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The second cohomology with symplectic coefficients of the moduli space of smooth projective curves

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Abstract. Each finite dimensional irreducible rational representation V of the symplectic group $\mathrm{Sp}_{2g}(\mathbb{Q})$ determines a generically defined local system $\mathbb{V}$ over the moduli space $\mathcal{M}_g$ of genus g smooth projective curves. We study $H^2(\mathcal{M}_g;\mathbb{V})$ and the mixed Hodge structure on it. Specifically, we prove that if $g \geqslant 6$, then the natural map $IH^2(\widetilde{\mathcal{M}}_g;\mathbb{V}) \to H^2(\mathcal{M}_g;\mathbb{V})$ is an isomorphism where $\widetilde{\mathcal{M}}_g$ is the Satake compactification of $\mathcal{M}_g$. Using the work of Saito we conclude that the mixed Hodge structure on $H^2(\mathcal{M}_g;\mathbb{V})$ is pure of weight $2+r$ if $\mathbb{V}$ underlies a variation of Hodge structure of weight r. We also obtain estimates on the weight of the mixed Hodge structure on $H^2(\mathcal{M}_g;\mathbb{V})$ for $3 \leqslant g < 6$. Results of this article can be applied in the study of relations in the Torelli group T_g.

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Introduction

The moduli space $\mathcal{M}_g$ of smooth projective curves of genus g is a quasi-projective variety over $\mathbb{C}$. Its points correspond to isomorphism classes of smooth projective complex curves of genus g. It has only finite quotient singularities, and therefore behaves like a smooth variety.

This space has several natural compactifications. In this article we will be interested in the so called Satake compactification $\widetilde{\mathcal{M}}_g$ of $\mathcal{M}_g$. The period map determines an inclusion of $\mathcal{M}_g$ into $\mathcal{A}_g$, the moduli space of principally polarized abelian varieties. The Satake compactification $\widetilde{\mathcal{M}}_g$ is the closure of $\mathcal{M}_g$ inside $\overline{\mathcal{A}}_g$, the Satake compactification of $\mathcal{A}_g$. It has quite complicated singularities at its boundary $\widetilde{\mathcal{M}}_g - \mathcal{M}_g$.

Each representation of the algebraic group Sp_{2g} gives rise to an orbifold local system over $\mathcal{M}_g$. To explain this we introduce the mapping class group Γ_g. It is the group of connected components of the group of the orientation preserving diffeomorphisms of a compact orientable surface S of genus g. The group Γ_g is the

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orbifold fundamental group of $\mathcal{M}_g$, and representations of Γ_g give rise to orbifold local systems over $\mathcal{M}_g$. There is a natural surjective map

$$\Gamma_g \to \operatorname{Aut}(H_1(S;\mathbb{Z}),\cap),$$

where $\cap$ is determined by the intersection pairing. The right-hand group is isomorphic to $\mathrm{Sp}_{2g}(\mathbb{Z})$. So each finite dimensional rational representation V of an algebraic group Sp_{2g} gives rise to a *symplectic* orbifold local system $\mathbb{V}$ over $\mathcal{M}_g$.

Since $\mathbb{V}$ is generically defined over $\widetilde{\mathcal{M}}_g$, one can consider the intersection cohomology groups $IH^\bullet(\widetilde{\mathcal{M}}_g;\mathbb{V})$. There is a natural restriction map

$$IH^\bullet(\widetilde{\mathcal{M}}_g;\mathbb{V}) \to H^\bullet(\mathcal{M}_g;\mathbb{V}).$$

The main result of this article is

THEOREM (cf. Th. 4.1). *The natural restriction map*

$$IH^k(\widetilde{\mathcal{M}}_g;\mathbb{V}) \to H^k(\mathcal{M}_g;\mathbb{V})$$

is an isomorphism when $k=1$ for all $g \geqslant 3$, and when $k=2$ for all $g \geqslant 6$.

The group $H^1(\mathcal{M}_g;\mathbb{V})$ is easily computed when $g \geqslant 3$ for all symplectic local systems $\mathbb{V}$ using Johnson's fundamental work [23]) (cf. [14]).

Let X be an algebraic variety. From Saito's work [37], [38] we know that $H^\bullet(X;\mathbb{V})$ has natural mixed Hodge structure (MHS) if $\mathbb{V}\to X$ is an admissible polarized variation of Hodge structure, and $IH^\bullet(X;\mathbb{V})$ has natural mixed Hodge structure if $\mathbb{V}$ is a generically defined admissible polarized variation of Hodge structure over X. Further if X is compact and $\mathbb{V}$ is pure of weight r, then $IH^\bullet(X;\mathbb{V})$ is pure of weight $k+r$.

THEOREM (cf. Cor. 5.1, Cor. 5.3). *If $g \geqslant 6$ and $\mathbb{V} \to \mathcal{M}_g$ is a variation of Hodge structure of weight r whose underlying local system is symplectic, then the natural mixed Hodge structure on $H^2(\mathcal{M}_g;\mathbb{V})$ is pure of weight $2+r$. If $3 \leqslant g < 6$, then the weights of the mixed Hodge structure on $H^2(\mathcal{M}_g;\mathbb{V})$ lie in $\{2+r, 3+r\}$.*

Each symplectic local system $\mathbb{V}$ associated to an irreducible representation V of Sp_{2g} underlies a variation of Hodge structure over $\mathcal{M}_g$ which is unique up to Tate twist. It is convenient to fix the weight of the variation of Hodge structure $\mathbb{V}(\lambda)$ associated to a dominant integral weight λ. Fix fundamental weights $\lambda_1,\lambda_2,\ldots,\lambda_g$ of Sp_{2g}. If $\lambda = a_1\lambda_1 + a_2\lambda_2 + \cdots + a_g\lambda_g$, define $|\lambda| = a_1 + 2a_2 + \cdots + ga_g$. This is the smallest integer r such that $V(\lambda) \subseteq H_1(S)^{\otimes r}$. (A good reference is [11].) Then $\mathbb{V}(\lambda)$ can be realized uniquely as a variation of Hodge structure of weight $|\lambda|$.

Harer proved in [17] that the cohomology $H^k(\mathcal{M}_g;\mathbb{Z})$ stabilizes when $g \geqslant 3k$, and Ivanov later improved the range of stability [21, 22]. He showed that $H^k(\mathcal{M}_g;\mathbb{Z})$ stabilizes when $g \geqslant 2k+2$. In [22] Ivanov also proved that

$H^k(\mathcal{M}_{g,1}; \mathbb{V}(\lambda))$ is independent of g when $g \geqslant 2k + 2 + |\lambda|$. (The space $\mathcal{M}_{g,1}$ is the moduli space of curves with a marked non-zero tangent vector.) In [28] Looijenga calculated the stable cohomology groups of $\mathcal{M}_g$ with symplectic coefficients as a module over stable cohomology groups of $\mathcal{M}_g$ with trivial coefficients. In particular, this implies that $H^k(\mathcal{M}_g; \mathbb{V}(\lambda))$ is independent of g when $g \geqslant 2k + 2 + 2|\lambda|$.

Looijenga's result also provides very specific information about the MHS on $H^k(\mathcal{M}_g; \mathbb{V}(\lambda))$. Combined with computations of $H^k(\mathcal{M}_g; \mathbb{Q})$ in low dimensions due to Harer [16, 19, 20], it implies that $H^k(\mathcal{M}_g; \mathbb{V}(\lambda))$ is pure of weight $k + |\lambda|$ when $k \leqslant 4$ and g is in the stability range. In particular, $H^2(\mathcal{M}_g; \mathbb{V}(\lambda))$ is pure of weight $2 + |\lambda|$ when $g \geqslant 6 + 2|\lambda|$. Recently, Pikaart proved in [34] that the stable cohomology $H^k(\mathcal{M}_g; \mathbb{Q})$ is pure of weight k. Combined with Looijenga's computations, this shows that $H^k(\mathcal{M}_g; \mathbb{V}(\lambda))$ is pure of weight $k + |\lambda|$ whenever $g \geqslant 2k + 2 + 2|\lambda|$.

Unlike the stability range, our purity range is *independent* of $|\lambda|$. This is important for the following application which was the motivation for this article.

The Torelli group T_g is the kernel of the surjective homomorphism $\Gamma_g \to \mathrm{Sp}_{2g}(\mathbb{Z})$. One can consider the Malcev Lie algebra $\mathfrak{t}_g$ associated to T_g. (For definitions see [13]). This Lie algebra is an analogue of the Lie algebra associated to the pure braid group on m strings, which is important in the study of Vassiliev invariants and conformal field theory. By a result of Johnson [23], T_g is finitely generated when $g \geqslant 3$. Thus, $\mathfrak{t}_g$ is also finitely generated when $g \geqslant 3$. It is not known for any $g \geqslant 3$ whether T_g is finitely presented or not.

In [15] Hain gives an explicit presentation of $\mathfrak{t}_g$ for $g \geqslant 3$. More specifically, he proves that for each choice of $x_0 \in \mathcal{M}_g$ there is a canonical MHS on $\mathfrak{t}_g$ which is compatible with the bracket. Thus,

$$\mathfrak{t}_g \otimes \mathbb{C} \cong \prod_m \mathrm{Gr}^W_{-m} \mathfrak{t}_g \otimes \mathbb{C},$$

where $\mathrm{Gr}^W_\bullet$ are the graded quotients of the MHS associated to a choice of x_0. Hain proves that for all $g \geqslant 3$

$$\mathrm{Gr}^W_\bullet \mathfrak{t}_g = \mathbb{L}(H_1(\mathfrak{t}_g))/(R_g),$$

where $\mathbb{L}$ stands for the free Lie algebra, and R_g is a set of relations. According to a result of Johnson [23] $H_1(\mathfrak{t}_g)$ is isomorphic as an Sp_{2g}-module to $V(\lambda_3)$.

Using the above theorem about the MHS on $H^2(\mathcal{M}_g; \mathbb{V})$ Hain proves that the relations R_g are quadratic when $g \geqslant 6$, and quadratic and possibly cubic when $g = 3, 4, 5$. Moreover, he explicitly calculates all quadratic relations. This implies that $\mathfrak{t}_g$ is finitely presented for all $g \geqslant 3$.

We shall outline the proof of the first theorem above. There are three main steps in the proof. The first step is to notice that if $g \geqslant 3$, then the boundary $\widetilde{\mathcal{M}}_g - \mathcal{M}_g$ of the Satake compactification has one irreducible component of codimension two,

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and all other irreducible components have codimension three. This immediately implies that $H^1(\mathcal{M}_g;\mathbb{V}) \cong IH^1(\widetilde{\mathcal{M}}_g;\mathbb{V})$.

The codimension two irreducible component of $\widetilde{\mathcal{M}}_g - \mathcal{M}_g$ has a Zariski open subset isomorphic to $\mathcal{M}_1 \times \mathcal{M}_{g-1}$. We denote it by X. (In the paper we work with a smooth Zariski open subset of X. However this is just a technical detail, and we do not want to draw an attention to it here.) Let N^* be the link bundle of X in $\widetilde{\mathcal{M}}_g$. We denote by π the corresponding projection. Then there is an exact sequence

$$0 \to IH^2(\widetilde{\mathcal{M}}_g;\mathbb{V}) \to H^2(\mathcal{M}_g;\mathbb{V}) \to H^0(X;R^2\pi_*\mathbb{V}),$$

and the last morphism factors through the edge homomorphism

$$\psi\colon H^2(N^*;\mathbb{V}) \to H^0(X;R^2\pi_*\mathbb{V})$$

of the Leray–Serre spectral sequence of π. Therefore it suffices to show that ψ is the trivial homomorphism.

The second step is to understand the link bundle N^*. Let L be the pull-back under $pr_2\colon X \to \mathcal{M}_{g-1}$ of the unit relative tangent bundle over $\mathcal{M}_{g-1}$, and $\widetilde{\pi}$ be the corresponding projection $L \to X$. We show that L is a two-to-one unramified covering of N^*. (This is done in Sect. 3.) Here we need to assume that $g \geqslant 4$. Denote by $\widetilde{\mathbb{V}}$ the pull-back of the local system $\mathbb{V}$ to L, and by $\widetilde{\psi}$ the edge homomorphism $H^2(L;\widetilde{\mathbb{V}}) \to H^0(X;R^2\widetilde{\pi}_*\widetilde{\mathbb{V}})$ of the Leray–Serre spectral sequence of $\widetilde{\pi}$. There is a commutative diagram

$$\begin{tikzcd}
H^2(L;\widetilde{\mathbb{V}}) \arrow[r, "\widetilde{\psi}"] & H^0(X;R^2\widetilde{\pi}_*\widetilde{\mathbb{V}}) \\
H^2(N^*;\mathbb{V}) \arrow[r, "\psi"] \arrow[u] & H^0(X;R^2\pi_*\mathbb{V}), \arrow[u]
\end{tikzcd}$$

where both vertical maps are inclusions. This implies that ψ is trivial, if $\widetilde{\psi}$ is trivial.

The third step is to show that $\widetilde{\psi}$ is trivial. The local system $\widetilde{\mathbb{V}}$ extends to the stratum $X \cong \mathcal{M}_1 \times \mathcal{M}_{g-1}$, and splits over it according to the branching rule for the standard inclusion of $\mathrm{Sl}_2 \times \mathrm{Sp}_{2g-2}$ into Sp_{2g}. The bundle map $\widetilde{\pi}$ respects this splitting. Thus, it suffices to show that $\widetilde{\psi}$ is trivial for each irreducible symplectic local system $\overline{\mathbb{V}}$ over X. We complete the computation using Schur's lemma and the fact, due to Harer [20], that $H^2(\Gamma_{g,1};H^1(S))$ is trivial when $g \geqslant 4$. (One can also use a result from [19, Sect. 7] that $H^2(\Gamma_{g,1};H^1(S))$ is trivial when $g \geqslant 9$.)

1. Basic facts about the moduli space of curves

In this section we recall the definitions and basic properties of the moduli spaces of curves, and the corresponding mapping class groups.

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The moduli space $\mathcal{M}_{g,r}^s$ parameterizes the isomorphism classes of smooth complex projective curves of genus g with s marked points and r marked nonzero holomorphic tangent vectors. The existence of such moduli spaces follows from geometric invariant theory. These moduli spaces are known to be normal quasi-projective varieties [30, Th. 5.11, Th. 7.13].

One can also construct $\mathcal{M}_{g,r}^s$ using Teichmüller theory. This approach allows us to establish the relation between the moduli spaces and the corresponding mapping class groups.

Let S denote a smooth compact orientable surface of genus g. Fix $s+r$ distinct points $p_1, \ldots, p_{r+s}$ on S, and r non-zero tangent vectors $v_1, \ldots, v_r$ at points $p_1, \ldots, p_r$ respectively. One can consider triples

$$(C, (q_1, \ldots, q_{r+s}, w_1, \ldots, w_r), [f]),$$

where C is a smooth projective genus g curve, $q_1, \ldots, q_{r+s}$ are distinct points on C, $w_1, \ldots, w_r$ are non-zero holomorphic tangent vectors at $q_1, \ldots, q_r$ respectively, and $f\colon C \to S$ is an orientation preserving diffeomorphism such that $f(q_i) = p_i$ and $f_*(w_i) = v_i$ (we use the canonical identification of the holomorphic tangent space with the underlying real tangent space). We denote by $[f]$ the isotopy class of f relative to $\{q_1, \ldots, q_{r+s}, w_1, \ldots, w_r\}$. Two triples

$$(C_j, (q_1^j, \ldots, q_{r+s}^j, w_1^j, \ldots, w_r^j), [f_j]), \quad j = 1, 2,$$

are called *equivalent* if there exists a biholomorphism $h\colon C_1 \to C_2$ such that $h(q_i^1) = q_i^2$, $h_*(w_i^1) = w_i^2$, and $[f_2 \circ h] = [f_1]$ where the isotopy is required to preserve the marked points and tangent vectors. The space of equivalence classes $\mathcal{T}_{g,r}^s$ is called the *Teichmüller space* [18], [19, p. 26]. It is known that $\mathcal{T}_{g,r}^s$ is a contractible complex manifold of dimension $3g-3+s+2r$ when $2g-2+s+2r>0$.

The *mapping class group* $\Gamma_{g,r}^s$ is defined to be $\mathrm{Diff}^+(S)/\mathrm{Diff}_0^+(S)$, where $\mathrm{Diff}^+(S)$ is the group of orientation preserving diffeomorphisms of S, which leave the marked points $p_1, \ldots, p_{r+s}$ and marked tangent vectors $v_1, \ldots, v_r$ fixed, and $\mathrm{Diff}_0^+(S)$ is the connected component of the identity. If $g>0$, then the group $\Gamma_{g,r}^s$ is torsion free when either $r>0$, or $s>2g+2$.

The group $\Gamma_{g,r}^s$ acts on $\mathcal{T}_{g,r}^s$ as follows. If $g \in \Gamma_{g,r}^s$, then

$$g(C, (q_1, \ldots, w_r), [f]) = (C, (q_1, \ldots, w_r), [g \circ f]).$$

The quotient space $\Gamma_{g,r}^s \setminus \mathcal{T}_{g,r}^s$ is the moduli space $\mathcal{M}_{g,r}^s$ of curves with s marked points and r marked tangent vectors. The group $\Gamma_{g,r}^s$ acts on $\mathcal{T}_{g,r}^s$ by biholomorphisms, and this action is properly discontinuous and virtually free. It follows that $\mathcal{M}_{g,r}^s$ is a complex analytic variety with only finite quotient singularities. This analytic structure agrees with the one coming from geometric invariant theory. If $\Gamma_{g,r}^s$ is torsion free, then the action is free, and $\mathcal{M}_{g,r}^s$ is smooth.

Notation. We shall omit indices r and s from $\mathcal{T}^s_{g,r}$, $\Gamma^s_{g,r}$, and $\mathcal{M}^s_{g,r}$ when they are equal to zero. We shall use both $\mathcal{M}^1_1$ and $\mathcal{M}_1$ to denote the moduli space of elliptic curves.

Remark. One can also consider $\mathcal{M}^{[s]}_g$, the moduli space of genus g curves with a marked set of cardinality s. It is the quotient of $\mathcal{M}^s_g$ by the natural action of the symmetric group on s letters. This action permutes the marked points.

The singular locus of $\mathcal{M}_g$ is contained in the locus of curves with non-trivial automorphisms. When $g \geqslant 3$, we denote by $^\circ\mathcal{M}_g$ the locus of curves with only trivial automorphisms. This is a smooth Zariski open subset of $\mathcal{M}_g$ whose complement has codimension $g-2$.

There are natural surjective morphisms between different moduli spaces which correspond to forgetting marked points and marked tangent vectors [25]. We will consider the morphisms $\mathcal{M}^1_g \to \mathcal{M}_g$ and $\mathcal{M}_{g,1} \to \mathcal{M}^1_g$. The first morphism $\mathcal{M}^1_g \to \mathcal{M}_g$ is called the 'universal curve' [10, p. 218]. Its fiber over a point $[C] \in \mathcal{M}_g$ is $C/\operatorname{Aut} C$. On the level of the mapping class groups there is a corresponding short exact sequence [4]

$$1 \to \pi_1(S) \to \Gamma^1_g \to \Gamma_g \to 1.$$

The morphism $\mathcal{M}_{g,1} \to \mathcal{M}^1_g$ 'forgets' the tangent vector, but remembers its base point. When $g \geqslant 2$ it is the frame bundle of the relative holomorphic tangent bundle to the universal curve. On the level of the mapping class groups there is a corresponding short exact sequence [4]

$$1 \to \mathbb{Z} \to \Gamma_{g,1} \to \Gamma^1_g \to 1.$$

The composition of the two morphisms discussed above is the morphism $\mathcal{M}_{g,1} \to \mathcal{M}_g$ obtained by forgetting the tangent vector. If C is a curve without non-trivial automorphisms, then the fiber over $[C] \in \mathcal{M}_g$ is isomorphic to $T^u C$, the frame bundle of the holomorphic tangent bundle of the curve C. The corresponding homomorphism of the mapping class groups is $\Gamma_{g,1} \to \Gamma_g$.

One can also consider finite index level subgroups $\Gamma^s_{g,r}[l]$ of $\Gamma^s_{g,r}$ for each integer l. The level l subgroup is defined to be the subgroup of $\Gamma^s_{g,r}$ which acts trivially on $H_1(S; \mathbb{Z}/l\mathbb{Z})$. Consequently, one has a short exact sequence

$$1 \to \Gamma^s_{g,r}[l] \to \Gamma^s_{g,r} \to \mathrm{Sp}_{2g}(\mathbb{Z}/l\mathbb{Z}) \to 1.$$

The quotient $\Gamma^s_{g,r}[l] \backslash \mathcal{T}^s_{g,r}$ is isomorphic to $\mathcal{M}^s_{g,r}[l]$, the moduli space of smooth projective curves with a level l structure which is defined in Section 2.

It is well-known that for all $g \geqslant 1$ and $l \geqslant 3$, the group $\Gamma^s_{g,r}[l]$ acts freely on $\mathcal{T}^s_{g,r}$. Thus for each $l \geqslant 3$ the moduli space $\mathcal{M}^s_{g,r}[l]$ is a smooth finite cover of $\mathcal{M}^s_{g,r}$.

When $\mathcal{M}^s_{g,r}$ is different from $\mathcal{M}_1$ and $\mathcal{M}_2$ each representation of $\Gamma^s_{g,r}$ determines an orbifold local system over $\mathcal{M}^s_{g,r}$. When $\mathcal{M}_g$ is either $\mathcal{M}_1$ or $\mathcal{M}_2$ we consider only such representations of Γ_g that for each $[C] \in \mathcal{M}_g$ represented by a curve with only two automorphisms, the stabilizer of $(C, [f]) \in \mathcal{T}_g$ acts trivially on the representation space. These representations give rise to orbifold local systems over $\mathcal{M}_1$ and $\mathcal{M}_2$.

Let V be a representation of $\Gamma^s_{g.r}$ on a rational vector space, and let $\mathbb{V}$ be the associated orbifold local system over $\mathcal{M}^s_{g,r}$. The contractibility of the Teichmüller space implies that for all $g \geqslant 1$

$$H^\bullet(\Gamma^s_{g,r}; V) \cong H^\bullet(\mathcal{M}^s_{g,r}; \mathbb{V}) \cong H^\bullet(\mathcal{M}^s_{g,r}[l]; \mathbb{V}[l])^{\mathrm{Sp}_{2g}(\mathbb{Z}/l\mathbb{Z})}.$$

2. Compactifications of the moduli space of curves

In this section we recall some basic properties of the Satake compactification and the Deligne–Mumford compactification of the moduli spaces of curves.

We start with the Deligne–Mumford compactification of $\mathcal{M}^s_g$. A *stable curve* is a reduced connected curve which has only nodes as singularities, and a finite automorphism group [8]. The *Deligne–Mumford compactification* $\overline{\mathcal{M}}^s_g$ of $\mathcal{M}^s_g$ is the moduli space of stable projective curves. It is a normal projective variety in which $\mathcal{M}^s_g$ is a Zariski open subset [8], [31, Th. 5.1]. The singularities of $\overline{\mathcal{M}}^s_g$ are contained in the locus of stable curves with non-trivial automorphisms [10, p. 218].

We will describe the boundary $\overline{\mathcal{M}}^s_g - \mathcal{M}^s_g$ in the case when $s = 0$. The boundary $\overline{\mathcal{M}}_g - \mathcal{M}_g$ is the union of irreducible divisors

$$\bigcup_{i=0}^{[g/2]} \Delta_i,$$

where each divisor Δ_i has the following property. When $i = 0$ there is birational morphism $\overline{\mathcal{M}}^{[2]}_{g-1} \to \Delta_0$; when $1 \leqslant i < g - i$ there is birational morphism $\overline{\mathcal{M}}^1_i \times \overline{\mathcal{M}}^1_{g-i} \to \Delta_i$; and when $i = g - i$ there is a birational morphism from the $\mathbb{Z}/2\mathbb{Z}$-quotient of $\overline{\mathcal{M}}^1_i \times \overline{\mathcal{M}}^1_i$ to Δ_i.

DEFINITION 2.1 (cf. [36, Def. 10.5]). A *level l structure* on a stable curve C is a symplectic monomorphism $H^1(C; \mathbb{Z}/l\mathbb{Z}) \to (\mathbb{Z}/l\mathbb{Z})^{2g}$, where $(\mathbb{Z}/l\mathbb{Z})^{2g}$ has the standard symplectic structure.

Note that a level l structure on a smooth curve C is just a choice of a symplectic basis for $H^1(C; \mathbb{Z}/l\mathbb{Z})$, or, equivalently, for $H_1(C; \mathbb{Z}/l\mathbb{Z})$ because the symplectic form determines the canonical identification between homology and cohomology. The same is true for a singular stable curve C whose dual graph is a tree.

From now on we assume that $l \geqslant 3$. Denote by $\mathcal{M}_g[l]$ the moduli space of smooth curves with a level l structure. It is isomorphic to the quotient of $\mathcal{T}_g$ by the action of $\Gamma_g[l]$ (cf. Sect. 1). The moduli space $\mathcal{M}_g[l]$ is a smooth quasi-projective variety,

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and the forgetful morphism $\mathcal{M}_g[l] \to \mathcal{M}_g$ is a Galois covering [8, Prop. 5.8], [33, Thm. 1.8].

When $g \geqslant 2$ and $l \geqslant 3$ there exists the moduli space of stable curves with a level l structure $\overline{\mathcal{M}}_g[l]$, which is a compactification of $\mathcal{M}_g[l]$ [8, p. 106], [29, Bem. 1], [35, Rem. 2.3.7]. This is a projective variety according to [32, Thm. 4, III.8], and there is a finite morphism $\overline{\mathcal{M}}_g[l] \to \overline{\mathcal{M}}_g$ determined by forgetting a level l structure.

In [29] Mostafa proves that $\overline{\mathcal{M}}_g[l]$ is not smooth, at least when $g \geqslant 3$. However, in this article we are interested in particular strata of the boundary of $\overline{\mathcal{M}}_g[l]$. The irreducible component Δ_1 of the boundary of $\overline{\mathcal{M}}_g$ contains a Zariski open subset isomorphic to $\mathcal{M}_1^1 \times \mathcal{M}_{g-1}^1$. Consider the inverse image of this subset under the finite morphism above. It is a finite disjoint union of locally closed subvarieties of codimension one each of which is isomorphic to $\mathcal{M}_1^1[l] \times \mathcal{M}_{g-1}^1[l]$. According to [29, Lem. 1], [27, p. 240] all points of this inverse image are smooth points of $\overline{\mathcal{M}}_g[l]$.

To introduce the Satake compactification $\widetilde{\mathcal{M}}_g$ of $\mathcal{M}_g$ we use the space $\mathcal{A}_g$, the moduli space of principally polarized abelian varieties of dimension g. It is the quotient of the Siegel upper-half space by the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$. The space $\mathcal{A}_g$ is a quasi-projective variety [30, Th. 7.10]. Among other compactifications, it admits the Satake compactification $\overline{\mathcal{A}}_g$ which is a projective variety [39].

The moduli space $\mathcal{M}_g$ is isomorphic to the image of the period map $\mathcal{M}_g \to \mathcal{A}_g$ which is a locally closed subvariety of $\mathcal{A}_g$ [33, Cor. 3.2]. The closure $\widetilde{\mathcal{M}}_g$ of $\mathcal{M}_g$ in the Satake compactification $\overline{\mathcal{A}}_g$ of $\mathcal{A}_g$ is called the *Satake compactification of* $\mathcal{M}_g$ (cf. [2]). There exists a birational morphism $\alpha: \overline{\mathcal{M}}_g \to \widetilde{\mathcal{M}}_g$ which is the identity on $\mathcal{M}_g$, and sends the point $[C]$ corresponding to a stable curve C to the polarized Jacobian of its normalization [26, p. 211].

The image of the boundary $\overline{\mathcal{M}}_g - \mathcal{M}_g$ under α is the boundary $\widetilde{\mathcal{M}}_g - \mathcal{M}_g$ of the Satake compactification. It follows that when $g \geqslant 3$ the boundary $\widetilde{\mathcal{M}}_g - \mathcal{M}_g$ has $[g/2]$ irreducible components each of which except one has codimension three in $\widetilde{\mathcal{M}}_g$. The irreducible component Φ_1 which is the image of $\Delta_1 \subset \overline{\mathcal{M}}_g$ has codimension two. It contains a Zariski open subset isomorphic to $\mathcal{M}_1 \times \mathcal{M}_{g-1}$.

One can also construct the *Satake compactification* $\widetilde{\mathcal{M}}_g[l]$ of $\mathcal{M}_g[l]$. Denote by $\mathcal{A}_g[l]$ the moduli space of principally polarized abelian varieties with a level l structure. A point in $\mathcal{A}_g[l]$ is represented by an abelian variety A of dimension g and a symplectic basis of $H_1(A; \mathbb{Z}/l\mathbb{Z})$. It is a quasi-projective variety [30, Th. 7.9], [33, Th. 1.8] which is smooth when $l \geqslant 3$. The space $\mathcal{A}_g[l]$ has the Satake compactification $\overline{\mathcal{A}}_g[l]$ which is a normal projective variety [36, p. 124], [39].

If $g = 1, 2$, then $\mathcal{M}_g[l]$ is isomorphic to a Zariski open subset of $\mathcal{A}_g[l]$, and we define the Satake compactification $\overline{\mathcal{A}}_g[l]$ of $\mathcal{A}_g[l]$ to be the Satake compactification of $\mathcal{M}_g[l]$. If $g \geqslant 3$, then the morphism $\mathcal{M}_g[l] \to \mathcal{A}_g[l]$ is not injective. In this case we define $\widetilde{\mathcal{M}}_g[l]$ to be the normalization of $\widetilde{\mathcal{M}}_g$ with respect to $\mathcal{M}_g[l]$.

It follows from this definition that $\widetilde{\mathcal{M}}_g[l]$ is a projective variety [32, III.8, Th. 4], and that the morphism $\mathcal{M}_g[l] \to \mathcal{M}_g$ extends to a finite morphism $\widetilde{\mathcal{M}}_g[l] \to \widetilde{\mathcal{M}}_g$. One can also show that there is a birational morphism $\alpha^l\colon \overline{\mathcal{M}}_g[l] \to \widetilde{\mathcal{M}}_g[l]$ with connected fibers which is the identity on $\mathcal{M}_g[l]$, and fits into the commutative diagram

$$\begin{tikzcd} \overline{\mathcal{M}}_g[l] \arrow[r, "\alpha^l"] \arrow[d] & \widetilde{\mathcal{M}}_g[l] \arrow[d] \\ \overline{\mathcal{M}}_g \arrow[r, "\alpha"] & \widetilde{\mathcal{M}}_g. \end{tikzcd}$$

The boundary $\widetilde{\mathcal{M}}_g[l] - \mathcal{M}_g[l]$ is the union of irreducible components each of which has codimension either two, or three in $\widetilde{\mathcal{M}}_g[l]$. The image of each component Φ_1^β of codimension two under the morphism $\widetilde{\mathcal{M}}_g[l] \to \widetilde{\mathcal{M}}_g$ is the codimension two component Φ_1 of $\widetilde{\mathcal{M}}_g - \mathcal{M}_g$. One can show that each Φ_1^β contains a Zariski open subset Z_β such that these subsets do not intersect each other, and each of them is isomorphic to a smooth Zariski open subset of $\mathcal{M}_1[l] \times \mathcal{M}_{g-1}[l]$.

3. Codimension two stratum of the Satake compactification

In this section we analyze the link of the codimension two boundary stratum Φ_1 inside the Satake compactification of the moduli space $\widetilde{\mathcal{M}}_g$. More precisely, we study the local links of the points in a smooth Zariski open subset of Φ_1, and we show that $\widetilde{\mathcal{M}}_g$ is equi-singular along this Zariski open subset. We will need this in Section 4. For the rest of this section we assume that $g \geqslant 4$.

Recall that Φ_1 contains a Zariski open subset X isomorphic to $\mathcal{M}_1 \times \mathcal{M}_{g-1}$. We identify it with $\mathcal{M}_1 \times \mathcal{M}_{g-1}$. Then a point in X is represented by a pair of isomorphism classes of curves $([C_1],[C_2])$. Let X° be a Zariski open subset of X defined as follows. Recall that in Section 1 we defined ${}^\circ\mathcal{M}_g$ to be the locus of curves with only trivial automorphisms when $g \geqslant 3$. We define ${}^\circ\mathcal{M}_1$ to be the locus of elliptic curves with exactly two automorphisms. Then X° is the subset of X corresponding to ${}^\circ\mathcal{M}_1 \times {}^\circ\mathcal{M}_{g-1}$. In this section we study the link of X° in $\mathcal{M}_g \cup X^\circ \subset \widetilde{\mathcal{M}}_g$.

Let N be a regular neighborhood of X° in $\mathcal{M}_g \cup X^\circ$. The complement $N^* = N - X^\circ$ is a deleted regular neighborhood of X°.

Recall that $\mathcal{M}_{g-1,1} \to \mathcal{M}_{g-1}$ is a surjective morphism defined by forgetting the holomorphic tangent vector. Let L_2 be the inverse image of ${}^\circ\mathcal{M}_{g-1}$ in $\mathcal{M}_{g-1,1}$, and $\widetilde{\pi}_2\colon L_2 \to {}^\circ\mathcal{M}_{g-1}$ be the corresponding map. The fiber of $\widetilde{\pi}_2$ over $[C_2] \in {}^\circ\mathcal{M}_{g-1}$

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is T^uC_2, the punctured holomorphic tangent bundle. Denote by L the product ${}^\circ\mathcal{M}_1 \times L_2$, and by $\widetilde{\pi}$ the pull-back of $\widetilde{\pi}_2$ to X°

$$\begin{tikzcd} L = {}^\circ\mathcal{M}_1 \times L_2 \arrow[r, "pr_2"] \arrow[d, "\widetilde{\pi}"'] & L_2 \arrow[d, "\widetilde{\pi}_2"'] \\ X^\circ = {}^\circ\mathcal{M}_1 \times {}^\circ\mathcal{M}_{g-1} \arrow[r, "pr_2"] & {}^\circ\mathcal{M}_{g-1}. \end{tikzcd}$$

LEMMA 3.1. *The bundle $\widetilde{\pi}\colon L = {}^\circ\mathcal{M}_1 \times L_2 \to X^\circ$ is a two-to-one unramified cover of the punctured regular neighborhood N^*. The corresponding fix point free action of $\mathbb{Z}/2\mathbb{Z}$ on L sends a vector v to $-v$.*

Proof. The morphism $\mathcal{M}_{g-1,1} \to \mathcal{M}_{g-1}$ factors as

$$\mathcal{M}_{g-1,1} \to \mathcal{M}^1_{g-1} \to \mathcal{M}_{g-1}.$$

Denote by Y_2 the inverse image of ${}^\circ\mathcal{M}_{g-1}$ under the second morphism. Then the commutative diagram above factors as

$$\begin{tikzcd} L = {}^\circ\mathcal{M}_1 \times L_2 \arrow[r, "pr_2"] \arrow[d, "\pi^c"'] & L_2 \arrow[d, "\pi^c_2"'] \\ Y = {}^\circ\mathcal{M}_1 \times Y_2 \arrow[r, "pr_2"] \arrow[d, "\bar{\pi}"'] & Y_2 \arrow[d, "\bar{\pi}_2"'] \\ X^\circ = {}^\circ\mathcal{M}_1 \times {}^\circ\mathcal{M}_{g-1} \arrow[r, "pr_2"] & {}^\circ\mathcal{M}_{g-1}, \end{tikzcd}$$

where π^c_2 (resp. $\bar{\pi}_2$) is the restriction of $\mathcal{M}_{g-1,1} \to \mathcal{M}^1_{g-1}$ (resp. $\mathcal{M}^1_{g-1} \to \mathcal{M}_{g-1}$) to L_2 (resp. Y_2), and π^c (resp. $\bar{\pi}$) is its pull-back along pr_2.

At the same time $Y = {}^\circ\mathcal{M}_1 \times Y_2$ is isomorphic to a smooth Zariski open subset of the boundary component Δ_1 in the Deligne–Mumford compactification. We identify Y with this Zariski open subset. Then the morphism $\bar{\pi}\colon Y \to X^\circ$ is the restriction of the morphism $\alpha\colon \overline{\mathcal{M}}_g \to \widetilde{\mathcal{M}}_g$ to Y.

The morphism α is the identity when restricted to $\mathcal{M}_g$. Therefore a deleted regular neighborhood N^* of X° in $\mathcal{M}_g \cup X^\circ$ and a deleted regular neighborhood of the divisor Y in $\mathcal{M}_g \cup Y \subset \overline{\mathcal{M}}_g$ can be chosen to be the same.

The deleted neighborhood of Y is homeomorphic to the punctured normal bundle of Y in $\mathcal{M}_g \cup Y$. Note that the only non-trivial automorphism of a pair $(C_1, x_1), (C_2, x_2)$ representing a point in Y is induced by the elliptic involution of (C_1, x_1). It follows that $\mathbb{Z}/2\mathbb{Z}$ acts on the space of versal deformations of the stable curve $(C_1, x_1), (C_2, x_2)$, and this action fixes the divisor that is the locus of the singular curves [1, Chap. 13, Lem. (1.6)]. Therefore the fiber of the normal bundle of $Y \subset \Delta_1$ at the point $[(C_1, x_1), (C_2, x_2)]$ is isomorphic to the $\mathbb{Z}/2\mathbb{Z}$ quotient of $T_{x_1}C_1 \otimes T_{x_2}C_2$, where the generator of $\mathbb{Z}/2$ acts as $-id$. Thus N^* is the $\mathbb{Z}/2\mathbb{Z}$ quotient of the $\mathbb{C}^*$-bundle L' over Y whose fiber at $[(C_1, x_1), (C_2, x_2)]$ is $T_{x_1}C_1 \otimes T_{x_2}C_2 - \{0\}$.

It is well-known that the moduli space of elliptic curves $\mathcal{M}_1$ is isomorphic to $\mathbb{C}$. It contains two distinguished points that correspond to the two elliptic curves with exceptional automorphisms. It follows that the space ${}^{\circ}\mathcal{M}_1$ is isomorphic to $\mathbb{C} - \{2 \text{ points}\}$. All line bundles over this space are trivial. Therefore the bundle L' is the pull-back of the punctured relative tangent bundle of the morphism $Y_2 \to {}^{\circ}\mathcal{M}_{g-1}$.

The punctured relative tangent bundle of the morphism $Y_2 \to {}^{\circ}\mathcal{M}_{g-1}$ is π_2^c: $L_2 \to Y_2$. Hence, one has a commutative diagram

$$\begin{array}{ccc} L' & \longrightarrow & L_2 \\ \downarrow & & \downarrow{\scriptstyle \pi_2^c} \\ Y = {}^{\circ}\mathcal{M}_1 \times Y_2 & \xrightarrow{pr_2} & Y_2, \end{array}$$

where L' is the pull-back of L_2. We conclude that the bundles L' and L are isomorphic, and N^* is the $\mathbb{Z}/2\mathbb{Z}$ quotient of L, where $\mathbb{Z}/2\mathbb{Z}$ action sends a vector in a fiber of π^c to its opposite. □

It follows from the lemma above that $\widetilde{\mathcal{M}}_g$ is equi-singular along X°. We expressed N^* as a bundle X° whose fiber over the point $([C_1, C_2])$ is equal to $T^u C_2$, the frame bundle of the holomorphic tangent bundle of C_2.

4. Main theorem

In this section we prove the main theorem of this article. The proof consists of a sequence of lemmas and propositions. We assume that the reader is familiar with intersection cohomology, and suggest the references [3, 5, 12].

Notation. For the rest of the paper we omit $R^{\bullet}$ from the notation for the derived functors. For example, if $f : X \to Y$ is a continuous map between topological spaces, then $f_* = R^{\bullet} f_*$.

As we mentioned before each representation of the mapping class group Γ_g, at least when $g \geqslant 3$, determines an orbifold local system over $\mathcal{M}_g$. In this section we consider only the *symplectic* local systems, that is local systems arising from finite dimensional rational representations of the algebraic group Sp_{2g}. We fix a symplectic representation V of Γ_g, and denote the corresponding orbifold local system by $\mathbb{V}$.

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THEOREM 4.1. *The natural map* $IH^k(\widetilde{\mathcal{M}}_g;\mathbb{V})\to H^k(\mathcal{M}_g;\mathbb{V})$ *induced by the inclusion is an isomorphism, when*

$$\begin{aligned} &k=0, \quad g\geqslant 1;\\ &k=1, \quad g\geqslant 3;\\ &k=2, \quad g\geqslant 6.\end{aligned}$$

The first statement is trivial and included only for the sake of completeness. The statement concerning the first cohomology is also rather simple. Indeed, in Section 2 we saw that if $g\geqslant 3$, then the boundary $\widetilde{\mathcal{M}}_g-\mathcal{M}_g$ of the Satake compactification has codimension two in $\widetilde{\mathcal{M}}_g$. This, and the properties of intersection cohomology immediately imply the statement of the theorem for $k=1$. The non-trivial part of this theorem concerns the second cohomology.

Remark. If $g\geqslant 3$, then the map $IH^1(\widetilde{\mathcal{M}}_g;\mathbb{V})\to H^1(\mathcal{M}_g;\mathbb{V})$ is an isomorphism for an arbitrary orbifold local system $\mathbb{V}$ determined by a representation of Γ_g on a rational vector space. This can be easily seen from the above argument.

Combining this with the computations of $H^1(\mathcal{M}_g;\mathbb{V})$ in [14, 23] one gets the following corollary.

COROLLARY 4.2. *If* $g\geqslant 3$ *and* $\mathbb{V}(\lambda)$ *is a generically defined local system corresponding to the representation of* Sp_{2g} *with the highest weight* λ*, then*

$$IH^1(\widetilde{\mathcal{M}}_g;\mathbb{V}(\lambda))\cong\begin{cases}\mathbb{Q} & \textit{when } \lambda=\lambda_3;\\ 0 & \textit{otherwise}.\end{cases}$$

□

The rest of this section is devoted to the proof of the isomorphism in second cohomology. We assume that $g\geqslant 4$. Recall that we denote by Φ_1 the codimension two irreducible component of the boundary of $\widetilde{\mathcal{M}}_g$, and by X° its Zariski open subset isomorphic to ${}^\circ\mathcal{M}_1\times{}^\circ\mathcal{M}_{g-1}$.

Notation. We denote by $\mathcal{S}^\bullet$ the intersection cohomology sheaf $\mathcal{IC}^\bullet(\mathbb{V})$ on $\widetilde{\mathcal{M}}_g$ corresponding to the local system $\mathbb{V}$. The following diagram defines the notation for the inclusions

$$\mathcal{M}_g \xhookrightarrow{i} \mathcal{M}_g\cup X^\circ \xhookleftarrow{j} X^\circ.$$

First, we use again that the boundary of $\widetilde{\mathcal{M}}_g$ has only one irreducible component of codimension two, namely Φ_1, and all other irreducible components have codimension three. This and the properties of intersection cohomology imply that the restriction

$$IH^2(\widetilde{\mathcal{M}}_g;\mathbb{V})\to IH^2(\mathcal{M}_g\cup X^\circ;\mathbb{V})$$

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is an isomorphism, and there is an exact sequence

$$\begin{aligned} 0 &\to IH^2(\mathcal{M}_g \cup X^\circ; \mathbb{V}) \to H^2(\mathcal{M}_g; \mathbb{V}) \xrightarrow{\phi} H^3(X^\circ; j^! \mathcal{S}^\bullet) \\ &\cong H^0(X^\circ; \mathcal{H}^3 j^! \mathcal{S}^\bullet). \end{aligned}$$

Therefore to prove the theorem it suffices to show that ϕ from the exact sequence above is the zero morphism.

The distinguished triangle

$$\begin{tikzcd} j^! \mathcal{S}^\bullet \arrow[rr] & & j^* \mathcal{S}^\bullet \arrow[dl] \\ & j^* i_* \mathbb{V} \arrow[ul, "{[1]}"] & \end{tikzcd}$$

implies that $\mathcal{H}^3 j^! \mathcal{S}^\bullet \cong \mathcal{H}^2 j^* i_* \mathbb{V}$. Then the morphism ϕ composed with this isomorphism can be factored as

$$H^2(\mathcal{M}_g; \mathbb{V}) \to H^2(X^\circ; j^* i_* \mathbb{V}) \xrightarrow{\psi} H^0(X^\circ; \mathcal{H}^2 j^* i_* \mathbb{V}).$$

The sheaf $j^* i_* \mathbb{V}$ is called the *local link cohomology functor* [9, p. 57]. It expresses the cohomology of N^*, the link of X° in $\mathcal{M}_g \cup X^\circ$. We denote by π the corresponding projection $N^* \to X^\circ$. Then the morphism ψ from the sequence above can be written as

$$\psi \colon H^2(N^*; \mathbb{V}) \to H^0(X^\circ; \mathcal{H}^2 \pi_* \mathbb{V}).$$

One can easily check that ψ is the edge homomorphism associated to the Leray–Serre spectral sequence determined by π.

In order to prove the theorem it is enough to show that ψ is the trivial homomorphism when $g \geqslant 6$, and the rest of this section deals with the proof of this fact.

First we want to understand the behavior of the local system $\mathbb{V}$ over N^*. We start with the following lemma.

LEMMA 4.3. *The orbifold local system* $\mathbb{V}$ *over* N^* *splits into a direct sum of symplectic orbifold local systems determined by rational representations of* $\mathrm{Sl}_2 \times \mathrm{Sp}_{2g-2}$.

Proof. Recall that a symplectic orbifold local system is determined by a representation of Γ_g which is the pull-back of an algebraic representation V of Sp_{2g}.

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Choose a level $l \geqslant 3$. The inverse image of $X^\circ \subset \widetilde{\mathcal{M}}_g$ in $\widetilde{\mathcal{M}}_g[l]$ has several connected components. Let N_l^* be a deleted regular neighborhood of one of them. Then one has a commutative diagram

$$\begin{array}{ccccc} N_l^* & \longrightarrow & \mathcal{M}_g[l] & \longrightarrow & \mathcal{A}_g[l] \\ \downarrow & & \downarrow & & \downarrow \\ N^* & \longrightarrow & \mathcal{M}_g & \longrightarrow & \mathcal{A}_g. \end{array}$$

(Recall that A_g stands for the moduli space of principally polarized abelian varieties.) Denote by $\mathbb{V}_l$ the pull-back of $\mathbb{V}$ to $\mathcal{M}_g[l]$, and by $\mathbb{V}_l'$ the local system over $A_g[l]$ determined by V. Both $\mathbb{V}_l$ and $\mathbb{V}_l'$ are genuine local systems, and $\mathbb{V}_l$ is the pull-back of $\mathbb{V}_l'$ under $\mathcal{M}_g[l] \to \mathcal{A}_g[l]$.

The product $\mathcal{A}_1 \times \mathcal{A}_{g-1}$ is canonically embedded in $\mathcal{A}_g$. Its inverse image under $\mathcal{A}_g[l] \to \mathcal{A}_g$ consists of several connected component, each of which is isomorphic to $\mathcal{A}_1[l] \times \mathcal{A}_{g-1}[l]$. The image of N_l^* in $\mathcal{A}_g[l]$ is contained in a tubular neighborhood of one of these connected components. The local system $\mathbb{V}_l'$, restricted to this connected component, splits according to the branching law of the inclusion $\mathrm{Sl}_2 \times \mathrm{Sp}_{g-2} \hookrightarrow \mathrm{Sp}_{2g}$. It follows that the local system $\mathbb{V}_l$ splits over N_l^* according to the same branching law. In addition, $\mathbb{V}_l$ is constant on the fibers of the composite

$$N_l^* \longrightarrow N^* \xrightarrow{\pi} X^\circ.$$

Thus the splitting of $\mathbb{V}_l$ over N_l^* descends to the splitting of $\mathbb{V}$ over N^*. $\square$

Our aim is to show that the morphism ψ is trivial. Therefore without loss of generality we can assume that $\mathbb{V}$ is a local system over N^* determined by an irreducible algebraic representation of $\mathrm{Sl}_2 \times \mathrm{Sp}_{2g-2}$ with highest weight (μ, ν). Note that μ is just a non-negative integer.

We consider two cases. First, assume that μ is odd. The morphism $N_l^* \to N^*$ is a Galois covering with the Galois group $\mathrm{Sl}_2(\mathbb{Z}/l\mathbb{Z}) \times \mathrm{Sp}_{2g-2}(\mathbb{Z}/l\mathbb{Z})$. The element $(-id, id)$ of this group leaves the fibers of $N_l^* \to N^* \to X^\circ$ fixed because it corresponds to the involution of the elliptic curve, and acts as $-id$ on the local system $\mathbb{V}_l$. It follows that $\mathcal{H}^2\pi_*\mathbb{V}$ is the trivial local system, and we have nothing more to prove.

Next, assume that μ is even. Then $(-id, id)$ acts trivially on the local system $\mathbb{V}_l$, and therefore the local system $\mathbb{V}$ extends to X°. This means that $\mathbb{V}$ is the restriction to N^* of a local system defined on the whole regular neighborhood N of X°. Denote the restriction of this local system to X° by $\overline{\mathbb{V}}$. Then $\overline{\mathbb{V}}$ is isomorphic to $\mathbb{W}_1(\mu) \boxtimes \mathbb{W}_2(\nu)$, the symplectic local system over ${}^\circ\mathcal{M}_1 \times {}^\circ\mathcal{M}_{g-1}$ determined by the highest weight (μ, ν). The local system $\mathbb{V}$ is the pull-back of $\overline{\mathbb{V}}$ under $\pi\colon N^* \to X^\circ$.

Lemma 3.1 says that N^* is the $\mathbb{Z}/2\mathbb{Z}$ quotient of the bundle L defined in Section 3, and π is induced by the projection $\widetilde{\pi}: L \to X^\circ$. We denote by $\widetilde{\mathbb{V}}$ be the pull-back of $\mathbb{V}$ to L. Then

$$\widetilde{\mathbb{V}} = \widetilde{\pi}^*\overline{\mathbb{V}} \cong \widetilde{\pi}^*(\mathbb{W}_1(\mu) \boxtimes \mathbb{W}_2(\nu)).$$

Let $B_\bullet$ be the Leray–Serre spectral sequence determined by π, and $A_\bullet$ be the Leray–Serre spectral sequence determined by $\widetilde{\pi}$. Let $\widetilde{\psi}$ be the edge homomorphism $H^2(L;\widetilde{\mathbb{V}}) \to H^0(X^\circ;\mathcal{H}^2\widetilde{\pi}_*\widetilde{\mathbb{V}})$ associated $A_\bullet$. The two-fold covering map $L \to N^*$ induces the map of the spectral sequences $B_\bullet \to A_\bullet$. One has for each q [6, p. 85]

$$\mathcal{H}^q\pi_*\mathbb{V} = (\mathcal{H}^q\widetilde{\pi}_*\widetilde{\mathbb{V}})^{\mathbb{Z}/2\mathbb{Z}}.$$

It follows that the induced map $B_2^{0,q} \to A_2^{0,q}$ is an inclusion of global $\mathbb{Z}/2\mathbb{Z}$ invariants. The homomorphism $H^2(N^*;\mathbb{V}) \to H^2(L;\widetilde{\mathbb{V}})$ is also an inclusion of $\mathbb{Z}/2\mathbb{Z}$ invariants, and one has a commutative diagram

$$\begin{tikzcd}
H^2(L;\widetilde{\mathbb{V}}) \arrow[r, "\widetilde{\psi}"] & H^0(X^\circ;\mathcal{H}^2\widetilde{\pi}_*\widetilde{\mathbb{V}}) \\
H^2(N^*;\mathbb{V}) \arrow[r, "\psi"] \arrow[u] & H^0(X^\circ;\mathcal{H}^2\pi_*\mathbb{V}), \arrow[u]
\end{tikzcd}$$

where both vertical maps are inclusions. It follows that if $\widetilde{\psi}$ is trivial, then ψ is trivial. We shall show that $\widetilde{\psi}$ is trivial.

Note that $\widetilde{\pi}_*\widetilde{\mathbb{V}}$ is quasi-isomorphic to $\widetilde{\pi}_*\mathbb{Q} \otimes \overline{\mathbb{V}}$. It follows that $\mathcal{H}^2\widetilde{\pi}_*\widetilde{\mathbb{V}}$ is isomorphic to $\mathcal{H}^2\widetilde{\pi}_*\mathbb{Q} \otimes \overline{\mathbb{V}}$.

LEMMA 4.4. *The local system $\mathcal{H}^2\widetilde{\pi}_*\mathbb{Q}$ over X° is isomorphic to* $\mathbb{W}_1(0) \boxtimes \mathbb{W}_2(\nu_1)$.

Proof. The bundle $\widetilde{\pi}$ is the pull-back of the bundle $\widetilde{\pi}_2: L_2 \to {}^\circ\mathcal{M}_{g-1}$ to X° (see Lemma 3.1). It follows that the local system $\mathcal{H}^2\widetilde{\pi}_*\mathbb{Q}$ is the exterior tensor product of the constant local system $\mathbb{W}_1(0)$ over ${}^\circ\mathcal{M}_1$ and $\mathcal{H}^2\widetilde{\pi}_{2*}\mathbb{Q}$.

Recall that $\widetilde{\pi}_2$ factors as

$$L_2 \xrightarrow{\pi_2^c} Y_2 \xrightarrow{\bar{\pi}_2} {}^\circ\mathcal{M}_{g-1},$$

where $\bar{\pi}_2$ is the restriction of the universal curve to Y_2, and π_2^c is a punctured relative tangent bundle to $\bar{\pi}_2$. Therefore we have a Gysin long exact sequence of local systems

$$\cdots \to \mathcal{H}^0\bar{\pi}_{2*}\mathbb{Q} \xrightarrow{e} \mathcal{H}^2\bar{\pi}_{2*}\mathbb{Q} \to \mathcal{H}^2\widetilde{\pi}_{2*}\mathbb{Q} \to \mathcal{H}^1\bar{\pi}_{2*}\mathbb{Q} \to 0, \tag{4.1}$$

where e is the multiplication by the Euler class. The Euler class is non-zero, because the genus of C_2 is greater than one. It follows that e is an isomorphism on rational cohomology. Thus we conclude that $\mathcal{H}^2\widetilde{\pi}_{2*}\mathbb{Q}$ is isomorphic to $\mathcal{H}^1\bar{\pi}_{2*}\mathbb{Q}$.

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The local system $\mathcal{H}^1\bar{\pi}_{2*}\mathbb{Q}$ is isomorphic to $\mathbb{W}_2(\nu_1)$, the local system corresponding to the standard representation of Sp_{2g-2}. It follows that

$$\mathcal{H}^2\pi^*\mathbb{Q} \cong \mathbb{W}_1(0) \boxtimes \mathbb{W}_2(\nu_1). \qquad \square$$

The lemma above shows that the edge homomorphism $\widetilde{\psi}$ is of the form

$$H^2(L; \widetilde{\pi}^*(\mathbb{W}_1(\mu) \boxtimes \mathbb{W}_2(\nu)))$$
$$\to H^0(X^\circ; \mathbb{W}_1(\mu) \boxtimes (\mathbb{W}_2(\nu_1) \otimes \mathbb{W}_2(\nu))).$$

LEMMA 4.5. *The space $H^0(X^\circ; \mathbb{W}_1(\mu) \boxtimes (\mathbb{W}_2(\nu_1) \otimes \mathbb{W}_2(\nu)))$ is isomorphic to $\mathbb{Q}$ if $\mu = 0$ and $\nu = \nu_1$, and zero otherwise.*

Proof. Applying the Künneth formula one gets

$$H^0(X^\circ; \mathbb{W}_1(\mu) \boxtimes (\mathbb{W}_2(\nu_1) \otimes \mathbb{W}_2(\nu)))$$
$$\cong H^0({}^\circ\mathcal{M}_1; \mathbb{W}_1(\mu)) \otimes H^0({}^\circ\mathcal{M}_{g-1}; \mathbb{W}_2(\nu_1) \otimes \mathbb{W}_2(\nu)).$$

The zero cohomology of a space with coefficients in a local system is equal to the space of global invariants of the local system. An irreducible symplectic local system has no global invariants unless it is constant. This implies that $H^0({}^\circ\mathcal{M}_1; \mathbb{W}_1(0)) \cong \mathbb{Q}$, and $H^0({}^\circ\mathcal{M}_1; \mathbb{W}_1(\mu)) = 0$ if $\mu \neq 0$.

Similarly, $H^0({}^\circ\mathcal{M}_{g-1}; \mathbb{W}_2(\nu_1) \otimes \mathbb{W}_2(\nu))$ is equal to zero, unless the local system $\mathbb{W}_2(\nu_1) \otimes \mathbb{W}_2(\nu)$ contains a constant local system as a direct summand. This occurs if and only if the tensor product $W_2(\nu_1) \otimes W_2(\nu)$ of irreducible representations of $\mathrm{Sp}_{2g-2}(\mathbb{Q})$ contains a copy of the trivial representation. It is known that all irreducible representations of the symplectic group are self-dual. Therefore the trivial part of that representation is equal to

$$(W_2(\nu_1) \otimes W_2(\nu))^{\mathrm{Sp}_{2g-2}(\mathbb{Q})}$$
$$= W_2(\nu_1) \otimes_{\mathrm{Sp}_{2g-2}(\mathbb{Q})} W_2(\nu) \cong \mathrm{Hom}_{\mathrm{Sp}_{2g-2}(\mathbb{Q})}(\mathbb{W}_2(\nu_1); \mathbb{W}_2(\nu)).$$

By Schur's lemma the latter term is isomorphic to $\mathbb{Q}$, if $\nu = \nu_1$, and 0 otherwise. $\square$

It follows that $\widetilde{\psi}$ is trivial unless $\overline{\mathbb{V}} \cong \mathbb{W}_1(0) \boxtimes \mathbb{W}_2(\nu_1)$. In the remaining part of this section we study this case. To simplify the notation we denote $\mathbb{W}_1(0) \boxtimes \mathbb{W}_2(\nu_1)$ by $\mathbb{W}_2(\nu_1)$.

LEMMA 4.6. *If $g \geqslant 6$, then the homomorphism*

$$\widetilde{\psi}: H^2(L; \widetilde{\pi}^*\mathbb{W}_2(\nu_1)) \to H^0(X^\circ; \mathbb{W}_2(\nu_1)^{\otimes 2})$$

is the zero map.

Proof. Note that $\widetilde{\psi}$ factors through the $A_\infty^{0,2}$ term of the spectral sequence

$$A_2^{p,q} = H^p(X^\circ; \mathcal{H}^q\widetilde{\pi}_*\mathbb{Q} \otimes \mathbb{W}_2(\nu_1)) \Rightarrow H^{p+q}(L; \widetilde{\pi}^*\mathbb{W}_2(\nu_1)).$$

Therefore, it suffices to prove that $A_\infty^{0,2} = 0$.

The morphism $\widetilde{\pi}_2: L_2 \to {}^\circ\mathcal{M}_{g-1}$ gives rise to the following Leray–Serre spectral sequence

$$C_2^{p,q} = H^p({}^\circ\mathcal{M}_{g-1}; \mathcal{H}^q\widetilde{\pi}_{2*}\mathbb{Q} \otimes \mathbb{W}_2(\nu_1)) \Rightarrow H^{p+q}(L_2; \widetilde{\pi}_2^*\mathbb{W}_2(\nu_1)).$$

The bundle $\widetilde{\pi}: L \to X^\circ$ is the pull-back of L_2, and the local system $\mathbb{W}_2(\nu_1)$ over X° is also the pull-back from the second factor. It follows that the morphism of spectral sequences $C_\bullet \to A_\bullet$ induced by the projection $pr_2: X^\circ \to {}^\circ\mathcal{M}_{g-1}$ is an inclusion of a direct summand. (Here we mean that for each (r, p, q) the term $C_r^{p,q}$ is a direct summand of $A_r^{p,q}$, and all differentials d_r respect this splitting.)

Note that $A_2^{0,2} \cong C_2^{0,2}$. Indeed,

$$\begin{aligned} A_2^{0,2} &= H^0(X^\circ; \mathbb{W}_2(\nu_1)^{\otimes 2}) \\ &\cong H^0({}^\circ\mathcal{M}_1; \mathbb{Q}) \otimes H^0({}^\circ\mathcal{M}_{g-1}; \mathbb{W}_2(\nu_1)^{\otimes 2}) \\ &\cong H^0({}^\circ\mathcal{M}_{g-1}; \mathbb{W}_2(\nu_1)^{\otimes 2}) = C_2^{0,2}, \end{aligned}$$

because $\mathcal{H}^2\widetilde{\pi}_{2*}\mathbb{Q} \cong \mathbb{W}_2(\nu_1)$ according to exact sequence (4.1). It follows that $A_\infty^{0,2} \cong C_\infty^{0,2}$.

The final step is to show that $C_\infty^{0,2} = 0$. There is a surjective homomorphism

$$H^2(L_2; \widetilde{\pi}_2^*\mathbb{W}_2(\nu_1)) \to C_\infty^{0,2},$$

associated to the spectral sequence $C_\bullet$. Therefore it suffices to show that

$$H^2(L_2; \widetilde{\pi}_2^*\mathbb{W}_2(\nu_1)) = 0.$$

The complement of the Zariski open subset ${}^\circ\mathcal{M}_{g-1}$ of $\mathcal{M}_{g-1}$ has complex codimension $g-3$ (cf. Sect. 1). It follows that L_2 also has complex codimension $g-3$ in $\mathcal{M}_{g-1,1}$. Thus

$$H^2(L_2; \widetilde{\pi}_2^*\mathbb{W}_2(\nu_1)) \cong H^2(\mathcal{M}_{g-1,1}; \mathbb{W}_2(\nu_1))$$

when $g-3 \geqslant 3$. The mapping class group of $\mathcal{M}_{g-1,1}$ is $\Gamma_{g-1,1}$, and their rational cohomology are the same. In particular,

$$\begin{aligned} H^2(\mathcal{M}_{g-1,1}; \mathbb{W}_2(\nu_1)) &\cong H^2(\Gamma_{g-1,1}; W_2(\nu_1)) \\ &= H^2(\Gamma_{g-1,1}; H^1(S; \mathbb{Q})), \end{aligned}$$

where S is a reference surface of genus $g-1$. In Lemma 4.7 below (based on a result of Harer) we show that $H^2(\Gamma_{g,1}; H^1(S; \mathbb{Q}))$ when $g \geqslant 5$. This implies that $C_\infty^{0,2} = 0$, and therefore both homomorphisms $\widetilde{\psi}$ and ψ are zero homomorphisms when $g \geqslant 6$. □

LEMMA 4.7. *If* $g \geqslant 5$, *then* $H^2(\Gamma_{g,1}; H^1(S;\mathbb{Q})) = 0$.

Proof. All cohomology groups are considered with rational coefficients. The homomorphism $\Gamma^1_{g,1} \to \Gamma_{g,1}$ is defined by forgetting a fixed point, and therefore is surjective. We can choose a fixed point in a neighborhood of the base point of a fixed tangent vector. This determines a splitting of the homomorphism above. As the associated spectral sequence has two rows, the existence of splitting implies that the spectral sequence degenerates at E_2. Hence, $H^2(\Gamma_{g,1}; H^1(S))$ is a direct summand of $H^3(\Gamma^1_{g,1})$ [19, Sect. 7]. Thus it suffices to prove that $H^3(\Gamma^1_{g,1}) = 0$.

There is a short exact sequence of groups

$$1 \to \mathbb{Z} \to \Gamma_{g,2} \to \Gamma^1_{g,1} \to 1.$$

It determines a Gysin long exact sequence

$$\cdots \to H^1(\Gamma^1_{g,1}) \to H^3(\Gamma^1_{g,1}) \to H^3(\Gamma_{g,2}) \to \cdots$$

We know that the last term is trivial according to Theorem 3.1 from [20]. The first term is trivial by [14, Prop. 5.2]. It follows that the middle term $H^3(\Gamma^1_{g,1})$ is also zero. □

Remark. In Theorem 3.1 from [20] Harer gives an explicit description of a basis of $H_3(\Gamma_{4,2}) \cong \mathbb{Q}$. Using this one can deduce that $H_3(\Gamma^1_{4,1})$ is trivial, and therefore that $H^2(\Gamma_{4,1}; H^1(S;\mathbb{Q}))$ is trivial.

Recall that each irreducible symplectic local system over $\mathcal{M}_g$ is determined by its highest weight λ. If $\lambda_1, \dots, \lambda_g$ is a set of fundamental weights of Sp_{2g}, then λ is uniquely expressed as $\sum_{i=1}^g a_i\lambda_i$ for some non-negative integers a_i. We defined $|\lambda|$ to be $\sum_{i=1}^g ia_i$.

DEFINITION 4.8. We say that an irreducible symplectic local system over $\mathcal{M}_g$ determined by the highest weight λ is *even* if $|\lambda|$ is even, and it is *odd* if $|\lambda|$ is odd.

The following corollary is a consequence of the proof of the main theorem.

COROLLARY 4.9. *If* $\mathbb{V}$ *is an even local system, then the natural map*

$$IH^2(\widetilde{\mathcal{M}}_g; \mathbb{V}) \to H^2(\mathcal{M}_g; \mathbb{V})$$

is an isomorphism when $g \geqslant 4$.

Proof. The estimate $g \geqslant 6$, rather than $g \geqslant 4$, appears in the proof of Lemma 4.6. This lemma deals with the case when a symplectic local system $\mathbb{V}$ restricted to N^* has a direct summand isomorphic to the irreducible local system $\pi^*(\mathbb{W}_1(0) \boxtimes \mathbb{W}_2(\nu_1))$. Note that $\mathbb{V}$ contains such direct summand if and only if the corresponding algebraic representation V of Sp_{2g} restricted to the subgroup $\mathrm{Sl}_2 \times \mathrm{Sp}_{2g-2}$ contains a copy of $W_1(0) \boxtimes W_2(\nu_1)$. The branching rule of Sp_{2g} over $\mathrm{Sl}_2 \times \mathrm{Sp}_{2g-2}$ respects even and odd components. Therefore if V is even, then its restriction cannot contain $W_1(0) \boxtimes W_2(\nu_1)$. This implies that in this case $\widetilde{\psi}$ is trivial for all $g \geqslant 4$. □

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5. Mixed Hodge theory

In this section we consider the mixed Hodge structure on $H^2(\mathcal{M}_g;\mathbb{V})$ where $\mathbb{V}$ is an irreducible symplectic local system. We prove that the mixed Hodge structure on $H^2(\mathcal{M}_g;\mathbb{V})$ is pure when $g \geqslant 6$. We also prove that if $g = 3, 4, 5$, then the mixed Hodge structure on $H^2(\mathcal{M}_g;\mathbb{V})$ has at most two weights. In this section we assume that $g \geqslant 3$.

We use results of the theory of mixed Hodge modules developed by M. Saito. For definitions and results we refer the reader to [37, 38]. In this paper we use only the formal properties of mixed Hodge modules.

Notation. Let $H = (H_{\mathbb{Q}}, H_{\mathbb{C}}, W_{\bullet}, F^{\bullet})$ be a rational mixed Hodge structure where $\mathbb{W}_{\bullet}$ denotes the weight filtration, and $F^{\bullet}$ denotes the Hodge filtration. Denote the graded quotient $W_k H_{\mathbb{Q}}/W_{k-1}H_{\mathbb{Q}}$ by $\mathrm{Gr}^W_k H$. We shall say that an integer m is a *weight* of a mixed Hodge structure H if $\mathrm{Gr}^W_m H \neq 0$. We use abbreviations: MHS for mixed Hodge structure, and MHM for mixed Hodge module.

In [7] Deligne proved that the rational cohomology of every quasi-projective variety possesses a natural MHS. In [38] Saito proved that the cohomology and intersection cohomology of an algebraic variety with coefficients in an admissible variation of MHS carry MHSs. The definition of an admissible variation of MHS is given for curves in [40], and in general in [24] (also see [37, 2.1]). There is a strong belief that when both MHSs of Deligne and Saito exist they are the same.

Let $\mathbb{V}$ be an irreducible symplectic local system over $\mathcal{M}_g$ determined by highest weight λ. This is clear that the restriction of the local system $\mathbb{V}$ to ${}^{\circ}\mathcal{M}_g$ underlies a polarized variation of Hodge structure of geometric origin. Therefore the restriction of $\mathbb{V}$ to ${}^{\circ}\mathcal{M}_g$ is an admissible variation of Hodge structure. The local system $\mathbb{V}$ is irreducible, therefore the corresponding variation of Hodge structure is unique up to Tate twist [14, Prop. 8.1]. We fix $\mathbb{V}$ as a variation of Hodge structure by decreeing its weight to be $|\lambda|$.

According to the theory of MHMs both $IH^q(\widetilde{\mathcal{M}}_g;\mathbb{V})$ and $H^q(\mathcal{M}_g;\mathbb{V})$ carry natural MHSs [37, pp. 146–147]. The MHS on $H^q(\mathcal{M}_g;\mathbb{V})$ can be defined using either the smooth covers $\mathcal{M}_g[l]$ for $l \geqslant 3$, or the isomorphism $H^q(\mathcal{M}_g;\mathbb{V}) \cong IH^q(\mathcal{M}_g;\mathbb{V})$ where in the second term we consider the restriction of $\mathbb{V}$ to ${}^{\circ}\mathcal{M}_g$. This is easy to check that all these ways lead to the same MHS.

THEOREM 5.1. *If $g \geqslant 6$, or if $g \geqslant 4$ and $\mathbb{V}$ is an even local system, then the mixed Hodge structure on $H^2(\mathcal{M}_g;\mathbb{V}(\lambda))$ is pure of weight $2 + |\lambda|$.*

Proof. The theory of MHM implies that the restriction

$$IH^2(\widetilde{\mathcal{M}}_g;\mathbb{V}) \to H^2(\mathcal{M}_g;\mathbb{V})$$

is a morphism of MHSs, and according to Theorem 4.1 and Corollary 4.9 this is an isomorphism. The space $\widetilde{\mathcal{M}}_g$ is a projective variety. It follows that the MHS on $IH^2(\widetilde{\mathcal{M}}_g;\mathbb{V})$ is pure of weight $2 + |\lambda|$ [38, pp. 221–222]. Thus the MHS on $H^2(\mathcal{M}_g;\mathbb{V})$ is also pure of weight $2 + |\lambda|$. □

In the rest of this section we deal with the MHS on $H^2(\mathcal{M}_g[l];\mathbb{V})$ where $\mathcal{M}_g[l]$ is the moduli space of curves with a level l structure, and $\mathbb{V}$ is a symplectic local system $\mathbb{V}(\lambda)$ which underlies a variation of Hodge structure of weight $|\lambda|$. We assume that $l \geqslant 3$, and therefore $\mathcal{M}_g[l]$ is smooth and $\mathbb{V}$ is a genuine (not only orbifold) local system. There exists a natural MHS on $H^q(\mathcal{M}_g[l];\mathbb{V})$ for each $q \geqslant 0$.

THEOREM 5.2. *If $l \geqslant 3$ and $g \geqslant 3$, then $\mathrm{Gr}_k^W H^2(\mathcal{M}_g[l];\mathbb{V}) = 0$ for $k > 3 + |\lambda|$ and $k < 2 + |\lambda|$.*

Proof. In the beginning we recall some facts from Section 2. The moduli space $\mathcal{M}_g[l]$ has the Satake compactification $\widetilde{\mathcal{M}}_g[l]$ which is a projective variety. The boundary $\widetilde{\mathcal{M}}_g[l] - \mathcal{M}_g[l]$ has codimension two in $\widetilde{\mathcal{M}}_g[l]$, and each codimension two irreducible component Φ_1^β has a Zariski open subset Z_β such that the subsets Z_β do not intersect each other, and each of them is isomorphic to a smooth Zariski open subset of $\mathcal{M}_1[l] \times \mathcal{M}_{g-1}[l]$.

Notation. We denote by $\mathcal{S}^\bullet$ the intersection cohomology sheaf $\mathcal{IC}^\bullet(\mathbb{V})$ on $\widetilde{\mathcal{M}}_g[l]$. The following diagrams defines the notation for the inclusions

$$\mathcal{M}_g[l] \xhookrightarrow{i} \mathcal{M}_g[l] \cup (\cup_\beta Z_\beta) \xhookleftarrow{j} \cup_\beta Z_\beta,$$

and we denote by j_β the restriction of j to Z_β. This notation is similar to that in Section 4.

It follows that one has an exact sequence

$$0 \to IH^2(\widetilde{\mathcal{M}}_g[l];\mathbb{V}) \to H^2(\mathcal{M}_g[l];\mathbb{V}) \to H^3(\cup_\beta Z_\beta; j^!\mathcal{S}^\bullet),$$

in the category of MHSs. Taking graded quotients with respect to weight filtration is an exact functor. Therefore for every k there is an exact sequence

$$\begin{aligned} 0 &\to \mathrm{Gr}_k^W IH^2(\widetilde{\mathcal{M}}_g[l];\mathbb{V}) \\ &\to \mathrm{Gr}_k^W H^2(\mathcal{M}_g[l];\mathbb{V}) \to \mathrm{Gr}_k^W H^3(\cup_\beta Z_\beta; j^!\mathcal{S}^\bullet). \end{aligned}$$

Since the space $\widetilde{\mathcal{M}}_g[l]$ is a projective variety, and $\mathbb{V}$ is a polarized variation of Hodge structure of geometric origin of weight $|\lambda|$, the intersection cohomology $IH^2(\widetilde{\mathcal{M}}_g[l];\mathbb{V})$ has a pure MHS of weight $2 + |\lambda|$. To prove the theorem we will show that $\mathrm{Gr}_k^W H^3(\cup_\beta Z_\beta; j^!\mathcal{S}^\bullet) = 0$ unless $k = 3 + |\lambda|$.

As the sets Z_β are disjoint it suffices to show that each $H^3(Z_\beta; j_\beta^!\mathcal{S}^\bullet)$ has a pure MHS of weight $3 + |\lambda|$. From now on we fix an arbitrary index β, and omit β from the notation for Z_β and j_β.

The sheaf $j^!\mathcal{S}^\bullet$ is constructible, and $\widetilde{\mathcal{M}}_g[l]$ is equi-singular along Z. Therefore $\mathcal{H}^3 j^!\mathcal{S}^\bullet$ is a local system over Z. The standard argument implies that there is an isomorphism of MHSs

$$H^3(Z; j^!\mathcal{S}^\bullet) \cong H^0(Z; \mathcal{H}^3 j^!\mathcal{S}^\bullet)$$

and there is an isomorphism of MHMs

$$\mathcal{H}^3 j^! \mathcal{S}^\bullet \cong \mathcal{H}^2 j^* i_* \mathbb{V}. \tag{5.1}$$

We will show that these MHMs are pure of weight $3 + |\lambda|$.

Recall that $j^* i_* \mathbb{V}$ expresses cohomology of the link of Z in $\mathcal{M}_g[l] \cup Z$. The inverse image of Z under the birational morphism $\alpha^l : \overline{\mathcal{M}}_g[l] \to \widetilde{\mathcal{M}}_g[l]$ is a smooth locally closed divisor. We denote it by Y. Then the link of Z in $\mathcal{M}_g[l] \cup Z$ is the same as the link of Y in $\mathcal{M}_g[l] \cup Y$. We use this to find the weights on $\mathcal{H}^2 j^* i_* \mathbb{V}$.

The following commutative diagram introduces the notation

$$\begin{tikzcd}
\mathcal{M}_g[l] \arrow[r, "\kappa"] \arrow[d, "="'] & \mathcal{M}_g[l] \cup Y \arrow[d, "\alpha^l"'] & Y \arrow[l, "\mu"'] \arrow[d, "\bar{\pi}"'] \\
\mathcal{M}_g[l] \arrow[r, "i"] & \mathcal{M}_g[l] \cup Z & Z. \arrow[l, "j"']
\end{tikzcd}$$

The local link cohomology functor of Y is $\mu^* \kappa_* \mathbb{V}$. Therefore one expects that $j^* i_* \mathbb{V} \simeq \bar{\pi}_* \mu^* \kappa_* \mathbb{V}$. (The sign $\simeq$ denotes an isomorphism in the derived category of MHMs.) Indeed, both α^l and $\bar{\pi}$ are proper maps, therefore $\alpha^l_* = \alpha^l_!$ and $\bar{\pi}_* = \bar{\pi}_!$. It follows that for an arbitrary sheaf $\mathcal{F}^\bullet$ on $\mathcal{M}_g[l] \cup Y$ one has that $j^* \alpha^l_* \mathcal{F}^\bullet \simeq \bar{\pi}_* \mu^* \mathcal{F}^\bullet$ [5, Prop. 10.7]. Therefore

$$j^* i_* \mathbb{V} \simeq j^* \alpha^l_* \kappa_* \mathbb{V} \simeq \bar{\pi}_* \mu^* \kappa_* \mathbb{V}. \tag{5.2}$$

Thus $\mathcal{H}^2 j^* i_* \mathbb{V} \cong \mathcal{H}^2 \bar{\pi}_* \mu^* \kappa_* \mathbb{V}$ is an isomorphism of MHMs.

The variation of Hodge structure $\mathbb{V}$ on $\mathcal{M}_g[l]$ extends to a variation of Hodge structure on $\mathcal{M}_g[l] \cup Y$ because $\mathbb{V}$ is pulled back from $\mathcal{A}_g[l]$. We denote its restriction to Y by $\overline{\mathbb{V}}$. Then $\mu^* \kappa_* \mathbb{V} \simeq \mu^* \kappa_* \mathbb{Q} \otimes \overline{\mathbb{V}}$ where $\mathbb{Q}$ denotes the constant variation of Hodge structure of weight zero with the fiber isomorphic to $\mathbb{Q}$.

Denote by $D\mathcal{F}^\bullet$ the dual of $\mathcal{F}^\bullet$ in the derived category of MHMs. The spaces $\mathcal{M}_g[l] \cup Y$ and Y are smooth, therefore we have $D\mathbb{Q} \simeq \mathbb{Q}[2n](n)$ and $D\mathbb{Q}_Y \simeq \mathbb{Q}_Y[2n-2](n-1)$. It follows that there is a string of isomorphisms in the derived category of MHMs

$$\begin{aligned}
\mu^! \mathbb{Q} &\simeq D_Y(\mu^* D\mathbb{Q}) \simeq D_Y(\mu^* \mathbb{Q}[2n](n)) \simeq D_Y(\mathbb{Q}_Y[2n](n)) \\
&\simeq (D_Y \mathbb{Q}_Y)[-2n](-n) \simeq \mathbb{Q}_Y[-2](-1).
\end{aligned}$$

Using this and the distinguished triangle

$$\begin{tikzcd}
\mu^! \mathbb{Q} \arrow[rr] & & \mathbb{Q} \arrow[dl] \\
& \mu^* \kappa_* \mathbb{Q}, \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

one can deduce that

$$\mathcal{H}^0\mu^*\kappa_*\mathbb{Q} \cong \mathcal{H}^0\mathbb{Q}, \qquad \mathcal{H}^1\mu^*\kappa_*\mathbb{Q} \cong \mathcal{H}^2\mu^!\mathbb{Q} \quad \text{and} \quad \mathcal{H}^q\mu^*\kappa_*\mathbb{Q} = 0$$

for $q \geqslant 2$. It follows that $\mathcal{H}^0\mu^*\kappa_*\mathbb{V}$ is a pure Hodge module of weight $|\lambda|$, and $\mathcal{H}^1\mu^*\kappa_*\mathbb{V}$ is a pure Hodge module of weight $2+|\lambda|$.

According to [37, 1.20], there is a (perverse) spectral sequence in the category of MHMs

$$E_2^{p,q} = \mathcal{H}^p\bar{\pi}_*(\mathcal{H}^q\mu^*\kappa_*\mathbb{V}) \Rightarrow \mathcal{H}^{p+q}\bar{\pi}_*\mu^*\kappa_*\mathbb{V}.$$

As all spaces involved are smooth the perverse spectral sequence coincides with the ordinary one. It has only two non-zero rows. Thus there is an exact sequence of MHMs

$$\mathcal{H}^2\bar{\pi}_*(\mathcal{H}^0\mu^*\kappa_*\mathbb{V}) \to \mathcal{H}^2\bar{\pi}_*\mu^*\kappa_*\mathbb{V} \to \mathcal{H}^1\bar{\pi}_*(\mathcal{H}^1\mu^*\kappa_*\mathbb{V}).$$

The map $\bar{\pi}$ is proper. Therefore $\mathcal{H}^2\bar{\pi}_*(\mathcal{H}^0\mu^*\kappa_*\mathbb{V})$ is pure of weight $2+|\lambda|$, and $\mathcal{H}^1\bar{\pi}_*(\mathcal{H}^1\mu^*\kappa_*\mathbb{V})$ is pure of weight $3+|\lambda|$. Consequently, we have that

$$\mathrm{Gr}_k^W\mathcal{H}^2\bar{\pi}_*\mu^*\kappa_*\mathbb{V} = 0,$$

for $k > 3+|\lambda|$ and $k < 2+|\lambda|$.

Since $\mathbb{V}$ is a variation of Hodge structure of geometric origin of weight $|\lambda|$, the intersection cohomology sheaf $\mathcal{S}^\bullet$ underlies a pure Hodge module of weight $|\lambda|$. Therefore $j^!\mathcal{S}^\bullet$ is a MHM of weight $\geqslant |\lambda|$ [37, Prop. 1.7]. It follows that $\mathrm{Gr}_k^W\mathcal{H}^3j^!\mathcal{S}^\bullet = 0$ for $k < 3+|\lambda|$. Combining the last two paragraphs, and isomorphisms (5.1), (5.2) one gets that $\mathcal{H}^3j^!\mathcal{S}^\bullet$ is pure of weight $3+|\lambda|$. □

COROLLARY 5.3. *Let $\mathbb{V}(\lambda)$ be a symplectic local system over $\mathcal{M}_g$ underlying a variation of Hodge structure of weight $|\lambda|$. If $g \geqslant 3$, then $\mathrm{Gr}_k^W H^2(\mathcal{M}_g;\mathbb{V}) = 0$ for $k > 3+|\lambda|$ and $k < 2+|\lambda|$.*

Proof. Choose $l \geqslant 3$. One has as isomorphism

$$H^2(\mathcal{M}_g;\mathbb{V}) \cong H^2(\mathcal{M}_g[l];\mathbb{V})^{\mathrm{Sp}_{2g}(\mathbb{Z}/l\mathbb{Z})}$$

in the category of MHSs. The weights of the right-hand side are $2+|\lambda|$ and $3+|\lambda|$ according to the theorem above. Therefore the same is true for the left-hand side. □

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