(\mathrm{s}(y_{a(s)\ s} \circ_{\mu(s)} y_{a(\mu(s))}) \wedge \textstyle\bigwedge_{v \in V \setminus \{a(s), a(\mu(s))\}} \mathrm{s} y_v \Big) \otimes \bigotimes_{f \in I \amalg J} h_f \\ y_{v\ i} \circ_j y_{v'} &\coloneqq \mathrm{s}^{-m}(\xi_{v\ i} \circ_j \xi_{v'}) \otimes (h_{v\ i} \circ_j h_{v'}), \end{aligned}$$

where $\delta_s \in \{\pm 1\}$ is some sign. We note that $d^G_s(\Phi^G_{I,J}(x)) \coloneqq \gamma(G)\varepsilon(x) d^G_s(y)$ is equal to some sign $\delta'_s \in \{\pm 1\}$ times the image under $\Phi_{I,J}$ of the element $[z]$ represented by the directed (I,J)-graph $\mathrm{ctr}_s(G)$ decorated by $\xi_{a(s)\ s} \circ_{\mu(s)} \xi_{a(\mu(s))}$ at the collapsed vertex $\mathrm{pr}(a(s)) = \mathrm{pr}(a(\mu(s)))$ and by ξ_v at all other vertices v. (Note that the family of basis elements $h_\bullet$ obtained for $\mathrm{ctr}_s(G)$ does not agree with the family obtained for the noncontracted flags of G. However, they are equivalent under the Γ_i-action.)

We will now determine the sign δ'_s in a special case. To this end, let $s \in S \cap F$ and $t \coloneqq \mu(s)$ such that $a(s) \neq a(t)$. Assume that $v \coloneqq a(s)$ and $v' \coloneqq a(t)$ are, in this order, the first two elements of N. Moreover, assume that $a \colon F \to N$ is order-preserving and that s is the last element of $a^{-1}(v)$ and that t is the first element of $a^{-1}(v')$. (Note that any directed (I,J)-graph is isomorphic to one of this form.) We have

$$\begin{aligned} \delta'_s &= \gamma(G)\varepsilon(x)\langle h_{v';t}, h_{v;s} \rangle (-1)^c \gamma(\mathrm{ctr}_s(G))\varepsilon(z) \\ c &\coloneqq b_{v,v'}(x) + a^{s,t}_{v,v'}(x) \end{aligned}$$

by definition. We first note that $\langle h_{v';t}, h_{v;s} \rangle = 1$, that $\gamma(G)\gamma(\mathrm{ctr}_s(G)) = (-1)^{m|N|+n}$, and that

$$\begin{aligned} c &= |\mathrm{s} y_v| + \big(|h_{v;s}| - m\big)(|h_{v'}| - m) + |h_{v';t}||h_{v';t+1,\deg v'}| + 1 \\ &= (1 - m + \mathrm{w}(v)) + (n-m)(\mathrm{w}(v') - m) + (m-n)(\mathrm{w}(v') - (m-n)) + 1, \end{aligned}$$

which reduces the question to determining $\varepsilon(x)\varepsilon(z)$. To this end, we factor the permutation defining ε into three parts: firstly permuting the h_f among each other, secondly interleaving the $\mathrm{s}^{-m}\xi_v$ and h_v and

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thirdly interleaving the s1 and y_v. The deviation between $\varepsilon(x)$ and $\varepsilon(z)$ for the first part is given by -1 to the power of

$$m\big(|v|^{\mathrm{out}} - 1\big) + m(\mathrm{w}(v) - n),$$

for the second part, it is -1 to the power of

$$(|h_s| + |h_t|)(|N| - 2)(-m) + |h_v|(-m) \equiv |N|m + \mathrm{w}(v)m (\mathrm{mod}\ 2),$$

and for the the third part, it is -1 to the power of

$$(-m + |h_s| + |h_t|)|N| + |y_v| = -m + \mathrm{w}(v),$$

so that, putting everything together, we obtain $\delta'_s = (-1)^{m(|v|^{\mathrm{out}}-1)}$.

Lastly, we show that, for any isomorphism of directed (I,J)-graphs $\chi\colon G \to G'$, we have

$$[d^G_s(\Phi^G_{I,J}(x))] = [d^{G'}_{\chi(s)}(\Phi^{G'}_{I,J}(\chi\,.\,x))],$$

which will finish the identification of the differential. The statement is clear up to sign, so it is enough to show that those agree as well. We write $\chi\,.\,x = \alpha x'$, where $\alpha \in \{\pm 1\}$ and x' is the element represented by G' and the labels $(\chi\,.\,\xi_{\chi^{-1}(w)})_{w\in N'}$. Lastly, we define y' analogously to (3.6), so that $\Phi^{G'}_{I,J}(x') = \gamma(G')\varepsilon(x')y'$.

By definition, we have $[\gamma(G)\varepsilon(x)y] = \Phi_{I,J}([x]) = \Phi_{I,J}([\chi\,.\,x]) = [\alpha\gamma(G')\varepsilon(x')y']$. Noting that $\gamma(G) = \gamma(G')$, it follows from the definitions that $\alpha\gamma(G)\varepsilon(x)\gamma(G')\varepsilon(x') = \alpha\varepsilon(x)\varepsilon(x')$ is the sign induced by permuting

$$\bigotimes_{v\in N} \mathrm{s}\left(\mathrm{s}^{-m}\xi_v \otimes \bigotimes_{f\in a^{-1}(v)} h_f\right) \quad \text{into} \quad \bigotimes_{w\in N'} \mathrm{s}\left(\mathrm{s}^{-m}\xi_{\chi^{-1}(w)} \otimes \bigotimes_{g\in a'^{-1}(w)} h_{\chi^{-1}(g)}\right)$$

(here, we use that the families $(h_f)_{f\in F}$ and $(h'_{\chi(f)})_{f\in F}$ are equivalent under the action of Γ_i for large enough i; hence, we can assume that they are equal, which we will do from now on). This sign is the product of the sign α' incurred by permuting $\bigotimes_{v\in N} \mathrm{s}y_v$ into $\bigotimes_{w\in N'} \mathrm{s}y_{\chi^{-1}(w)}$ and the signs α_v incurred by permuting $\bigotimes_{f\in a^{-1}(v)} h_f$ into $\bigotimes_{g\in a'^{-1}(\chi(v))} h_{\chi^{-1}(g)}$.

It remains to evaluate the sign difference between $d^G_s(y)$ and $d^{G'}_{\chi(s)}(y')$. For this, write $t := \mu(s)$, and $\{v,v'\} := \{a(s),a(t)\}$ such that $v < v'$. Also let $\{w,w'\} := \{\chi(v),\chi(v')\}$ such that $w < w'$. Then the relevant summands of $d_1(\tilde{y})$ and $d_1(\tilde{y}')$ are the ones indexed by (v,v') and (w,w'). Note that -1 to the power of $b_{v,v'} + b'_{w,w'}$, the sign difference between $\bigwedge_{\tilde{v}\in V\setminus\{v,v'\}} \mathrm{s}y_{\tilde{v}}$ and $\bigwedge_{\tilde{w}\in V'\setminus\{w,w'\}} \mathrm{s}y'_{\tilde{w}}$, and α' and $\alpha_{\tilde{v}}$ for $\tilde{v} \in V\setminus\{v,v'\}$ cancel each other out, except for, if χ swaps the order of v and v', a sign of -1 to the power of $1 + |y'_w||y'_{w'}|$. Now note that, in the case that χ swaps the order of v and v', we have $[y'_{w'}, y'_w] = (-1)^{1+|y'_w||y'_{w'}|}[y'_w, y'_{w'}]$ in the summand of $d_1(\tilde{y}')$. Hence, the only sign that has not yet canceled is $\alpha_v\alpha_{v'}$.

Thus, it remains to show that the summand of $[y_v, y_{v'}]$ corresponding to (s,t) equals the summand of $[y'_{\chi(v)}, y'_{\chi(v')}]$ corresponding to $(\chi(s),\chi(t))$, up to a sign of $\alpha_v\alpha_{v'}$. (For ease of notation, we assume from now on that $a(s) = v$, and we set $w := \chi(v)$, $w' := \chi(v')$, $s' := \chi(s)$, and $t' := \chi(t)$, possibly changing notation from before.) These two summands agree in $(\mathrm{s}^{-m}\mathscr{C})[H_i]$ up to sign. The difference in signs is given by -1 to the power of $a^{s,t}_{v,v'} + a^{s',t'}_{w,w'}$ times the sign incurred by permuting

$$h_v\ {}_s\tilde{\circ}_t\ h_{v'} := h_{v;1,s-1} \otimes h_{v';t+1,|v'|} \otimes h_{v';1,t-1} \otimes h_{v;s+1,|v|}$$

into $h'_w\ {}_{s'}\tilde{\circ}_{t'}\ h'_{w'}$. Now we note that -1 to the power of $a^{s,t}_{v,v'} + 1 + m + m(|h_{v;s}| + |h_{v'}|)$ is precisely the sign incurred by permuting $h_v \otimes h_{v'}$ into $(h_v\ {}_s\tilde{\circ}_t\ h_{v'}) \otimes h_{v';t} \otimes h_{v;s}$ and analogously for $a^{s',t'}_{w,w'}$.

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Noting that $|h_{v;s}| = |h'_{w;s'}|$ and $|h_{v'}| = |h'_{w'}|$, this implies the claim since $\alpha_v \alpha_{v'}$ is the sign incurred by permuting $h_v \otimes h_{v'}$ into $h'_w \otimes h'_{w'}$.

Now we prove that $\Phi_{\varnothing,\varnothing}$ is a map of graded coalgebras and that $\Phi_{I,J}$ is compatible with the comodule structures over these coalgebras. It is clear that $\Phi_{\varnothing,\varnothing}$ is compatible with the counits. We will now identify the comodule structure map ρ of $\mathfrak{DG}^m(\mathscr{C})_{I,J}$ obtained by pulling back the one of the Chevalley–Eilenberg chains. When $I = J = \varnothing$, this also deals with the comultiplication Δ of the coalgebra structure since, in this case, we have $\Delta = \rho$ on both sides (see Remark 3.23). We continue to use the notation of (3.5) and (3.6). We have

$$\rho(y) \coloneqq \sum_{N' \subseteq N} \delta(N') \cdot \Big(\bigwedge_{v \in N'} \mathrm{s} y_v\Big) \otimes \Big(\Big(\bigwedge_{w \in N \setminus N'} \mathrm{s} y_w\Big) \otimes \bigotimes_{f \in I \amalg J} h_f\Big), \tag{3.8}$$

where the sum runs over all subsets N' of N and $\delta(N') \in \{\pm 1\}$ is the sign incurred by permuting $\bigotimes_{v \in N} \mathrm{s} y_v$ into $\bigotimes_{v \in N'} \mathrm{s} y_v \otimes \bigotimes_{w \in N \setminus N'} \mathrm{s} y_w$. Note that (the equivalence class of) $\bigwedge_{v \in N'} \mathrm{s} y_v$ is trivial except when $\mu(S \cap a^{-1}(N')) = T \cap a^{-1}(N')$. Hence, we can also consider the sum in (3.8) to run only over all neighbor-closed vertex sets N' of G. For such an N', we set

$$\begin{aligned} x_{N'} &\coloneqq (\mathrm{s}1)^{\otimes N'} \otimes \bigotimes_{v \in N'} \mathrm{s}^{-m} \xi_v \otimes \mathrm{s}^{m|S'|} 1 \\ x'_{N'} &\coloneqq (\mathrm{s}1)^{\otimes N \setminus N'} \otimes \bigotimes_{v \in N \setminus N'} \mathrm{s}^{-m} \xi_v \otimes \mathrm{s}^{m|S \setminus S'|} 1, \end{aligned}$$

where $S' \coloneqq S \cap a^{-1}(N')$. We denote the element of $\mathfrak{DG}^m(\mathscr{C})$ represented by $\mathrm{ind}_G(N')$ and $x_{N'}$ by $[x_{N'}]$, and the element of $\mathfrak{DG}^m(\mathscr{C})_{I,J}$ represented by $\mathrm{coind}_G(N')$ and $x'_{N'}$ by $[x'_{N'}]$. We have

$$\begin{aligned} \Phi_{\varnothing,\varnothing}([x_{N'}]) &= \gamma(\mathrm{ind}_G(N'))\varepsilon(x_{N'}) \cdot \textstyle\bigwedge_{v \in N'} \mathrm{s} y_v \\ \Phi_{I,J}([x'_{N'}]) &= \gamma(\mathrm{coind}_G(N'))\varepsilon(x'_{N'}) \cdot \Big(\textstyle\bigwedge_{v \in N \setminus N'} \mathrm{s} y_v \otimes \bigotimes_{f \in I \amalg J} h_f\Big) \end{aligned}$$

so that it only remains to see that

$$\gamma(G)\varepsilon(x)\delta(N')\gamma(\mathrm{ind}_G(N'))\varepsilon(x_{N'})\gamma(\mathrm{coind}_G(N'))\varepsilon(x'_{N'})$$

is equal to the sign in the definition of the comodule structure of $\mathfrak{DG}^m(\mathscr{C})_{I,J}$. Noting that

$$\gamma(G)\gamma(\mathrm{ind}_G(N'))\gamma(\mathrm{coind}_G(N')) = (-1)^{m|N'||N \setminus N'|},$$

this is clear from the definitions. □

3.5. Coinvariants of the truncation

In this subsection, our goal is to obtain a description of the differential graded $(\Sigma_I \times \Sigma_J)$-module

$$\operatorname*{colim}_{i \in \mathbb{N}} \Big(\mathrm{C}^{\mathrm{CE}}_*\big((\mathrm{s}^{-m}\mathscr{C})[H_i]^+\big) \otimes V_i^{\otimes I} \otimes W_i^{\otimes J}\Big)_{\Gamma_i}$$

in terms of a graph complex (including the coalgebra/comodule structures, as before). Here, $\mathscr{C}$ is a cyclic operad, H_i, V_i, W_i and Γ_i are as in the preceding subsections, and $(-)^+$ denotes positive truncation (see Definition 2.27). We begin with the following useful definition.

Definition 3.31. Let I and J be finite linearly ordered sets. A directed (I,J)-graph G is *truncated* if it has no vertices of valence zero or one, and each vertex of valence two has only incoming edges. We denote by

$$\mathbf{DirGraph}^{\mathbf{trunc}}_{\mathrm{id}_I,\mathrm{id}_J} \subset \mathbf{DirGraph}_{\mathrm{id}_I,\mathrm{id}_J}$$

the full subgroupoid spanned by the truncated directed (I,J)-graphs.

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We also denote by $\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})_{I,J} \subseteq \mathfrak{DG}^m(\mathscr{C})_{I,J}$ the differential graded $(\Sigma_I \times \Sigma_J)$-submodule spanned by the elements represented by truncated decorated (I,J)-graphs. When $I = J = \varnothing$, we omit them from the notation.

Remark 3.32. When $I = J = \varnothing$, we have that $\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C}) \subseteq \mathfrak{DG}^m(\mathscr{C})$ is a subcoalgebra. For general I and J, the $\mathfrak{DG}^m(\mathscr{C})$-comodule structure of $\mathfrak{DG}^m(\mathscr{C})_{I,J}$ restricts to a $\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})$-comodule structure on $\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})_{I,J}$.

Corollary 3.33. *Let I and J be finite linearly ordered sets and let $\mathscr{C}$ be a cyclic operad concentrated in nonnegative even degrees such that $\mathscr{C}(\!(0)\!)$, $\mathscr{C}(\!(1)\!)$ and $\mathscr{C}(\!(2)\!)$ are concentrated in degree 0. Assume that $2n < m < 3n$ and set*

$$\alpha := \begin{cases} \frac{2}{3}, & \text{if } m = 3n-1. \\ 1, & \text{otherwise.} \end{cases}$$

Then, for all $i \in \mathbb{N}_{\geq 1}$, there is, in homological degrees $\leq \alpha \dim H_i$, an isomorphism of differential graded $(\Sigma_I \times \Sigma_J)$-modules

$$\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})_{I,J} \cong \left(\mathrm{C}^{\mathrm{CE}}_*\big((\mathrm{s}^{-m}\mathscr{C})[H_i]^+\big) \otimes V_i^{\otimes I} \otimes W_i^{\otimes J}\right)_{\Gamma_i} \tag{3.9}$$

that is compatible with the stabilization maps on the right-hand side. When $I = J = \varnothing$, it is an isomorphism of differential graded coalgebras. For general I and J, it is compatible with the comodule structures over these coalgebras.

In particular, after stabilizing, we obtain an isomorphism

$$\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})_{I,J} \cong \operatorname*{colim}_{i \in \mathbb{N}} \left(\mathrm{C}^{\mathrm{CE}}_*\big((\mathrm{s}^{-m}\mathscr{C})[H_i]^+\big) \otimes V_i^{\otimes I} \otimes W_i^{\otimes J}\right)_{\Gamma_i}$$

of differential graded $(\Sigma_I \times \Sigma_J)$-modules. When $I = J = \varnothing$, it is an isomorphism of differential graded coalgebras. For general I and J, it is compatible with the comodule structures over these coalgebras.

Proof. The condition $2n < m < 3n$ implies that

$$(\mathrm{s}^{-m}\mathscr{C})[H_i]^+ = \big(\mathrm{s}^{-m}\mathscr{C}(\!(2)\!) \otimes_{\Sigma_2} (W_i \otimes W_i)\big) \oplus \bigoplus_{k \geq 3} \mathrm{s}^{-m}\mathscr{C}(\!(k)\!) \otimes_{\Sigma_k} H_i^{\otimes k} \subseteq (\mathrm{s}^{-m}\mathscr{C})[H_i],$$

where W_i is the degree $m-n$ part of H_i. Then the explicit description of the isomorphism of Theorem 3.29, together with Remark 3.30, implies that there is an isomorphism from the differential graded $(\Sigma_I \times \Sigma_J)$-submodule D'_* of $\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})_{I,J}$ spanned by the decorated graphs with at most $\dim H_i - |I| - |J|$ internal flags to the differential graded $(\Sigma_I \times \Sigma_J)$-submodule C'_* of the right-hand side of (3.9) spanned by the elementary tensors involving at most $\dim H_i$ elements of H_i (including V_i and W_i).

We will now prove that C'_p is equal to degree p of the right-hand side of (3.9) when $p \leq \alpha \dim H_i$. To this end, we take elements

$$\begin{aligned} y &:= x \otimes v \otimes w && \in \mathrm{C}^{\mathrm{CE}}_*\big((\mathrm{s}^{-m}\mathscr{C})[H_i]^+\big) \otimes V_i^{\otimes I} \otimes W_i^{\otimes J} \\ x &:= \textstyle\bigwedge_{v \in N} \mathrm{s}\Big(\mathrm{s}^{-m}\xi_v \otimes \bigotimes_{j=1}^{k_v} h_{v,j}\Big) && \in \mathrm{C}^{\mathrm{CE}}_*\big((\mathrm{s}^{-m}\mathscr{C})[H_i]^+\big) \end{aligned}$$

and note that $|x| = \sum_{v \in N}(1 - m + |\xi_v| + \mathrm{w}(v))$, where $\mathrm{w}(v) := \sum_{j=1}^{k_v} |h_{v,j}|$. Due to the truncation, we have that

$$\mathrm{w}(v) \begin{cases} = 2(m-n), & \text{if } k_v = 2 \\ \geq k_v n, & \text{if } k_v \geq 3 \end{cases}$$

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and that $k_v \geq 2$. Setting $N_2 \coloneqq \{v \in N \mid k_v = 2\}$ and similarly for $N_{\geq 3}$, we thus obtain

$$|x| \geq |N_2|(1 - m + 2(m - n)) + \sum_{v \in N_{\geq 3}} (1 - m + k_v n)$$

since $|\xi_v| \geq 0$ by assumption. We note $1 - m + 2(m - n) \geq 2$ and, when $k_v \geq 3$, that

$$\frac{3}{1 - m + 3n}(1 - m + k_v n) \geq k_v$$

since $\frac{3n}{1-m+3n} \geq 1$ as $-3n < 1 - m < 0$. Setting

$$\beta \coloneqq \max\Big(1, \frac{3}{1 - m + 3n}\Big) = \alpha^{-1},$$

we thus obtain

$$\beta|x| \geq |N_2|2 + \sum_{v \in N_{\geq 3}} k_v = \sum_{v \in N} k_v$$

and hence that $\beta|x|+|I|+|J|$ is larger or equal to the number of elements of H_i involved in y. Furthermore,

$$\beta|x| + |I| + |J| = \beta|y| + (1 - \beta n)|I| + (1 - \beta(m - n))|J| \leq \beta|y|$$

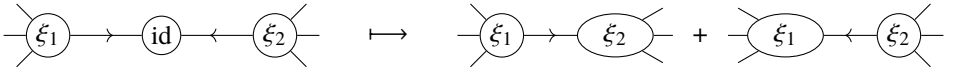
since $|v| = n|I|$ and $|w| = (m - n)|J|$. Hence, $\beta|y| = \alpha^{-1}|y| \leq \dim H_i$ implies $[y] \in C'_*$.

Using the same argument, we can show that $D'_p = \mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})_p$ for $p \leq \alpha \dim H_i$. (A combinatorial argument using graphs would produce a better bound here; however, we will not need this.) □

4. The undirected graph complex

Throughout this section, we let $m \in \mathbb{N}_0$ be a natural number and $\mathscr{C}$ a cyclic operad concentrated in nonnegative even degrees such that $\mathscr{C}(\!(2)\!) = \mathbb{Q}\langle \mathrm{id} \rangle$ with the trivial Σ_2-action. (As in the previous section, the condition that $\mathscr{C}$ is concentrated in even degrees is purely for the convenience of avoiding more signs.)

Our goal is to obtain a simpler description of the homology of the truncated directed graph complex $\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})$ of Definition 3.31. In fact, we will obtain a description in terms of *undirected* graphs. The rough idea is as follows. Restricting the differential of $\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})$ to the two edges incident to a vertex of valence two (which must be labeled by the identity operation by our assumption on $\mathscr{C}$) yields the following picture.



Thus, intuitively, taking homology should kill the difference between the two orientations of an edge. We formalize this using a spectral sequence argument: filtering $\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{L}\mathrm{ie})$ by the number of vertices of valence two yields a spectral sequence, for which we prove that its first page is concentrated in a single row. We moreover show that this row is isomorphic as a chain complex to a (well-known) undirected version of the graph complex, which we denote by $\mathfrak{UG}^m(\mathscr{C})$. The spectral sequence then implies that there is an isomorphism $\mathrm{H}_*(\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{L}\mathrm{ie})) \cong \mathrm{H}_*(\mathfrak{UG}^m(\mathscr{C}))$. (This argument is similar to a proof sketched by Willwacher [Wil14, Appendix K].)

Remark 4.1. There should also be a version of everything we do in this section in the more general situation of the truncated directed (I, J)-graph complex $\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})_{I,J}$. We will not elaborate on this here though. It might appear in future work.

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We begin by noting that the degree k part of the graded vector space $\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})$ splits as a direct sum

$$\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})_k = \bigoplus_{l\in\mathbb{N}_0} \mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})_{k,l},$$

where $\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})_{k,l}$ is the subspace generated by elements represented by truncated decorated graphs with exactly l vertices of valence two. Note that a truncated decorated graph with N vertices and E edges represents an element of homological degree $k \geq N(1-m)+Em$ (since $\mathscr{C}$ is concentrated in nonnegative degrees). Since all of its vertices are at least bivalent we have $E \geq N$ and hence $k \geq N$ (since $m \geq 0$). Thus, in the above decomposition, a summand is trivial if $l > k$, and hence, the sum is finite.

The differential restricts to a map

$$d\colon \mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})_{k,l} \xrightarrow{(d^1,d^2)} \mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})_{k-1,l} \oplus \mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})_{k-1,l-1}$$

so that regrading via $C_{p,q} \coloneqq \mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})_{p+q,q}$ yields a double complex $C_{*,*}$ with a differential d^1 of bidegree $(-1,0)$ and a differential d^2 of bidegree $(0,-1)$ – that $d^1 \circ d^2 = -d^2 \circ d^1$ follows immediately from $d \circ d = 0$. The total complex of $C_{*,*}$ is canonically isomorphic to $\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})$. Moreover, $C_{p,q}$ can only be nontrivial if $0 \leq q \leq p+q$. Hence, we obtain a convergent spectral sequence of the following form:

$$E^1_{p,q} = \mathrm{H}_q\left(C_{p,*}, d^2\right) \;\Rightarrow\; \mathrm{H}_{p+q}\left(\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})\right) \tag{4.1}$$

(see, for example, [Wei95, §5.6]).

We will use this spectral sequence to identify the homology of $\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})$. However, we will need a few preliminaries before we can do so, beginning with an undirected version of Definition 3.16.

Definition 4.2. We denote by **Graph** the following groupoid:

- Objects are tuples (F, N, μ, a) with F and N finite linearly ordered sets, $a\colon F \to N$ a map of sets, and μ a matching of F (i.e., μ is a fixed-point-free bijection $F \to F$ of order two).
- A morphism $(F, N, \mu, a) \to (F', N', \mu', a')$ is a pair of bijections $(f\colon F \to F', g\colon N \to N')$ such that $f \circ \mu = \mu' \circ f$ and $g \circ a = a' \circ f$.

Moreover, we write $\mathbf{Graph}_{\geq 3}$ for the full subgroupoid of **Graph** spanned by those objects (F, N, μ, a) such that $|a^{-1}(v)| \geq 3$ for all $v \in N$.

Remark 4.3. Analogously to Remark 3.17, we can think of **Graph** as the groupoid of *undirected graphs* and of $\mathbf{Graph}_{\geq 3}$ as the groupoid of undirected graphs such that each vertex has at least valence 3.

Given $G = (F, N, \mu, a) \in \mathbf{Graph}$, the set F specifies the set of flags, N is the set of vertices (or nodes), the map a specifies to which vertex a flag is incident, and the matching μ specifies how the flags are connected to form edges. In particular, an *edge* of G is an orbit of μ (i.e., a subset $\{f, f'\} \subseteq F$ such that $\mu(f) = f'$ and $\mu(f') = f$).

Definition 4.4. For $G = (F, N, \mu, a) \in \mathbf{Graph}$, we write $\mathrm{Edge}(G)$ for the set of edges of G. We equip it with the linear order pulled back along the injection $\mathrm{Edge}(G) \to F$ given by $\{f, f'\} \mapsto \min(f, f')$. This construction yields a functor $\mathrm{Edge}\colon \mathbf{Graph} \to \mathbf{\Sigma}$.

Definition 4.5. Let $D = (F, N, S, T, \mu, a)$ be a directed graph. Its *underlying undirected graph* is the undirected graph (F, N, μ', a), where $\mu'\colon F \to F$ is the matching given by $\mu'(s) = \mu(s)$ for $s \in S \subseteq F$ and $\mu'(t) = \mu^{-1}(t)$ for $t \in T \subseteq F$.

Definition 4.6. For a truncated directed graph D, we define its *underlying graph* $\mathrm{U}(D) \in \mathbf{Graph}_{\geq 3}$ to be the underlying undirected graph of D, except that all vertices of valence two together with their two incident edges are replaced by a single edge.

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Remark 4.7. To be entirely precise, we would need to specify a linear order on the set of vertices and the set of flags of $\mathrm{U}(D)$. This choice will not matter, however, and so we can make it arbitrarily.

We will now study (the homology of) the chain complexes $C^2_{p,*} \coloneqq (C_{p,*}, d^2)$ occurring in the spectral sequence (4.1). To this end, let D be a truncated directed graph. Note that applying d^2 to an element represented by D yields a sum of elements each of which is represented by a directed graph D' such that $\mathrm{U}(D') \cong \mathrm{U}(D)$. Hence, there is a splitting of chain complexes

$$C^2_{p,*} \cong \bigoplus_{[G]} C^G_{p,*},$$

where $[G]$ runs over all isomorphism classes of $\mathbf{Graph}_{\geq 3}$ and $C^G_{p,*}$ is the subcomplex of $C^2_{p,*}$ generated by the elements represented by directed graphs D with $\mathrm{U}(D) \cong G$. Thus, we can restrict our attention to the chain complexes $C^G_{p,*}$.

To understand (the homology of) $C^G_{p,*}$, we will construct a functor $\widetilde{C}_{\bullet,*}$ from $\mathbf{Graph}_{\geq 3}$ to chain complexes in graded vector spaces (here, the first grading $\bullet$ is the one of the graded vector space, and the second grading $*$ is the one of the chain complex) such that

$$\left(\widetilde{C}_{\bullet,*}(G)\right)_{\mathrm{Aut}(G)} \cong C^G_{\bullet,*} \coloneqq \bigoplus_p C^G_{p,*} \tag{4.2}$$

for all G. Since $\mathrm{Aut}(G)$ is finite and, in characteristic 0, quotienting by a finite group commutes with taking homology, we will then be able to deduce the homology of $C^G_{\bullet,*}$ from the homology of $\widetilde{C}_{\bullet,*}(G)$ together with its $\mathrm{Aut}(G)$-action.

More precisely, we will take a 'coordinate-free' approach to the isomorphism (4.2) and actually construct an isomorphism

$$\operatorname*{colim}_{\mathbf{Graph}_{\geq 3}} \widetilde{C}_{\bullet,*} \cong C^2_{\bullet,*} \coloneqq \left(\bigoplus_p C_{p,*}, d^2\right)$$

of chain complexes in graded vector spaces. This will be the more convenient approach as it does not require us to chose representatives of isomorphism classes of $\mathbf{Graph}_{\geq 3}$. We need a few preliminaries before we can construct $\widetilde{C}_{\bullet,*}$.

Definition 4.8. Let S be a set. We write $\mathrm{dir}(S)$ for the set of functions $S \to \{-, 0, +\}$. For an element $o \in \mathrm{dir}(S)$, we set its *degree* to be $|o| \coloneqq |o^{-1}(0)|$.

Moreover, for $? \in \{+, -\}$, $o \in \mathrm{dir}(S)$, and $s \in S$ such that $o(s) = 0$, we write $o^?_s \in \mathrm{dir}(S)$ for the function with $o^?_s(s) = ?$ and $o^?_s(s') = o(s')$ for all $s' \in S \setminus \{s\}$.

Definition 4.9. When G is an undirected graph, we write $\mathrm{dir}(G) \coloneqq \mathrm{dir}(\mathrm{Edge}(G))$. This forms a (covariant) functor dir from **Graph** to the category of sets by letting an isomorphism $\chi\colon G \to G'$ act on $o \in \mathrm{dir}(G)$ such that

$$(\chi \,.\, o)(e) \coloneqq \begin{cases} o(\chi^{-1}(e)), & \text{if } \chi^{-1}(f_1) < \chi^{-1}(f_2) \\ -o(\chi^{-1}(e)), & \text{if } \chi^{-1}(f_1) > \chi^{-1}(f_2) \end{cases}$$

for each edge $e = \{f_1 < f_2\}$ of G'. (Here, we set $-- = +$, $-0 = 0$, and $-+ = -$, as expected.)

Remark 4.10. We think of the elements of $\mathrm{dir}(G)$ as all possible ways to, for each edge of G, either chose an orientation or leave it undirected.

Definition 4.11. Let $e = \{f < f'\}$ be an edge of an undirected graph G. We set the vector space $\mathrm{Or}(e)$ of orientations of e to be $\mathbb{Q}$ (in degree 0). An isomorphism $\chi\colon G \to G'$ of undirected graphs induces a map $\mathrm{Or}(e) \to \mathrm{Or}(\chi(e))$ by setting it to be $\mathrm{id}_{\mathbb{Q}}$ if $\chi(f) < \chi(f')$ and $-\mathrm{id}_{\mathbb{Q}}$ otherwise.

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Definition 4.12. Let $G \coloneqq (F, N, \mu, a) \in \mathbf{Graph}_{\geq 3}$ and $o \in \mathrm{dir}(G)$. We write $\mathrm{D}(G, o)$ for the truncated directed graph obtained from G by, for each edge $e = \{f < f'\}$, doing the following:

- if $o(e) = +$, we orient it from f to f' (i.e. $f \in S$ and $f' \in T$).
- if $o(e) = -$, we orient it from f' to f (i.e. $f' \in S$ and $f \in T$).
- if $o(e) = 0$, we replace e by a new vertex v_e with two incoming edges (s_e^+, t_e^+) and (s_e^-, t_e^-) such that the flag s_e^+ is incident to $a(f')$, the flag s_e^- is incident to $a(f)$, and the flags t_e^+ and t_e^- are incident to v_e.

The linear orders on the set of vertices and the set of flags of $\mathrm{D}(G, o)$ are inherited from G such that, for $e = \{f < f'\}$ with $o(e) = 0$, the new vertex v_e is inserted as the successor of $a(f)$ (if there are multiple such vertices, they are inserted according to the order of the edges e), the two new flags s_e^+ and t_e^+ are successively inserted in this order instead of f', and similarly for s_e^-, t_e^-, and f.

This construction yields an equivalence of categories D from the Grothendieck construction $\int_{\mathbf{Graph}_{\geq 3}} \mathrm{dir}$ to $\mathbf{DirGraph}^{\mathbf{trunc}}$.

Remark 4.13. Note that $\mathrm{U}(\mathrm{D}(G, o))$ is canonically isomorphic to G for all o.

Definition 4.14. We say that a collection of decorations ξ_v of the vertices v of G *represents* the element of $\mathrm{Dec}^m_{\mathscr{C}}(\mathrm{D}(G, o))$ that is represented by the ξ_v and $\mathrm{id} \in \mathscr{C}(\!(2)\!)$ at all of the added vertices of valence two.

We are now ready to construct the functor $\widetilde{C}_{\bullet,*}$ of (4.2). As a functor to bigraded vector spaces, we set

$$\widetilde{C}(G) \coloneqq \bigoplus_{o \in \mathrm{dir}(G)} \mathrm{Dec}^m_{\mathscr{C}}(\mathrm{D}(G, o)) \tag{4.3}$$

bigraded by $(k - |o|, |o|)$, where k is the homological degree. An isomorphism $\chi \colon G \to G'$ acts on this in the canonical way (i.e., it maps $\mathrm{Dec}^m_{\mathscr{C}}(\mathrm{D}(G, o))$ to $\mathrm{Dec}^m_{\mathscr{C}}(\mathrm{D}(G, \chi \,.\, o))$ via the map induced by $\mathrm{D}(\chi, o)$). This is precisely the following left Kan extension

$$\begin{array}{ccc} \int_{\mathbf{Graph}_{\geq 3}} \mathrm{dir} & \xrightarrow[\simeq]{\mathrm{D}} & \mathbf{DirGraph}^{\mathbf{trunc}} \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow{\scriptstyle \mathrm{Dec}^m_{\mathscr{C}}} \\ \mathbf{Graph}_{\geq 3} & \xrightarrow{\widetilde{C}} & \mathbf{BiGrVect} \end{array}$$

of $\mathrm{Dec}^m_{\mathscr{C}} \circ \mathrm{D}$ along the projection pr.

Since D is an equivalence of categories and since, by definition of C^2, there is an isomorphism of bigraded vector spaces $\mathrm{colim}_{\mathbf{DirGraph}^{\mathbf{trunc}}} \mathrm{Dec}^m_{\mathscr{C}} \cong C^2$, we obtain an isomorphism $\mathrm{colim}_{\mathbf{Graph}_{\geq 3}} \widetilde{C} \cong C^2$ of bigraded vector spaces. (This uses that the colimit of a left Kan extension is isomorphic to the colimit of the original functor.)

To construct the differential of $\widetilde{C}(G)$, we will use the technical lemma following the next definition.

Definition 4.15. Our standing assumption that $\mathscr{C}(\!(2)\!) = \mathbb{Q}\langle \mathrm{id} \rangle$ implies that the vector spaces $\mathrm{Dec}^m_{\mathscr{C}}(\mathrm{D}(G, o))$ are independent of o up to isomorphism (when $o(e) = 0$, the extra vertex v_e is labeled by an element of $\mathscr{C}(\!(2)\!) \cong \mathbb{Q}$). We denote by

$$c_{G,e} \colon \mathrm{Dec}^m_{\mathscr{C}}(\mathrm{D}(G, o_e^+)) \xrightarrow{\cong} \mathrm{Dec}^m_{\mathscr{C}}(\mathrm{D}(G, o_e^-))$$

the canonical such isomorphism (by a slight abuse, we do not include o in the notation). It is induced by the canonical bijection σ between the set of initial flags of $\mathrm{D}(G, o_e^+)$ and the set of initial flags of $\mathrm{D}(G, o_e^-)$ (i.e., it is given by $\mathrm{id} \otimes (\mathrm{s}^m \mathbb{Q})^{\otimes \sigma}$).

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Lemma 4.16. *There exist, for all* $G \in \mathbf{Graph}_{\geq 3}$, $e \in \mathrm{Edge}(G)$, *and* $o \in \mathrm{dir}(G)$ *such that* $o(e) = 0$, *two isomorphisms*

$$d^{+}_{G,e} \colon \mathrm{Dec}^{m}_{\mathscr{C}}(\mathrm{D}(G,o)) \longrightarrow \mathrm{Dec}^{m}_{\mathscr{C}}(\mathrm{D}(G,o^{+}_{e}))$$
$$d^{-}_{G,e} \colon \mathrm{Dec}^{m}_{\mathscr{C}}(\mathrm{D}(G,o)) \longrightarrow \mathrm{Dec}^{m}_{\mathscr{C}}(\mathrm{D}(G,o^{-}_{e}))$$

such that the following properties hold for all elements $? \in \{\pm\}$ *and* $x \in \mathrm{Dec}^{m}_{\mathscr{C}}(\mathrm{D}(G,o))$*:*

1. *Let* $\chi \colon G \to G'$ *be an isomorphism of* $\mathbf{Graph}_{\geq 3}$ *and set* $?' \coloneqq (\chi \,.\, o^{?}_{e})(\chi(e))$. *Then* $\chi \,.\, d^{?}_{G,e}(x) = d^{?'}_{G',\chi(e)}(\chi \,.\, x)$.
2. *Let* $e' \neq e$ *be another edge of* G *such that* $o(e') = 0$, *and let* $?' \in \{\pm\}$. *Then* $d^{?'}_{G,e'}(d^{?}_{G,e}(x)) = -d^{?}_{G,e}(d^{?'}_{G,e'}(x))$.
3. *We have* $r(d^{?}_{G,e}(x)) = d^{\mathrm{D}(G,o)}_{s^{?}_{e}}(r(x))$, *where* r *denotes the canonical map from* $\mathrm{Dec}^{m}_{\mathscr{C}}(\mathrm{D}(G,o))$ *respectively* $\mathrm{Dec}^{m}_{\mathscr{C}}(\mathrm{D}(G,o^{?}_{e}))$ *to* C^{2}.
4. *We have that* $d^{-}_{G,e} \circ (d^{+}_{G,e})^{-1} \colon \mathrm{Dec}^{m}_{\mathscr{C}}(\mathrm{D}(G,o^{+}_{e})) \to \mathrm{Dec}^{m}_{\mathscr{C}}(\mathrm{D}(G,o^{-}_{e}))$ *is equal to* $(-1)^{m} c_{G,e}$.

Assuming this lemma, we can define a differential on $\widetilde{C}(G)$ of bidegree $(0,-1)$ by

$$d \coloneqq \sum_{e \in \mathrm{Edge}(G)} (d^{+}_{G,e} + d^{-}_{G,e}) \tag{4.4}$$

(here $d^{?}_{G,e}$ is understood to be zero on $\mathrm{Dec}^{m}_{\mathscr{C}}(\mathrm{D}(G,o))$ when $o(e) \neq 0$). By the second property above, we have $d \circ d = 0$ and hence that $\widetilde{C}_{\bullet,*}(G) \coloneqq (\widetilde{C}(G), d)$ is a chain complex. Moreover, the first property implies functoriality of $\widetilde{C}_{\bullet,*}$, and the third property implies that the isomorphism $\mathrm{colim}_{\mathbf{Graph}_{\geq 3}} \widetilde{C} \cong C^{2}$ is compatible with the differentials.

Proof of Lemma 4.16. It will be convenient to construct the isomorphisms $d^{+}_{G,e}$ and $d^{-}_{G,e}$ simultaneously for all elements (G,e) of some fixed isomorphism class of the category of undirected graphs with a specified edge. To this end, we fix some pair $(G_0 = (F_0, N_0, \mu_0, a_0), e_0)$ in this isomorphism class such that the one or two vertices incident to e_0 are the first elements of N_0, such that a_0 is order preserving and such that the two flags making up e_0 come first in the linear orders of their respective vertices (if they are incident to the same vertex, we ask these two flags to be the first two elements).

Now let $o \in \mathrm{dir}(G_0)$ such that $o(e_0) = 0$ and let $x \in \mathrm{Dec}^{m}_{\mathscr{C}}(\mathrm{D}(G_0,o))$ be the element represented by some decorations $(\xi_v)_{v \in N_0}$. Then we set $d^{?}_{G_0,e_0}(x)$ to be $(-1)^{|N|m+n+b_?}$ times the element of $\mathrm{Dec}^{m}_{\mathscr{C}}(\mathrm{D}(G_0, o^{?}_{e_0}))$ represented by the ξ_v. Here,

$$b_{-} \coloneqq 0$$
$$b_{+} \coloneqq \begin{cases} m, & \text{if } e_0 \text{ is a loop} \\ m|v_{1,o}|^{\mathrm{out}}, & \text{otherwise,} \end{cases}$$

where $v_{1,o}$ is the first element of N_0 considered as a vertex of $\mathrm{D}(G_0,o)$.

We extend this to all pairs (G,e) by choosing some isomorphism $\chi \colon G \to G_0$ such that $\chi(e) = e_0$, and setting for $o \in \mathrm{dir}(G)$, $x \in \mathrm{Dec}^{m}_{\mathscr{C}}(\mathrm{D}(G,o))$, and $? \in \{\pm\}$,

$$d^{?}_{G,e}(x) \coloneqq \mathrm{D}(\chi, o^{?}_{e})^{-1} \,.\, d^{?'}_{G_0,e_0}(\mathrm{D}(\chi,o) \,.\, x), \tag{4.5}$$

where $?' \coloneqq (\chi \,.\, o^{?}_{e})(e_0)$. To see that this is well defined, we need to show that equation (4.5) holds when $(G,e) = (G_0,e_0)$ and χ is any automorphism of G_0 such that $\chi(e_0) = e_0$. This follows directly from comparing the signs induced by $\mathrm{D}(\chi,o)$ and $\mathrm{D}(\chi, o^{?}_{e_0})$ (distinguishing the cases of e_0 being a loop or

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not and χ flipping the two flags making up e_0 or not). The first property in the statement of the lemma is thus true by definition.

To prove the third property, it is enough, by equation (4.5) and the fact that $r(\chi \,.\, x) = r(x)$, to do so for the pair (G_0, e_0). In this case, the equality of the signs involved can be deduced easily from Definitions 4.12 and 3.24.

The second property follows from similar arguments: by equation (4.5), it is enough to check this for one member of each of the isomorphism classes of the category of undirected graphs with two distinct specified edges. Distinguishing various cases, corresponding to the different ways the vertices incident to the two edges can coincide or not, we choose representatives of the isomorphism classes that behave nicely with our construction above. In these cases, it is then straightforward (but tedious) to check the equality directly from the definition.

To show the fourth property, we first note that, for all isomorphisms $\chi\colon G \to G'$, the following diagram of isomorphisms commutes:

$$\begin{array}{ccc} \mathrm{Dec}^m_{\mathscr{C}}(\mathrm{D}(G, o^+_e)) & \xrightarrow{c_{G,e}} & \mathrm{Dec}^m_{\mathscr{C}}(\mathrm{D}(G, o^-_e)) \\ \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \chi} \\ \mathrm{Dec}^m_{\mathscr{C}}(\mathrm{D}(G', o^{?}_{\chi(e)})) & \xleftarrow{c_{G',\chi(e)}} & \mathrm{Dec}^m_{\mathscr{C}}(\mathrm{D}(G', o^{-?}_{\chi(e)})), \end{array}$$

where $? := (\chi \,.\, o)(\chi(e))$. Hence, it is enough to prove this property in the cases of the pairs (G_0, e_0). There, it is true by definition since $(-1)^{b_+ + b_- + m}$ is (regardless of whether e_0 is a loop or not) precisely the m-th power of the sign of the canonical bijection between the set of initial flags of $\mathrm{D}(G_0, o^+_{e_0})$ and the set of initial flags of $\mathrm{D}(G_0, o^-_{e_0})$. □

We now claim that, for any G, the homology of $\widetilde{C}_{\bullet,*}(G)$ is concentrated in bidegrees $(p, 0)$ (which implies that the spectral sequence (4.1) collapses on the second page). This follows immediately from the following more general lemma.

Lemma 4.17. *Let S be a finite set and $(V_o)_{o\in\mathrm{dir}(S)}$ a family of vector spaces. Furthermore, let there be, for each $s \in S$ and $o \in \mathrm{dir}(S)$ such that $o(s) = 0$, two isomorphisms $d^+_s\colon V_o \to V_{o^+_s}$ and $d^-_s\colon V_o \to V_{o^-_s}$. Assume that, for all $?, ?' \in \{\pm\}$ and $s \neq s' \in S$ such that $o(s) = 0 = o(s')$, we have an equality $d^?_s \circ d^{?'}_{s'} = -d^{?'}_{s'} \circ d^?_s$ of maps $V_o \to V_{o^{??'}_{ss'}} = V_{o^{?'?}_{s's}}$.*

Let $C := \bigoplus_{o\in\mathrm{dir}(S)} V_o$ be graded by the degree of o and equip it with the differential given by $d := \sum_{s\in S}(d^+_s + d^-_s)$ (here, we set $d^?_s(V_o) = 0$ for all o with $o(s) \neq 0$). Then C_* is a chain complex with homology concentrated in degree 0.

Proof. First note that the anticommutativity of the partial differentials implies that $d \circ d = 0$, so we only need to show the claim about the homology. Now choose some $s \in S$. Then C can be bigraded via

$$C_{p,0} := \bigoplus_{\substack{o\in\mathrm{dir}(S)\\ |o^{-1}(0)|=p\\ o(s)\neq 0}} V_o \qquad \text{and} \qquad C_{p,1} := \bigoplus_{\substack{o\in\mathrm{dir}(S)\\ |o^{-1}(0)|=p+1\\ o(s)=0}} V_o$$

and $C_{p,q} := 0$ for $q \notin \{0, 1\}$. The two differentials $d^1 := \sum_{s'\in S\setminus\{s\}}(d^+_{s'} + d^-_{s'})$ and $d^2 := d^+_s + d^-_s$ give this the structure of a first-quadrant double complex with total complex isomorphic to C. Hence, there is a convergent spectral sequence

$$E^1_{p,q} = \mathrm{H}_q(C_{p,*}, d^2) \;\Rightarrow\; \mathrm{H}_{p+q}(C_*) \tag{4.6}$$

with the differential on the first page induced by d^1 (see e.g. [Wei95, §5.6]).

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Note that $C_{p,*}$ splits as $\bigoplus_o C^o_{p,*}$ where o runs over those elements of $\mathrm{dir}(S)$ of degree $p+1$ such that $o(s) = 0$, and $C^o_{p,*}$ is the subcomplex

$$V_o \xrightarrow{(d^+_s, d^-_s)} V_{o^+_s} \oplus V_{o^-_s}$$

of $C_{p,*}$. Since d^+_s and d^-_s are both isomorphisms, the homology of $C^o_{p,*}$ is concentrated in degree 0. Moreover, the inclusion $V_{o^+_s} \to C^o_{p,*}$ induces an isomorphism on homology (where the former is considered as a chain complex concentrated in degree 0). Hence, setting

$$C^+ \coloneqq \bigoplus_{\substack{o \in \mathrm{dir}(S) \\ o(s) = +}} V_o$$

grading it by the degree of o and equipping it with d^1 as a differential, we obtain an isomorphism of chain complexes $C^+_\bullet \to \big(\mathrm{H}_0(C_{\bullet,*}, d^2), d^1\big)$. Since the E_1-page of the spectral sequence (4.6) is concentrated in the row $q = 0$, there thus is an isomorphism $\mathrm{H}_*(C) \cong \mathrm{H}_*(C^+)$. Hence, it is enough to show that the homology of C^+ is concentrated in degree 0.

To see this, we note that C^+ is a chain complex as described in the statement of this lemma, but with base set $S \setminus \{s\}$ instead of S. Thus, we can, by induction, reduce to the case where S is empty. In that case, the statement is trivially true, as any such chain complex is concentrated in degree 0. □

Let us recap what we did so far. We constructed the spectral sequence (4.1)

$$E^1_{p,q} = \mathrm{H}_q\Big(C^2_{p,*}\Big) \;\Rightarrow\; \mathrm{H}_{p+q}\big(\mathfrak{DGC}^m_{\mathrm{trunc}}(\mathscr{C})\big)$$

and, in (4.3) and (4.4), a functor $\widetilde{C}_{\bullet,*}$ from $\mathbf{Graph}_{\geq 3}$ to chain complexes in graded vector spaces such that there is an isomorphism $\mathrm{colim}\, \widetilde{C}_{\bullet,*} \cong C^2_{\bullet,*}$. Then we proved, in Lemma 4.17, that the pointwise homology $\mathrm{H}_* \circ \widetilde{C}_{\bullet,*}$ is concentrated in bidegrees $(p, 0)$.

Now note that $\mathbf{Graph}_{\geq 3}$ is a disjoint union of groupoids equivalent to finite groups, and hence that taking rational homology commutes with $\mathbf{Graph}_{\geq 3}$-colimits (since rational homology commutes with direct sums and quotients by finite groups). Hence, there are isomorphisms

$$\mathrm{H}_*\Big(C^2_{\bullet,*}\Big) \cong \mathrm{H}_*\Big(\operatorname*{colim}_{\mathbf{Graph}_{\geq 3}} \widetilde{C}_{\bullet,*}\Big) \cong \operatorname*{colim}_{\mathbf{Graph}_{\geq 3}} \Big(\mathrm{H}_* \circ \widetilde{C}_{\bullet,*}\Big) \tag{4.7}$$

of bigraded vector spaces. In particular, the homology of $C^2_{\bullet,*}$ is also concentrated in bidegrees $(p, 0)$. The spectral sequence then implies that the differential d^1 of the double complex $C_{*,*}$ induces a differential on $\mathrm{H}_0(C^2_{\bullet,*})$ such that there is an isomorphism

$$\mathrm{H}_\bullet\big(\mathfrak{DGC}^m_{\mathrm{trunc}}(\mathscr{C})\big) \cong \mathrm{H}_\bullet\Big(\mathrm{H}_0(C^2_{\bullet,*}), d^1\Big) \tag{4.8}$$

of graded vector spaces (Lemma 2.10 provides an explicit description of this isomorphism, which we will use later). Pulling back the differential d^1 along the isomorphism (4.7) hence yields a differential on the graded vector space $\mathrm{colim}\, (\mathrm{H}_0 \circ \widetilde{C}_{\bullet,*})$ such that the homology is isomorphic to the homology of $\mathfrak{DGC}^m_{\mathrm{trunc}}(\mathscr{C})$. This colimit, equipped with the pulled-back differential, is the undirected graph complex we are looking for. To prove this, we will first give an explicit description of the functor $\mathrm{H}_0 \circ \widetilde{C}_{\bullet,*}$ and then also identify the differential in these terms.

Definition 4.18. For an undirected graph $G \coloneqq (F, N, \mu, a) \in \mathbf{Graph}_{\geq 3}$, we set

$$\mathrm{UDec}^m_{\mathscr{C}}(G) \coloneqq (\mathrm{s}\mathbb{Q})^{\otimes N} \otimes (\mathrm{s}^{-m}\mathscr{C})(a) \otimes \bigotimes_{e \in \mathrm{Edge}(G)} \mathrm{s}^m\Big(\mathrm{Or}(e)^{\otimes m+1}\Big)$$

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and call it the *(graded) vector space of $\mathscr{C}$-decorations on G*. This yields a functor $\mathrm{UDec}^m_{\mathscr{C}}\colon \mathbf{Graph}_{\geq 3} \to \mathbf{GrVect}$. Moreover, we set

$$\mathfrak{UG}^m(\mathscr{C}) \coloneqq \operatorname*{colim}_{\mathbf{Graph}_{\geq 3}} \mathrm{UDec}^m_{\mathscr{C}}$$

and call it the *(graded) vector space of $\mathscr{C}$-decorated undirected graphs*.

Given a family $\left(\xi_v \in \mathscr{C}(a^{-1}(v))\right)_{v\in N}$ of elements of $\mathscr{C}$, we will call

$$\mathrm{s}^{|N|}1 \otimes \bigotimes_{v\in N} \mathrm{s}^{-m}\xi_v \otimes \mathrm{s}^{m|\mathrm{Edge}(G)|}1 \in \mathrm{UDec}^m_{\mathscr{C}}(G)$$

the element *represented* by (G and) the decorations ξ_v. By a slight abuse of terminology, we will use the same name for its image in $\mathfrak{UG}^m(\mathscr{C})$.

Lemma 4.19. *The following composition*

$$\mathrm{UDec}^m_{\mathscr{C}}(G) \xrightarrow{\cong} \mathrm{Dec}^m_{\mathscr{C}}(\mathrm{D}(G, \mathrm{const}_+)) \xrightarrow{\cong} \mathrm{H}_0(\widetilde{C}_{\bullet,*}(G))$$

is a natural isomorphism $\mathrm{UDec}^m_{\mathscr{C}} \to \mathrm{H}_0 \circ \widetilde{C}_{\bullet,*}$ of functors $\mathbf{Graph}_{\geq 3} \to \mathbf{GrVect}$. Here, the left-hand map is given by sending the element represented by G and some decorations ξ_v to the element represented by $\mathrm{D}(G, \mathrm{const}_+)$ and the same decorations ξ_v. The right-hand map is the composite of the canonical inclusion of the middle term into $\widetilde{C}_{\bullet,0}(G)$ followed by the projection onto the zeroth homology.

Proof. That the left-hand map is an isomorphism is clear. Hence, it is enough to show that the right-hand map is one as well, and that their composition is natural. We begin with the former.

By definition, we have

$$\mathrm{H}_0(\widetilde{C}_*(G)) = \left(\bigoplus_{\substack{o\in \mathrm{dir}(G)\\ |o|=0}} \mathrm{Dec}^m_{\mathscr{C}}(\mathrm{D}(G,o)) \right)/\sim,$$

where $\sim$ is generated by the relations $d^+_{G,e}(x) \sim -d^-_{G,e}(x)$ for each edge e of G, element $o \in \mathrm{dir}(S)$ such that $o^{-1}(0) = \{e\}$, and $x \in \mathrm{Dec}^m_{\mathscr{C}}(\mathrm{D}(G,o))$. The relation $\sim$ can be equivalently described as being generated by $x \sim -d^-_{G,e}\left((d^+_{G,e})^{-1}(x)\right)$ for each edge e, element $o \in \mathrm{dir}(S)$ with $|o| = 0$ and $o(e) = +$, and $x \in \mathrm{Dec}^m_{\mathscr{C}}(\mathrm{D}(G,o))$.

By the last property of Lemma 4.16, we have an equality of isomorphisms $-d^-_{G,e} \circ (d^+_{G,e})^{-1} = (-1)^{m+1}c_{G,e}$. We also have, by definition, that $c_{G,e} \circ c_{G,e'} = c_{G,e'} \circ c_{G,e}$. This implies that each of the canonical maps $\mathrm{Dec}^m_{\mathscr{C}}(\mathrm{D}(G,o)) \to \mathrm{H}_0(\widetilde{C}_*(G))$ is an isomorphism.

We will now identify the functorial structure of $\mathrm{H}_0 \circ \widetilde{C}_{\bullet,*}$ in these terms, which will imply the desired naturality. Let $\chi\colon G \to G'$ be an isomorphism and let $x \in \mathrm{Dec}^m_{\mathscr{C}}(\mathrm{D}(G, \mathrm{const}_+))$ be represented by some decorations $(\xi_v)_{v\in N}$, where N is the set of vertices of G. Then $\chi \,.\, x$ is $\varepsilon \in \{\pm 1\}$ times the element y of $\mathrm{Dec}^m_{\mathscr{C}}(\mathrm{D}(G', \chi \,.\, \mathrm{const}_+))$ represented by the decorations $(\chi \,.\, \xi_{\chi^{-1}(v)})_{v\in N'}$, where N' is the set of vertices of G'. This element y is equivalent under $\sim$ to

$$(-1)^{k(m+1)}\left(c^{-1}_{G',e_k} \circ \cdots \circ c^{-1}_{G',e_1}\right)(y) \in \mathrm{Dec}^m_{\mathscr{C}}(\mathrm{D}(G', \mathrm{const}_+)),$$

where $\{e_1, \ldots, e_k\} \coloneqq \{e \in \mathrm{Edge}\}(\mathrm{G'}) \mid (\chi \,.\, \mathrm{const}_+)(\mathrm{e}) \neq +$. We hence obtain that $\chi \,.\, x$ is equivalent under $\sim$ to $\varepsilon(-1)^{k(m+1)}\,\mathrm{sgn}(\sigma)^m$ times the element of $\mathrm{Dec}^m_{\mathscr{C}}(\mathrm{D}(G', \mathrm{const}_+))$ represented by the decorations $(\chi \,.\, \xi_{\chi^{-1}(v)})_{v\in N'}$, where σ is the canonical bijection between the set of initial flags of $\mathrm{D}(G', \chi \,.\, \mathrm{const}_+)$ and the set of initial flags of $\mathrm{D}(G', \mathrm{const}_+)$. Finally, we note that, under the canonical

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identification of $\mathrm{Edge}(G)$ with the initial flags S of $\mathrm{D}(G, \mathrm{const}_+)$ and the analogous identification for G' (both of which are order preserving), the composition $\sigma \circ \mathrm{D}(\chi, \mathrm{const}_+)|_S$ corresponds precisely to the map $\mathrm{Edge}(G) \to \mathrm{Edge}(G')$ induced by χ. This finishes the proof. □

Combining Lemma 4.19 with the isomorphism (4.7), we obtain an isomorphism

$$\mathfrak{UG}^m(\mathscr{C}) = \operatorname*{colim}_{\mathbf{Graph}_{\geq 3}} \mathrm{UDec}^m_{\mathscr{C}} \cong \operatorname*{colim}_{\mathbf{Graph}_{\geq 3}} (\mathrm{H}_0 \circ \widetilde{C}_{\bullet,*}) \cong \mathrm{H}_0(C^2_{\bullet,*}) \tag{4.9}$$

of graded vector spaces. As explained after (4.8), we will now identify the result of pulling back the differential d^1 of $\mathrm{H}_0(C^2_{\bullet,*})$ to $\mathfrak{UG}^m(\mathscr{C})$ along the isomorphism (4.9). Moreover, note that, via the isomorphisms (4.8) and (4.9), the graded coalgebra structure of $\mathrm{H}_\bullet\big(\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})\big)$ determines a graded coalgebra structure on $\mathrm{H}_\bullet(\mathfrak{UG}^m(\mathscr{C}))$. We will also describe this graded coalgebra structure explicitly.

Definition 4.20. Let $G = (F, N, \mu, a) \in \mathbf{Graph}_{\geq 3}$ be an undirected graph. A *neighbor-closed vertex set* of G is a subset $N' \subseteq N$ such that $\mu(a^{-1}(N')) = a^{-1}(N')$. A *connected component* of G is a neighbor-closed vertex set that is inclusion minimal among nonempty neighbor-closed vertex sets.

A neighbor-closed vertex set N' of G determines an undirected graph

$$\mathrm{ind}_G(N') \coloneqq (F', N', \mu|_{F'}, a|_{F'}) \in \mathbf{Graph}_{\geq 3},$$

where $F' \coloneqq a^{-1}(N')$.

Remark 4.21. When $N' \subseteq N$ is a neighbor-closed vertex set of an undirected graph $G = (F, N, \mu, a)$, then $N \setminus N'$ is one as well. Moreover, any neighbor-closed vertex set of G is a disjoint union of a unique set of connected components of G.

Definition 4.22. We equip the graded vector space $\mathfrak{UG}^m(\mathscr{C})$ of $\mathscr{C}$-decorated undirected graphs with the structure of a cocommutative differential graded coalgebra in the following way. The differential is given by

$$d([x]) \coloneqq \sum_{e \in \mathrm{Edge}(G)} d^G_e(x),$$

where $G = (F, N, \mu, a) \in \mathbf{Graph}_{\geq 3}$ is an undirected graph and $x \in \mathrm{UDec}^m_{\mathscr{C}}(G)$. Here, the $d^G_e(x)$ are elements of $\mathfrak{UG}^m(\mathscr{C})$ uniquely determined by the following properties:

- For every isomorphism $\chi\colon G \to G'$ of undirected graphs holds $d^G_e(x) = d^{G'}_{\chi(e)}(\chi \,.\, x)$.
- If e is a loop, then $d^G_e(x) = 0$.
- If e is not a loop, let $f < f'$ be the two flags making up e and assume that $v \coloneqq a(f)$ and $v' \coloneqq a(f')$ are, in this order, the first two elements of N. Moreover, assume that a is order-preserving and that f is the first element of F (in particular, e is the first element of $\mathrm{Edge}(G)$). Then d^G_e applied to the element represented by G and some decorations $(\xi_w)_{w \in N}$ is the element represented by the graph

$$\mathrm{ctr}_e(G) \coloneqq \Big(F \setminus e, N/(v \sim v'), \mu|_{F\setminus e}, a' \coloneqq \mathrm{pr} \circ a|_{F\setminus e}\Big)$$

(obtained from G by contracting e) decorated by $\omega \,.\, (\xi_v \;{}_f\circ_{f'}\; \xi_{v'})$ at the collapsed vertex $\bar{v} \coloneqq \mathrm{pr}(v) = \mathrm{pr}(v')$ and by ξ_w at all other vertices w. Here, ω is the canonical bijection between the linearly ordered set $a^{-1}(v) \;{}_f\circ_{\mu(f)}\; a^{-1}(v')$ of Remark 2.32 and $a'^{-1}(\bar{v}) \subseteq F \setminus e$.

The counit $\varepsilon\colon \mathfrak{UG}^m(\mathscr{C}) \to \mathbb{Q}$ sends the element represented by the empty graph to 1 and an element represented by any other graph to 0. The comultiplication $\Delta\colon \mathfrak{UG}^m(\mathscr{C}) \to \mathfrak{UG}^m(\mathscr{C}) \otimes \mathfrak{UG}^m(\mathscr{C})$ is given as follows: for an element $x \in \mathfrak{UG}^m(\mathscr{C})$ represented by $G = (F, N, \mu, a)$ and decorations $(\xi_v)_{v \in N}$,

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we set

$$\Delta(x) \coloneqq \sum_{N'} (-1)^{m|N'||N\setminus N'|} \delta(N') x_{N'} \otimes x_{N\setminus N'},$$

where the sum runs over all neighbor-closed vertex subsets N' of G and x_A is the element of $\mathfrak{UG}^m(\mathscr{C})$ represented by $\mathrm{ind}_G(A)$ and $(\xi_v)_{v\in A}$. The sign $\delta(N') \in \{\pm 1\}$ is the sign incurred by permuting

$$(\mathrm{s}1)^{\otimes N} \otimes \bigotimes_{v\in N} \mathrm{s}^{-m}\xi_v \otimes (\mathrm{s}^m 1)^{\otimes \mathrm{Edge}(G)}$$

into

$$(\mathrm{s}1)^{\otimes N'} \otimes \bigotimes_{v\in N'} \mathrm{s}^{-m}\xi_v \otimes (\mathrm{s}^m 1)^{\otimes \mathrm{Edge}(G_1)} \otimes (\mathrm{s}1)^{\otimes N\setminus N'} \otimes \bigotimes_{v\in N\setminus N'} \mathrm{s}^{-m}\xi_v \otimes (\mathrm{s}^m 1)^{\otimes \mathrm{Edge}(G_2)},$$

where $G_1 \coloneqq \mathrm{ind}_G(N')$ and $G_2 \coloneqq \mathrm{ind}_G(N \setminus N')$.

Remark 4.23. Note that it is easy to see that Δ is well defined (i.e., that $\Delta(\chi \,.\, x) = \Delta(x)$ for any isomorphism $\chi\colon G \to G'$). In the proof of Theorem 4.28, we will show that the elements $d_e^G(x)$ actually exist, which in particular implies that d is a well-defined map. Moreover, we will construct a surjection from a cocommutative differential graded coalgebra onto $\mathfrak{UG}^m(\mathscr{C})$ that is compatible with the differentials, the counits and the comultiplications. This implies that the maps above indeed equip $\mathfrak{UG}^m(\mathscr{C})$ with the structure of a cocommutative differential graded coalgebra.

Remark 4.24. The *(undirected) graph complex* $\mathfrak{UG}^m(\mathscr{C})$, equipped with the differential graded coalgebra structure of Definition 4.22, is a well-known object. Variants of it were first described by Kontsevich [Kon93, Kon94] and have, since then, appeared in the works of many different authors. Using the language of modular operads due to Getzler–Kapranov [GK98], one alternative description is

$$\mathfrak{UG}^m(\mathscr{C})^\vee \cong \mathsf{F}_{\mathfrak{T}^{1-m}}(\Sigma^{1-m}\mathscr{C}) \cong \Sigma^{m-1}\mathsf{F}_{\mathrm{Det}^{1-m}}(\mathscr{C}),$$

where F denotes the 'Feynman transform', $\mathfrak{T}$ and Det are certain 'hyperoperads', and Σ denotes a degree shift. Other descriptions have been given for example by Lazarev–Voronov [LV08, §3.1] (though with a different grading convention) and, in the case $m = 0$, by Conant–Vogtmann [CV03, §2.3].

The homology of $\mathfrak{UG}^m(\mathscr{C})$ is in general very complicated and in most cases of interest, only partial information is known. See, however, the following remarks for a couple of basic observations. Also see Remark 4.30 for an overview of what is known about this homology in the case of the cyclic Lie operad $\mathscr{C} = \mathscr{L}\mathrm{ie}$.

Remark 4.25. Note that, up to a regrading, the differential graded coalgebra $\mathfrak{UG}^m(\mathscr{C})$, and hence its homology, only depend on the parity of m.

Remark 4.26. It follows from the description of the coalgebra structure of $\mathfrak{UG}^m(\mathscr{C})$ that the cohomology $\mathrm{H}^*\big(\mathfrak{UG}^m(\mathscr{C})^\vee\big)$ of its linear dual is a free graded commutative algebra generated by the cohomology of the dual of the subcomplex $\mathfrak{UG}^m_{\mathrm{conn}}(\mathscr{C}) \subseteq \mathfrak{UG}^m(\mathscr{C})$ spanned by the connected graphs. This subcomplex splits as a direct sum

$$\mathfrak{UG}^m_{\mathrm{conn}}(\mathscr{C}) \cong \bigoplus_{g\in\mathbb{N}_0} \mathfrak{UG}^m_{\mathrm{conn};g}(\mathscr{C}),$$

where $\mathfrak{UG}^m_{\mathrm{conn};g}(\mathscr{C})$ denotes the subcomplex spanned by connected graphs of genus g (here, the *genus* of a graph is 1 plus the number of edges minus the number of vertices). Hence, understanding the homology of $\mathfrak{UG}^m(\mathscr{C})$ (or the cohomology of its dual) is equivalent to understanding the homology of $\mathfrak{UG}^m_{\mathrm{conn};g}(\mathscr{C})$ for all g (or the cohomologies of their duals).

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Remark 4.27. Let $G \in \mathbf{Graph}_{\geq 3}$ be a connected graph with N vertices, E edges and genus $g = 1-N+E$. Since every vertex is at least trivalent, we have $E \geq \frac{3}{2}N$, which implies $E \leq 3(g-1)$. Since G is nonempty, we have $N \geq 1$, which implies $g \leq E$. (In particular, there are no such graphs with $g \leq 1$.) Now assume that $\mathscr{C}$ is concentrated in degree 0. Then an element of $\mathfrak{UG}^m(\mathscr{C})$ represented by G has homological degree $p = (1-m)N + mE = E + (m-1)(g-1)$. This implies that, in this case, the subcomplex $\mathfrak{UG}^m_{\mathrm{conn};g}(\mathscr{C})$ of Remark 4.26 is concentrated in those homological degrees p such that $m(g-1)+1 \leq p \leq (m+2)(g-1)$.

Theorem 4.28. *Let $m \in \mathbb{N}_0$ be a natural number and $\mathscr{C}$ a cyclic operad concentrated in nonnegative even degrees such that $\mathscr{C}(\!(2)\!) = \mathbb{Q}\langle \mathrm{id}\rangle$ with the trivial Σ_2-action. Then there is an isomorphism of graded coalgebras*

$$\mathrm{H}_*(\mathfrak{UG}^m(\mathscr{C})) \cong \mathrm{H}_*\big(\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})\big),$$

where $\mathfrak{UG}^m(\mathscr{C})$ is equipped with the differential graded coalgebra structure of Definition 4.22.

Proof. We first show that there is an isomorphism $\mathfrak{UG}^m(\mathscr{C}) \to \mathrm{H}_0(C^2_{\bullet,*})$ of chain complexes, where the domain is equipped with the differential of Definition 4.22 and the target with the differential induced by d^1. Together with (4.8), this yields the desired isomorphism on the level of the underlying graded vector spaces.

The isomorphism $\Phi\colon \mathfrak{UG}^m(\mathscr{C}) \to \mathrm{H}_0(C^2_{\bullet,*})$ of (4.9) maps the element $[x] \in \mathfrak{UG}^m(\mathscr{C})$ represented by $G = (F, N, \mu, a)$ and decorations $(\xi_v)_{v\in N}$ to the element $[y]$ of $\mathrm{H}_0(C^2_{\bullet,*})$ represented by the directed graph $D_+(G) \coloneqq \mathrm{D}(G, \mathrm{const}_+)$ and the decorations ξ_v. By Definition 3.24, we have $d^1([y]) = \sum_{s\in S} d_s^{D_+(G)}(y)$, where S is the set of initial flags of $D_+(G)$. There is a canonical bijection $\sigma^G\colon \mathrm{Edge}(G) \to S$ (which is order preserving). We now define, for $e \in \mathrm{Edge}(G)$,

$$d_e^G(x) \coloneqq \Phi^{-1}\Big(\big[d^{D_+(G)}_{\sigma^G(e)}(\widetilde{\Phi}^G(x))\big]\Big),$$

where $\widetilde{\Phi}^G\colon \mathrm{UDec}^m_{\mathscr{C}}(G) \to \mathrm{Dec}^m_{\mathscr{C}}(D_+(G))$ is the isomorphism from Lemma 4.19. We will now prove that these elements $d_e^G(x)$ fulfill the properties of Definition 4.22. This implies that the map Φ is an isomorphism of chain complexes as desired.

First note that, if e is a loop, then $d^{D_+(G)}_{\sigma^G(e)}(\widetilde{\Phi}(x)) = 0$ and hence $d_e^G(x) = 0$. Hence, the second property of Definition 4.22 is fulfilled.

Now we show that $d_e^G(x) = d^{G'}_{\chi(e)}(\chi\,.\,x)$ for any isomorphism $\chi\colon G \to G'$ and edge $e = \{f_1 < f_2\} \in \mathrm{Edge}(G)$. This is clear when e is a loop. Hence, it is enough to consider the case where G fulfills that a is order preserving, that $v_1 \coloneqq a(f_1)$ and $v_2 \coloneqq a(f_2)$ are (in this order) the first two elements of N, and that f_1 is the last element of $a^{-1}(v_1)$ and f_2 the first of $a^{-1}(v_2)$. (This is enough since any isomorphism of undirected graphs with a distinguished non-loop edge factors through a graph of this form.) We need to show that

$$\big[d^{D_+(G)}_{\sigma^G(e)}(\widetilde{\Phi}^G(x))\big] = \big[d^{D_+(G')}_{\sigma^{G'}(\chi(e))}(\widetilde{\Phi}^{G'}(\chi\,.\,x))\big] \tag{4.10}$$

as elements of $\mathrm{H}_0(C^2_{\bullet,*})$. We first note that the left-hand side is equal to $(-1)^{m(|v_1|^{\mathrm{out}}-1)}$ times the equivalence class of the element y represented by $\mathrm{ctr}_{\sigma^G(e)}(D_+(G))$ and the decorations $\xi_{v_1}\,{}_{f_1}\circ_{f_2}\xi_{v_2}$ at the collapsed vertex $\mathrm{pr}(v_1) = \mathrm{pr}(v_2)$ and by ξ_w at all other vertices w. Now, we set $D' \coloneqq \mathrm{D}(G, \chi^{-1}\,.\,\mathrm{const}_+)$ and write $\psi\colon D' \to \mathrm{D}(G', \mathrm{const}_+) = D_+(G')$ for the isomorphism induced by χ. We further denote by S' the linearly ordered set of initial flags of D' and set $s' \coloneqq \psi^{-1}(\sigma^{G'}(\chi(e))) \in S'$. Then the right-hand side of (4.10) is equal to $[d^{D'}_{s'}(\psi^{-1}\,.\,\widetilde{\Phi}^{G'}(\chi\,.\,x))]$. It follows from the definitions that $\psi^{-1}\,.\,\widetilde{\Phi}^{G'}(\chi\,.\,x)$ is $\mathrm{sgn}(\omega)^m(-1)^{k(1+m)}$ times the element x' of $\mathrm{Dec}^m_{\mathscr{C}}(D')$ represented by the decorations $(\xi_w)_{w\in N}$, where k is the number of edges whose orientation is flipped by χ, and $\omega\colon S \to S'$ is the canonical bijection.

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There are now two cases: either χ preserves the orientation of e (i.e., we have $\chi(f_1) < \chi(f_2)$) or it reverses it. In both cases, we have that $d_{s'}^{D'}(x')$ is some sign $\varepsilon \in \{\pm 1\}$ times the element y' represented by $\mathrm{ctr}_{s'}(D')$ and the decorations $\xi_{v_1\ f_1 \circ f_2}\ \xi_{v_2}$ (in the first case) or $\xi_{v_2\ f_2 \circ f_1}\ \xi_{v_1}$ (in the second case) at the collapsed vertex $\mathrm{pr}(v_1) = \mathrm{pr}(v_2)$ and ξ_w at all other vertices w. In the first case, we have $\varepsilon = (-1)^{m(|v_1'|^{\mathrm{out}}-1)}$, where $v_1' := v_1$ considered as a vertex of D'. In the second case, we have $\varepsilon = (-1)^{1+m+m|v_1'|^{\mathrm{out}}}$ (to see this, one transforms D' into the form necessary to read off $d_{s'}^{D'}$ and then identifies the result of the contraction with $\mathrm{ctr}_{s'}(D')$). It follows from the proof of Lemma 4.19 that $[y'] = \delta[y] \in \mathrm{H}_0(C^2_{\bullet,*})$, where $\delta = \mathrm{sgn}(\hat{\omega})^m(-1)^{k(1+m)}$ in the first case, and $\delta = \mathrm{sgn}(\hat{\omega})^m(-1)^{(k-1)(1+m)}$ in the second. Here, $\hat{\omega}$ is the canonical bijection $S \setminus \{s\} \to S' \setminus \{s'\}$, and k is the number of edges flipped by χ, as above. Lastly, we note that commutativity of

$$\begin{array}{ccc} \{s\} \amalg (S \setminus \{s\}) & \xrightarrow{\cong} & S \\ {\scriptstyle \hat{\omega}}\downarrow & & \downarrow{\scriptstyle \omega} \\ \{s'\} \amalg (S' \setminus \{s'\}) & \xrightarrow{\cong} & S' \end{array}$$

implies that the product $\mathrm{sgn}(\omega)\,\mathrm{sgn}(\hat{\omega})$ equals $(-1)^{|v_1|^{\mathrm{out}}-1+|v_1'|^{\mathrm{out}}-1}$ in the first case, and $(-1)^{|v_1|^{\mathrm{out}}-1+|v_1'|^{\mathrm{out}}}$ in the second. This finishes the proof of (4.10).

We now prove that the elements $d_e^G(x)$ fulfill the third property of Definition 4.22. We assume that $e = \{f_1 < f_2\}$ is not a loop, that $v_1 := a(f_1)$ and $v_2 := a(f_2)$ are (in this order) the first two elements of N, that a is order-preserving, and that f_1 is the first element of F. We also assume that f_2 is the first element of $a^{-1}(v_2)$ (we can arrange this without introducing a sign). Now, we let D' be equal to $D_+(G)$ except that the linear order of F has been changed such that f_1 is the last element of $a^{-1}(v_1)$ instead of the first. Moreover, we denote by $\chi\colon D_+(G) \to D'$ the canonical isomorphism. We have $d_{f_1}^{D_+(G)}(\widetilde{\Phi}(x)) = d_{f_1}^{D'}(\chi\,.\,\widetilde{\Phi}(x))$ and that $\chi\,.\,\widetilde{\Phi}(x)$ is $(-1)^{m(|v_1|^{\mathrm{out}}-1)}$ times the element represented by D' and the decorations $(\xi_w)_{w\in N}$. By definition, we thus have that $d_{f_1}^{D'}(\chi\,.\,\widetilde{\Phi}(x))$ is the element represented by $\mathrm{ctr}_{f_1}(D')$ and the decorations $(\sigma|_{a^{-1}(v_1)}\,.\,\xi_{v_1})_{\ f_1\circ f_2}\ \xi_{v_2}$ at the collapsed vertex and ξ_w at all other vertices w. Here, σ is the canonical bijection from F to the linearly ordered set F' of flags of D'. Lastly, we note that $\mathrm{ctr}_{f_1}(D') = \mathrm{D}(\mathrm{ctr}_e(G), \mathrm{const}_+)$ and that $(\sigma|_{a^{-1}(v_1)}\,.\,\xi_{v_1})_{\ f_1\circ f_2}\ \xi_{v_2} = \omega\,.\,(\xi_{v_1\ f_1\circ f_2}\ \xi_{v_2})$, where ω is as in Definition 4.22. Thus, the third property is fulfilled as well.

It remains to prove that the isomorphism is compatible with the graded coalgebra structures. By Lemma 2.10, the isomorphism (4.8) is induced by the surjection of chain complexes

$$\Psi\colon \mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C}) \longrightarrow \left(\mathrm{H}_0(C^2_{\bullet,*}), d^1\right)$$

that is given by the canonical projections $\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})_{k,0} \to \mathrm{H}_0(C^2_{\bullet,*})$ and the trivial maps on $\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})_{k,l}$ for $l > 0$. Hence, it is enough to show that the surjection

$$\Theta := \Phi^{-1} \circ \Psi\colon \mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C}) \longrightarrow \mathfrak{UG}^m(\mathscr{C})$$

is a map of graded coalgebras.

It is clear that Θ is compatible with the counits. Now note that the comultiplication Δ of $\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})$ is compatible with its bigrading. In particular, $(\Psi\otimes\Psi)(\Delta(x))$ is trivial for an element $\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})_{k,l}$ with $l > 0$. Hence, it is enough to check that Θ is compatible with the comultiplication of $\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})_{*,0}$. Let $D = (F, N, S, T, \mu, a)$ be a truncated directed graph without vertices of valence two and let $x \in \mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})_{*,0}$ be the element represented by D and some decorations $(\xi_v)_{v\in N}$. Let $o_D \in \mathrm{dir}(\mathrm{U}(D))$ be defined by setting $o_D(\{f_1 < f_2\})$ to be $+$ if $f_1 \in S$ and $-$ otherwise. Then $\mathrm{D}(\mathrm{U}(D), o_D)$ is equal to D. By the proof of Lemma 4.19, this implies that $\Theta(x)$ is $\alpha(D) := \mathrm{sgn}(\sigma_D)^m(-1)^{k_D(1+m)}$ times the element of $\mathfrak{UG}^m(\mathscr{C})$ represented by $\mathrm{U}(D)$ and the decorations ξ_v. Here, $k_D := |o_D^{-1}(-)|$ and $\sigma_D\colon S \to \mathrm{Edge}(\mathrm{U}(D))$ is the canonical bijection.

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Recall that

$$\Delta(x) := \sum_{N'} (-1)^{m|N'||N\setminus N'|} \varepsilon(N') x_{N'} \otimes x_{N\setminus N'},$$

where the sum runs over all neighbor-closed vertex subsets N' of D and x_A is the element of $\mathfrak{DG}^m(\mathscr{C})$ represented by $\mathrm{ind}_D(A)$ and $(\xi_v)_{v\in A}$. The sign $\varepsilon(N')$ is specified in Definition 3.24. Applying $\Theta\otimes\Theta$ to the right-hand side, we obtain

$$\sum_{N'} (-1)^{m|N'||N\setminus N'|} \varepsilon(N') y_{N'} \otimes y_{N\setminus N'},$$

where y_A is $\alpha(\mathrm{ind}_D(A))$ times the element of $\mathfrak{UG}^m(\mathscr{C})$ represented by $\mathrm{U}(\mathrm{ind}_D(A)) = \mathrm{ind}_{\mathrm{U}(D)}(A)$ and $(\xi_v)_{v\in A}$. Note that a subset $N' \subseteq N$ is a neighbor-closed vertex subset of D if and only if it is a neighbor-closed vertex subset of $\mathrm{U}(D)$. Lastly, we observe that

$$\alpha(D)\alpha(\mathrm{ind}_D(N'))\alpha(\mathrm{ind}_D(N\setminus N'))\varepsilon(N') = \delta(N'),$$

where $\delta(N')$ is the sign of Definition 4.22. This finishes the proof. □

Remark 4.29. A proof of Theorem 4.28 for $\mathfrak{DG}^m(\mathscr{C})$ instead of $\mathfrak{DG}^m_{\mathrm{trunc}}(\mathscr{C})$ has been sketched by Willwacher [Wil14, Appendix K] (in the case of the commutative cyclic operad $\mathscr{C} = \mathscr{C}\mathrm{om}$). Other related statements have been proven in detail by Živković [Živ20] and Dolgushev–Rogers [DR19].

Remark 4.30. We quickly survey what is known about the homology of the undirected graph complex $\mathfrak{UG}^m(\mathscr{L}\mathrm{ie})$ associated to the cyclic Lie operad. By work of Kontsevich [Kon93] (see also Conant–Vogtmann [CV03, Theorem 2 and §3.1]) and Lazarev–Voronov [LV08, Corollary 3.13], there are, for $g \geq 2$, isomorphisms

$$\mathrm{H}^p\left(\mathfrak{UG}^m_{\mathrm{conn};g}(\mathscr{L}\mathrm{ie})^\vee\right) \cong \mathrm{H}_{(m+2)(g-1)-p}\left(\mathrm{Out}(\mathrm{F}_g); \widetilde{\mathbb{Q}}^{\otimes m}\right),$$

where $\mathfrak{UG}^m_{\mathrm{conn};g}(\mathscr{L}\mathrm{ie})$ is as in Remark 4.26. (Note that it follows from Remark 4.27 that $\mathfrak{UG}^m_{\mathrm{conn};g}(\mathscr{L}\mathrm{ie})$ is trivial for $g \leq 1$.) Here, $\mathrm{Out}(\mathrm{F}_g)$ is the group of outer automorphisms of the free group on g generators, and $\widetilde{\mathbb{Q}}$ is the one-dimensional $\mathrm{Out}(\mathrm{F}_g)$ representation given by

$$\mathrm{Out}(\mathrm{F}_g) \xrightarrow{\mathrm{ab}} \mathrm{Out}(\mathrm{F}_g^{\mathrm{ab}}) \cong \mathrm{GL}_g(\mathbb{Z}) \xrightarrow{\det} \mathrm{GL}_1(\mathbb{Z}) \longrightarrow \mathrm{GL}_1(\mathbb{Q})$$

(which is sometimes called the 'determinant representation').

The homology $\mathrm{H}_*(\mathrm{Out}(\mathrm{F}_g); \mathbb{Q})$, occurring when m is even, has been studied intensively. A recent state of the art is summarized in [CHKV16, §1], including certain stability results and computer calculations for small g. The only nontrivial homology (outside of degree 0) with $g \leq 4$ is $\mathrm{H}_4(\mathrm{Out}(\mathrm{F}_4); \mathbb{Q}) \cong \mathbb{Q}$. More recently, Borinsky–Vogtmann [BV23] obtained a formula and asymptotics for the Euler characteristic of $\mathrm{H}_*(\mathrm{Out}(\mathrm{F}_g); \mathbb{Q})$.

The homology $\mathrm{H}_*(\mathrm{Out}(\mathrm{F}_g); \widetilde{\mathbb{Q}})$, occurring when m is odd, is less well studied. The only structural results known to the author are Euler characteristic computations and asymptotics, again from [BV23], as well as that

$$\mathrm{H}_0\left(\mathrm{Out}(\mathrm{F}_g); \widetilde{\mathbb{Q}}\right) \cong \widetilde{\mathbb{Q}}_{\mathrm{Out}(\mathrm{F}_g)} \cong 0 \qquad \text{and} \qquad \mathrm{H}_*\left(\mathrm{Out}(\mathrm{F}_2); \widetilde{\mathbb{Q}}\right) \cong 0,$$

the latter of which follows from the fact that $\mathrm{Out}(\mathrm{F}_2) \cong \mathrm{GL}_2(\mathbb{Z})$ and the Eichler–Shimura isomorphism; see Conant–Hatcher–Kassabov–Vogtmann [CHKV16, §3.5]. Apart from this, there are computer calculations for small g by Brun–Willwacher [BW23, Figure 9]. Their work shows that the only nontrivial

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homology with $g \leq 4$ is $\mathrm{H}_3(\mathrm{Out}(\mathrm{F}_4); \widetilde{\mathbb{Q}}) \cong \mathbb{Q}$. Using Remark 4.27, this implies that $\mathrm{H}_p(\mathfrak{UG}^m(\mathscr{L}\mathrm{ie}))$ is trivial for $0 < p \leq \min(3m+2, 4m)$ when m is odd.

5. Connected sums of products of spheres

Throughout this section, we let $3 \leq k < l \leq 2k-2$ be two integers. We set

$$M_{g,r}^{k,l} := \#_g(\mathrm{S}^k \times \mathrm{S}^l) \setminus \coprod_r \mathring{\mathrm{D}}^{k+l};$$

that is, a g-fold connected sum of $\mathrm{S}^k \times \mathrm{S}^l$ with r disjoint $(k+l)$-dimensional open disks removed (we will often omit k and l from the notation and simply write $M_{g,r}$). We are interested in the stable cohomology of

$$\mathrm{Baut}_\partial(M_{g,1});$$

that is, of the classifying space of the group-like topological monoid of homotopy equivalences from $M_{g,1}$ to itself that fix the boundary pointwise. To make sense of this, we need to specify stabilization maps $\mathrm{Baut}_\partial(M_{g,1}) \to \mathrm{Baut}_\partial(M_{g+1,1})$. To this end, we note that there is a canonical inclusion

$$\iota_g \colon M_{g,1} = M_{g,0} \setminus \mathring{\mathrm{D}}^{k+l} \hookrightarrow M_{g,0} \# M_{1,1} = M_{g+1,1}$$

(where # denotes connected sum), which induces a map of topological monoids

$$\phi_g \colon \mathrm{aut}_\partial(M_{g,1}) \longrightarrow \mathrm{aut}_\partial(M_{g+1,1})$$

by extending an element of $\mathrm{aut}_\partial(M_{g,1})$ by the identity.

We will now introduce some related objects, which we will need for stating and proving our main result. We start by fixing some basepoints of S^k and S^l. Then, for $1 \leq i \leq g$, we denote by $\alpha_i^g \in \mathrm{H}_k(M_{g,1}; \mathbb{Z})$ the homology class represented by the canonical inclusion of S^k into the i-th summand, and by $(\alpha_i^g)^\# \in \mathrm{H}_l(M_{g,1}; \mathbb{Z})$ the class represented by the canonical inclusion of S^l into the i-th summand (we assume that these inclusions do not intersect the disks used to form the connected sum). These classes form bases of $\mathrm{H}_k(M_{g,1}; \mathbb{Z}) \cong \mathbb{Z}^g$ and $\mathrm{H}_l(M_{g,1}; \mathbb{Z}) \cong \mathbb{Z}^g$, respectively. We choose the orientations such that the intersection pairing fulfills $\langle \alpha_i^g, (\alpha_i^g)^\# \rangle_\cap = 1$. Also note that $\langle \alpha_i^g, (\alpha_j^g)^\# \rangle_\cap = 0$ when $i \neq j$. For $1 \leq i \leq g$, we moreover have $(\iota_g)_*(\alpha_i^g) = \alpha_i^{g+1}$ and $(\iota_g)_*\big((\alpha_i^g)^\#\big) = (\alpha_i^{g+1})^\#$.

Using this, we can define a stabilization map of groups

$$\psi_g \colon \mathrm{Aut}\Big(\widetilde{\mathrm{H}}_*(M_{g,1}; \mathbb{Z})\Big) \longrightarrow \mathrm{Aut}\Big(\widetilde{\mathrm{H}}_*(M_{g+1,1}; \mathbb{Z})\Big)$$

by letting $\psi_g(f)$ be the unique automorphism such that $\psi_g(f) \circ (\iota_g)_* = (\iota_g)_* \circ f$ and such that $\psi_g(f)$ is the identity on α_{g+1}^{g+1} and $(\alpha_{g+1}^{g+1})^\#$. (Here, Aut denotes automorphisms of graded abelian groups.) We now set

$$\Gamma_g := \mathrm{im}\Big(\pi_0(\mathrm{aut}_\partial(M_{g,1})) \to \mathrm{Aut}(\widetilde{\mathrm{H}}_*(M_{g,1}; \mathbb{Z}))\Big);$$

that is, it is the group of those automorphisms of the reduced integral homology of $M_{g,1}$ that can be realized by a homotopy automorphism that fixes the boundary pointwise. Together, the maps ϕ_g and ψ_g induce a stabilization map of groups $\gamma_g \colon \Gamma_g \to \Gamma_{g+1}$. The group Γ_g has been identified explicitly by Grey [Gre19].

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Proposition 5.1 (Grey). *We have*

$$\Gamma_g = \mathrm{Aut}\Big(\widetilde{\mathrm{H}}_*(M_{g,1};\mathbb{Z}), \langle -,-\rangle_\cap\Big) \subseteq \mathrm{Aut}\Big(\widetilde{\mathrm{H}}_*(M_{g,1};\mathbb{Z})\Big),$$

where $\langle -,-\rangle_\cap$ is the intersection pairing. Moreover, the canonical map

$$\pi_0\big(\mathrm{aut}_\partial(M_{g,1})\big) \longrightarrow \Gamma_g$$

has finite kernel.

Proof. This is [Gre19, Proposition 5.1]. (The extra structure Jq mentioned there is necessarily trivial. When $k+l$ is odd, it is trivial by definition. When $k+l$ is even, it is trivial since $\mathrm{H}_{(k+l)/2}(M_{g,1};\mathbb{Z}) \cong 0$ as $k \neq l$.) □

This implies, by the same arguments as in the proofs of Lemmas 3.1 and 3.2, that

$$\Gamma_g \cong \mathrm{Aut}\big(\mathrm{H}_k(M_{g,1};\mathbb{Z})\big)$$

acts on $\mathrm{H}_k(M_{g,1};\mathbb{Z})$ via the standard action and on $\mathrm{H}_l(M_{g,1};\mathbb{Z}) \cong \mathrm{H}_k(M_{g,1};\mathbb{Z})^\vee$ via its dual. The bases α_i^g of $\mathrm{H}_k(M_{g,1};\mathbb{Z})$ constructed above yield isomorphisms $\Gamma_g \cong \mathrm{GL}_g(\mathbb{Z})$ compatible with the γ_g and the standard inclusions $\mathrm{GL}_g(\mathbb{Z}) \to \mathrm{GL}_{g+1}(\mathbb{Z})$. Similarly, we obtain isomorphisms

$$\Gamma_g^{\mathbb{Q}} \coloneqq \mathrm{Aut}\Big(\widetilde{\mathrm{H}}_*(M_{g,1};\mathbb{Q}), \langle -,-\rangle_\cap\Big) \cong \mathrm{GL}_g(\mathbb{Q})$$

compatible with the stabilization maps.

Lastly, we set

$$\mathfrak{g}_g \coloneqq \mathrm{Der}_\omega\Big(\mathbb{L}(\mathrm{s}^{-1}\widetilde{\mathrm{H}}_*(M_{g,1};\mathbb{Q}))\Big)^+;$$

that is, it is the positive truncation of the graded Lie algebra of derivations that annihilate the element $\omega \coloneqq \sum_i [\beta_i, \beta_i^\#]$, where $\beta_1, \dots, \beta_g$ is some basis of $\mathrm{s}^{-1}\mathrm{H}_k(M_{g,1};\mathbb{Q})$ and $\beta_1^\#, \dots, \beta_g^\#$ is the basis of $\mathrm{s}^{-1}\mathrm{H}_l(M_{g,1};\mathbb{Q})$ dual to it with respect to the intersection pairing; see [BM20, §3.4 f.] for more details. We note that the canonical action of $\Gamma_g^{\mathbb{Q}}$ on $\mathrm{s}^{-1}\widetilde{\mathrm{H}}_*(M_{g,1};\mathbb{Q})$ preserves the element ω. Hence, we obtain an action of $\Gamma_g^{\mathbb{Q}}$ on $\mathfrak{g}_g$ by conjugation. Restricting along the canonical inclusion $\Gamma_g \to \Gamma_g^{\mathbb{Q}}$, we also obtain an action of Γ_g on $\mathfrak{g}_g$.

We now construct a stabilization map $\delta_g \colon \mathfrak{g}_g \to \mathfrak{g}_{g+1}$. For $\theta \in \mathfrak{g}_g$, the derivation $\delta_g(\theta)$ is uniquely determined by requiring it to restrict to θ along the map induced by ι_g and asking that $\delta_g(\theta)\Big(\mathrm{s}^{-1}\alpha_{g+1}^{g+1}\Big) = 0$ and $\delta_g(\theta)\Big(\mathrm{s}^{-1}(\alpha_{g+1}^{g+1})^\#\Big) = 0$. Here, α_i^{g+1} is the basis of $\mathrm{H}_k(M_{g+1,1};\mathbb{Z}) \subseteq \mathrm{H}_k(M_{g+1,1};\mathbb{Q})$ from above. It is clear from the definition that $\delta_g(\theta)(\omega) = 0$, so that δ_g indeed defines a map $\mathfrak{g}_g \to \mathfrak{g}_{g+1}$ of graded Lie algebras. Moreover, this becomes a map of $\Gamma_g^{\mathbb{Q}}$-modules when the target is equipped with the $\Gamma_g^{\mathbb{Q}}$-action obtained by restricting along γ_g.

Remark 5.2. The graded Lie algebra $\mathfrak{g}_g$ is a Lie model for the homotopy fiber of the map $\mathrm{Baut}_\partial(M_{g,1}) \longrightarrow \mathrm{B}\Gamma_g$, and one can think of the action of Γ_g on $\mathfrak{g}_g$ as modeling the action of $\Gamma_g \cong \pi_1(\mathrm{B}\Gamma_g)$ on this fiber; see [BM20, Proposition 5.6] and [BS] for precise statements (for non-relative self-equivalences this is [BZ22, Theorem 3.40]).

Berglund–Zeman [BZ22] recently developed methods for identifying the cohomology of a classifying space of self-equivalences $\mathrm{Baut}(X)$ in terms of certain group cohomology. The details of applying this to the relative situation will appear in the upcoming paper [BS] joint with Berglund. In particular, the following statement is proved there; one can think of it as a strengthening of collapse of the Serre spectral sequence associated to the fiber sequence of Remark 5.2.

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Proposition 5.3 (Berglund–Zeman, Berglund–Stoll). *There are, for any $g \in \mathbb{N}_0$ and Γ_g-module P, isomorphisms of graded vector spaces*

$$\Theta_g(P)\colon \mathrm{H}^*\big(\mathrm{Baut}_\partial(M_{g,1}); P\big) \xrightarrow{\cong} \mathrm{H}^*\big(\Gamma_g; \mathrm{H}^*_{\mathrm{CE}}(\mathfrak{g}_g) \otimes P\big)$$

that are compatible with the stabilization maps induced by ϕ_{g-1}, γ_{g-1}, and δ_{g-1}, in the sense that the diagram

$$\begin{array}{ccc}
\mathrm{H}^*\big(\mathrm{Baut}_\partial(M_{g,1}); P\big) & \xrightarrow{\Theta_g(P)} & \mathrm{H}^*\big(\Gamma_g; \mathrm{H}^*_{\mathrm{CE}}(\mathfrak{g}_g) \otimes P\big) \\
\downarrow{\scriptstyle \phi^*_{g-1}} & & \downarrow{\scriptstyle \gamma^*_{g-1}} \\
 & & \mathrm{H}^*\Big(\Gamma_{g-1}; \gamma^*_{g-1}(\mathrm{H}^*_{\mathrm{CE}}(\mathfrak{g}_g) \otimes P)\Big) \\
 & & \downarrow{\scriptstyle \delta^*_{g-1}} \\
\mathrm{H}^*\Big(\mathrm{Baut}_\partial(M_{g-1,1}); \gamma^*_{g-1}(P)\Big) & \xrightarrow{\Theta_{g-1}(\gamma^*_{g-1}(P))} & \mathrm{H}^*\Big(\Gamma_{g-1}; \mathrm{H}^*_{\mathrm{CE}}(\mathfrak{g}_{g-1}) \otimes \gamma^*_{g-1}(P)\Big)
\end{array}$$

commutes when $g \geq 1$. The map $\Theta_g(P)$ is an isomorphism of graded algebras when $P = \mathbb{Q}$ is trivial, and an isomorphism of modules over these algebras for general P.

Proof. For $P = \mathbb{Q}$ and without the compatibility with the stabilization maps, this is stated (without a detailed proof) as [BZ22, Theorem 4.40]. A full proof will appear in the upcoming paper [BS] joint with Berglund. □

The rest of this section is devoted to providing, for certain Γ_g-modules P, an explicit description of $\mathrm{H}^*(\Gamma_g; \mathrm{H}^*_{\mathrm{CE}}(\mathfrak{g}_g) \otimes P)$ in a stable range of degrees.

5.1. Reduction to invariants

In this subsection, our goal is to prove that there is an isomorphism of bigraded algebras

$$\mathrm{H}^p\Big(\Gamma_g; \mathrm{H}^q_{\mathrm{CE}}(\mathfrak{g}_g) \otimes P\Big) \cong \mathrm{H}^p(\Gamma_g; \mathbb{Q}) \otimes \mathrm{H}^q\Big(\big(\mathrm{C}^*_{\mathrm{CE}}(\mathfrak{g}_g) \otimes P\big)^{\Gamma_g^{\mathbb{Q}}}\Big)$$

in a stable range of degrees. This will mainly rely on a result of Li–Sun [LS19]. We need some preliminaries.

Definition 5.4. A *Schur bifunctor* is a functor $F\colon \mathbf{Vect} \times \mathbf{Vect} \to \mathbf{Vect}$ of the form

$$F(V_1, V_2) = \bigoplus_{n_1, n_2 \in \mathbb{N}_0} \mathscr{M}(n_1, n_2) \otimes_{\Sigma_{n_1} \times \Sigma_{n_2}} (V_1^{\otimes n_1} \otimes V_2^{\otimes n_2}),$$

where the $\mathscr{M}(n_1, n_2)$ are some right $(\Sigma_{n_1} \times \Sigma_{n_2})$-modules in **Vect**. We say F has *degree* $\leq d$ if $\mathscr{M}(n_1, n_2) \cong 0$ for all pairs (n_1, n_2) such that $n_1 + n_2 > d$. We say F is of *finite type* if each $\mathscr{M}(n_1, n_2)$ is finite dimensional.

Lemma 5.5. *Let F be a finite type Schur bifunctor of degree $\leq d$. Then the canonical inclusion*

$$F(V, V^\vee)^{\mathrm{GL}(V)} \longrightarrow F(V, V^\vee)^{\mathrm{SL}(V)}$$

is an isomorphism for all finite-dimensional vector spaces V such that $\dim V > d$. Here, $f \in \mathrm{GL}(V)$ acts on $V^\vee$ by $(f^{-1})^\vee$ and diagonally on $F(V, V^\vee)$.

Proof. Using, for G a group and W a G-module, the natural (in both W and G) isomorphism $(W_G)^\vee \cong (W^\vee)^G$, we can pass to the dual situation (i.e., take coinvariants instead of invariants). (To see that

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$F(V, V^\vee)^\vee$ is again a Schur bifunctor, we use that $\Sigma := \Sigma_{n_1} \times \Sigma_{n_2}$ is finite and that we are working in characteristic zero, so that $(M \otimes_\Sigma N)^\vee \cong M^\vee \otimes_\Sigma N^\vee$.)

Both the $\mathrm{GL}(V)$-action and the $\mathrm{SL}(V)$-action are compatible with the direct sum composition of $F(V, V^\vee)$, so that it is enough to prove for a single summand

$$W := \mathscr{M}(n_1, n_2) \otimes_{\Sigma_{n_1} \times \Sigma_{n_2}} \left(V^{\otimes n_1} \otimes (V^\vee)^{\otimes n_2}\right)$$

with $n_1 + n_2 < \dim V$ that $W_{\mathrm{SL}(V)} \to W_{\mathrm{GL}(V)}$ is an isomorphism. Denote by $e_1, \dots, e_g$ some basis of V. Then the equivalence relation defining $W_{\mathrm{GL}(V)}$ is generated by

$$m \otimes \bigotimes_{j=1}^{n_1} e_{i_j} \otimes \bigotimes_{j'=1}^{n_2} e^{\#}_{i'_{j'}} \sim m \otimes \bigotimes_{j=1}^{n_1} f(e_{i_j}) \otimes \bigotimes_{j'=1}^{n_2} (f^{-1})^\vee (e^{\#}_{i'_{j'}}) \tag{5.1}$$

for all $f \in \mathrm{GL}(V)$, $m \in \mathscr{M}(n_1, n_2)$ and $1 \le i_j, i'_{j'} \le g$. Choose some $1 \le i \le g$ such that $i \ne i_j, i'_{j'}$ for all j and j' (this is possible by our assumption that $\dim V > n_1 + n_2$). Now let $f' := f \circ h \in \mathrm{GL}(V)$, where h is defined by $h(e_i) = (\det f)^{-1} e_i$ and $h(e_j) = e_j$ for all $j \ne i$. Then $f' \in \mathrm{SL}(V)$ and it yields the same relation (5.1) as f did. Hence, $W_{\mathrm{SL}(V)} \to W_{\mathrm{GL}(V)}$ is injective. Since it is clearly surjective, it is thus an isomorphism. □

Now we are ready to prove the main result of this subsection. See [BM20, Theorem 8.4] for the analogous statement in the case $k = l$; the stable range occurring in their result has been improved by Krannich [Kra20] using an argument similar to the one we will employ.

Proposition 5.6. *Let Q be a finite type Schur bifunctor of degree $\le r$ and set $P := Q\big(\mathrm{H}_k(M_{g,1}; \mathbb{Q}), \mathrm{H}_l(M_{g,1}; \mathbb{Q})\big)$ as a $\Gamma_g^{\mathbb{Q}}$-module. Then the canonical map of bigraded vector spaces*

$$\mathrm{H}^p(\Gamma_g; \mathbb{Q}) \otimes \mathrm{H}^q\left(\left(\mathrm{C}^*_{\mathrm{CE}}(\mathfrak{g}_g) \otimes P\right)^{\Gamma_g^{\mathbb{Q}}}\right) \cong \mathrm{H}^p\left(\Gamma_g; \mathrm{H}^q\left(\left(\mathrm{C}^*_{\mathrm{CE}}(\mathfrak{g}_g) \otimes P\right)^{\Gamma_g^{\mathbb{Q}}}\right)\right) \longrightarrow \mathrm{H}^p\left(\Gamma_g; \mathrm{H}^q\left(\mathrm{C}^*_{\mathrm{CE}}(\mathfrak{g}_g) \otimes P\right)\right) \cong \mathrm{H}^p\left(\Gamma_g; \mathrm{H}^q_{\mathrm{CE}}(\mathfrak{g}_g) \otimes P\right)$$

is an isomorphism for $g > \max\left(p + 1, \frac{1}{2}(q+1) + r\right)$.

Proof. A result of Li–Sun [LS19, Example 1.10] implies that the canonical map

$$\mathrm{H}^p\left(\Gamma_g; \mathrm{H}^q\left(\mathrm{C}^*_{\mathrm{CE}}(\mathfrak{g}_g) \otimes P\right)^{\Gamma_g}\right) \longrightarrow \mathrm{H}^p\left(\Gamma_g; \mathrm{H}^q\left(\mathrm{C}^*_{\mathrm{CE}}(\mathfrak{g}_g) \otimes P\right)\right)$$

is an isomorphism for $g \ge p+2$. This uses that [Gre19, (Proof of) Proposition 7.1] implies that $\mathrm{C}^q_{\mathrm{CE}}(\mathfrak{g}_g)$, and hence also $\mathrm{H}^q\left(\mathrm{C}^*_{\mathrm{CE}}(\mathfrak{g}_g) \otimes P\right)$, is a finite dimensional rational representation of the reductive group GL_g and thus completely reducible (see, for example, [Mil17, Theorem 22.42]).

Now, by [Gre19, Lemma 3.4], the group Γ_g is rationally perfect (cf. [BM20, Definiton B.3]). Hence, taking Γ_g-invariants is exact when restricted to the category of finite dimensional representations. In particular, the canonical map

$$\mathrm{H}^q\left(\left(\mathrm{C}^*_{\mathrm{CE}}(\mathfrak{g}_g) \otimes P\right)^{\Gamma_g}\right) \longrightarrow \mathrm{H}^q\left(\mathrm{C}^*_{\mathrm{CE}}(\mathfrak{g}_g) \otimes P\right)^{\Gamma_g}$$

is an isomorphism.

Lastly we use that, by a result of Borel [Bor66, Theorem 1], the arithmetic subgroup $\mathrm{SL}_g(\mathbb{Z})$ is Zariski dense in $\mathrm{SL}_g(\mathbb{Q})$ (note that the condition of the theorem is fulfilled since SL_g is almost simple for $g \ge 2$). Hence, we have $V^{\mathrm{SL}_g(\mathbb{Z})} = V^{\mathrm{SL}_g(\mathbb{Q})}$ for any rational representation V of SL_g over $\mathbb{Q}$. In particular, for F

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a finite type Schur bifunctor of degree $\leq d$ and $V \coloneqq F(\mathbb{Q}^g, (\mathbb{Q}^g)^\vee)$, we thus have, when $g > d$, that

$$V^{\mathrm{GL}_g(\mathbb{Z})} = \left(V^{\mathrm{SL}_g(\mathbb{Z})}\right)^{\mathbb{Z}^\times} = \left(V^{\mathrm{SL}_g(\mathbb{Q})}\right)^{\mathbb{Z}^\times} = \left(V^{\mathrm{GL}_g(\mathbb{Q})}\right)^{\mathbb{Z}^\times} = V^{\mathrm{GL}_g(\mathbb{Q})}$$

where the third equality uses Lemma 5.5. Now we note that, by [Gre19, (Proof of) Proposition 7.1], there exists a finite type Schur bifunctor F_q of degree $\leq \frac{q}{2}$ such that there is an isomorphism

$$\mathrm{C}^q_{\mathrm{CE}}(\mathfrak{g}_g) \cong F_q\big(\mathrm{H}_k(M_{g,1};\mathbb{Q}), \mathrm{H}_l(M_{g,1};\mathbb{Q})\big)$$

of $\Gamma_g^{\mathbb{Q}}$-modules (where the action on the right-hand side is diagonally). Hence, $\mathrm{C}^q_{\mathrm{CE}}(\mathfrak{g}_g) \otimes P$ is, in the same way, the value of a Schur bifunctor of degree $\leq \frac{q}{2} + r$. This finishes the proof. For the condition on g given in the statement, note that an isomorphism of chain complexes in a range induces an isomorphism of homology in a range smaller by 1. □

5.2. The main theorem

We are now ready to deduce our main theorem from everything we have done so far.

Theorem 5.7. *Let $3 \leq k < l \leq 2k-2$ and $2 \leq g$ be integers and let I and J be finite linearly ordered sets. Furthermore, set $H_{g;i} \coloneqq \mathrm{H}_i(M^{k,l}_{g,1};\mathbb{Q})$ as a $\pi_0\Big(\mathrm{aut}_\partial(M^{k,l}_{g,1})\Big)$-module. Then there is, in cohomological degrees,*

$$\leq \begin{cases} g-2, & \text{when } |I|+|J| \leq \frac{1}{2}g \\ 2(g-|I|-|J|-1), & \text{otherwise} \end{cases}$$

an isomorphism of graded $(\Sigma_I \times \Sigma_J)$-modules

$$\mathrm{H}^*\Big(\mathrm{Baut}_\partial(M^{k,l}_{g,1}); H^{\otimes I}_{g;l} \otimes H^{\otimes J}_{g;k}\Big) \cong \\ \mathrm{s}^{(k-1)|I|+(l-1)|J|}\mathrm{H}^*(\mathrm{GL}(\mathbb{Z});\mathbb{Q}) \otimes \mathrm{H}^*\Big((\mathfrak{DG}^{k+l-2}_{\mathrm{trunc}}(\mathscr{L}\mathrm{ie})_{I,J})^\vee\Big) \otimes \mathrm{sgn}_I^{\otimes k-1} \otimes \mathrm{sgn}_J^{\otimes l-1}$$

compatible with the stabilization maps on the left-hand side (here, the shift refers to the cohomological grading). It is an isomorphism of algebras when $I = J = \varnothing$, and an isomorphism of modules over these algebras for general I and J. Here, $\mathrm{GL}(\mathbb{Z}) \coloneqq \mathrm{colim}_{g\in\mathbb{N}}\,\mathrm{GL}_g(\mathbb{Z})$ and sgn_S is the sign representation of Σ_S. Also, recall that $\mathfrak{DG}^{k+l-2}_{\mathrm{trunc}}(\mathscr{L}\mathrm{ie})_{I,J}$ is the truncated directed (I,J)-graph complex (see Definition 3.31) associated to the cyclic Lie operad.

In particular, after stabilizing, we obtain an isomorphism

$$\lim_{g\in\mathbb{N}} \mathrm{H}^*\Big(\mathrm{Baut}_\partial(M^{k,l}_{g,1}); H^{\otimes I}_{g;l} \otimes H^{\otimes J}_{g;k}\Big) \cong \\ \mathrm{s}^{(k-1)|I|+(l-1)|J|}\mathrm{H}^*(\mathrm{GL}(\mathbb{Z});\mathbb{Q}) \otimes \mathrm{H}^*\Big((\mathfrak{DG}^{k+l-2}_{\mathrm{trunc}}(\mathscr{L}\mathrm{ie})_{I,J})^\vee\Big) \otimes \mathrm{sgn}_I^{\otimes k-1} \otimes \mathrm{sgn}_J^{\otimes l-1}$$

of graded $(\Sigma_I \times \Sigma_J)$-modules. It is an isomorphism of algebras when $I = J = \varnothing$, and an isomorphism of modules over these algebras for general I and J.

Proof. Combining Propositions 5.3 and 5.6, we obtain, in the stable range

$$g > \max_{p+q=s} \max\Big(p+1, \tfrac{1}{2}(q+1) + |I| + |J|\Big) = \max\Big(s+1, \tfrac{1}{2}(s+1) + |I| + |J|\Big)$$

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an isomorphism of graded $(\Sigma_I \times \Sigma_J)$-modules

$$\mathrm{H}^s\Big(\mathrm{Baut}_\partial(M_{g,1}^{k,l}); H_{g;l}^{\otimes I} \otimes H_{g;k}^{\otimes J}\Big) \cong \bigoplus_{p+q=s} \mathrm{H}^p(\Gamma_g;\mathbb{Q}) \otimes \mathrm{H}^q\Big(\Big(\mathrm{C}^*_{\mathrm{CE}}(\mathfrak{g}_g) \otimes H_{g;l}^{\otimes I} \otimes H_{g;k}^{\otimes J}\Big)^{\Gamma_g^{\mathbb{Q}}}\Big),$$

which is compatible with the stabilization maps and the algebra/module structures. It follows from work of Borel [Bor74] (see also [KMP22, Theorem A][3]) that the canonical map $\mathrm{H}^p(\mathrm{GL}(\mathbb{Z});\mathbb{Q}) \to \mathrm{H}^p(\mathrm{GL}_g(\mathbb{Z});\mathbb{Q})$ is an isomorphism for $g \geq p+2$; in particular, we can replace $\mathrm{H}^p(\Gamma_g;\mathbb{Q}) \cong \mathrm{H}^p(\mathrm{GL}_g(\mathbb{Z});\mathbb{Q})$ with $\mathrm{H}^p(\mathrm{GL}(\mathbb{Z});\mathbb{Q})$ in the formula above.

Now, by [BM20, Proposition 6.6], there is an isomorphism of Lie algebras

$$\mathfrak{g}_g = \mathrm{Der}_\omega\Big(\mathbb{L}(\mathrm{s}^{-1}\widetilde{\mathrm{H}}_*(M_{g,1};\mathbb{Q}))\Big)^+ \cong (\mathrm{s}^{-(k+l-2)}\,\mathscr{L}\mathrm{ie})[\mathrm{s}^{-1}\widetilde{\mathrm{H}}_*(M_{g,1};\mathbb{Q})]^+$$

that is compatible with the $\Gamma_g^{\mathbb{Q}}$-actions and the stabilization maps. Here, the right-hand side is equipped with the graded Lie algebra structure of Lemma 2.34, where $\mathrm{s}^{-1}\widetilde{\mathrm{H}}_*(M_{g,1};\mathbb{Q})$ is equipped with the anti-symmetric pairing given by $\langle \mathrm{s}^{-1}a, \mathrm{s}^{-1}b\rangle = (-1)^{|a|+1}\langle a,b\rangle_\cap$. Then Corollary 3.33 implies that, in degrees $q \leq \frac{4}{3}g-1$ if $l = 2k-2$ or $q \leq 2g-1$ if $l \neq 2k-2$, there is an isomorphism of graded $(\Sigma_I \times \Sigma_J)$-modules

$$\mathrm{H}_q\Big(\Big(\mathrm{C}^{\mathrm{CE}}_*(\mathfrak{g}_g) \otimes (\mathrm{s}^{k-1}H_{g;k})^{\otimes I} \otimes (\mathrm{s}^{l-1}H_{g;l})^{\otimes J}\Big)_{\Gamma_g^{\mathbb{Q}}}\Big) \cong \mathrm{H}_q\Big(\mathfrak{DG}^{k+l-2}_{\mathrm{trunc}}(\mathscr{L}\mathrm{ie})_{I,J}\Big)$$

compatible with the stabilization maps on the left-hand side as well as the coalgebra/comodule structures. (Note that an isomorphism of chain complexes up to degree p induces an isomorphism on homology up to degree $p-1$.) Dualizing this yields the desired statement since there is an isomorphism

$$(\mathrm{s}^{k-1}H_{g;k})^{\otimes I} \otimes (\mathrm{s}^{l-1}H_{g;l})^{\otimes J} \cong \mathrm{s}^{(k-1)|I|+(l-1)|J|}H_{g;k}^{\otimes I} \otimes H_{g;l}^{\otimes J} \otimes \mathrm{sgn}_I^{\otimes k-1} \otimes \mathrm{sgn}_J^{\otimes l-1}$$

of $(\Gamma_g^{\mathbb{Q}} \times \Sigma_I \times \Sigma_J)$-modules and $H_{g;k}$ and $H_{g;l}$ are dual to each other. □

Remark 5.8. Via Schur–Weyl duality (see, for example, [FH04, §6.1] or [Eti+11, §5.19]), Theorem 5.7 can be used to deduce descriptions of $\mathrm{H}^*\Big(\mathrm{Baut}_\partial(M_{g,1}^{k,l}); P\Big)$ for more general $\Gamma_g^{\mathbb{Q}}$-modules P.

Corollary 5.9. *Let $3 \leq k < l \leq 2k-2$ and $2 \leq g$ be integers. Then there is, in cohomological degrees $\leq g-2$, an isomorphism of graded algebras*

$$\mathrm{H}^*\Big(\mathrm{Baut}_\partial(M_{g,1}^{k,l});\mathbb{Q}\Big) \cong \mathrm{H}^*(\mathrm{GL}(\mathbb{Z});\mathbb{Q}) \otimes \mathrm{H}^*\Big(\mathfrak{UG}^{k+l-2}(\mathscr{L}\mathrm{ie})^\vee\Big)$$

compatible with the stabilization maps on the left-hand side. (Recall that $\mathfrak{UG}^{k+l-2}(\mathscr{L}\mathrm{ie})$ denotes the undirected graph complex associated to the cyclic Lie operad; see Definition 4.22.)

In particular, after stabilizing, we obtain an isomorphism

$$\lim_{g\in\mathbb{N}} \mathrm{H}^*\Big(\mathrm{Baut}_\partial(M_{g,1}^{k,l});\mathbb{Q}\Big) \cong \mathrm{H}^*(\mathrm{GL}(\mathbb{Z});\mathbb{Q}) \otimes \mathrm{H}^*\Big(\mathfrak{UG}^{k+l-2}(\mathscr{L}\mathrm{ie})^\vee\Big)$$

of graded algebras.

Proof. This follows from Theorem 5.7 with $I = J = \varnothing$ and Theorem 4.28. □

Remark 5.10. It should be possible to obtain a version of this corollary for cohomology with nontrivial coefficients by generalizing the arguments of Section 4 (see Remark 4.1).

[3] They actually prove a slightly better range than Borel, which we will not need.

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Remark 5.11. Note that

$$\mathrm{H}^*(\mathrm{GL}(\mathbb{Z});\mathbb{Q}) \cong \Lambda\mathbb{Q}\langle x_i \mid i \in \mathbb{N}_{\geq 1}\rangle \qquad \text{where} \quad |x_i| = 4i+1$$

by a classical result of Borel [Bor74, p. 11.4]. The homology of $\mathfrak{UG}^{k+l-2}(\mathscr{L}\mathrm{ie})$ is more mysterious. See Remark 4.30 for a summary of what is known.

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