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MAXIMAL SUBSEMIGROUPS OF INFINITE SYMMETRIC GROUPS

SUZANA MENDES-GONÇALVES and R. P. SULLIVAN

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To the memory of Eckehart Hotzel, with respect and gratitude

Abstract

Brazil *et al*. ['Maximal subgroups of infinite symmetric groups', *Proc. Lond. Math. Soc. (3)* **68**(1) (1994), 77–111] provided a new family of maximal subgroups of the symmetric group $G(X)$ defined on an infinite set X. It is easy to see that, in this case, $G(X)$ contains subsemigroups that are not groups, but nothing is known about nongroup maximal subsemigroups of $G(X)$. We provide infinitely many examples of such semigroups.

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1. Introduction

Throughout this paper, X is an infinite set and $I(X)$ denotes the *symmetric inverse semigroup* on X, that is, the semigroup (under composition) consisting of all *one-to-one partial transformations* whose *domain*, dom α, and *range*, ran α, are subsets of X (see [2, Volume 1, page 29]). In addition, if $\alpha \in I(X)$, we write

$$g(\alpha) = |X \setminus \operatorname{dom} \alpha|, \quad d(\alpha) = |X \setminus \operatorname{ran} \alpha|$$

and refer to these cardinal numbers as the *gap* and *defect* of α, respectively.

In [11], the authors studied some algebraic properties of the semigroup defined by

$$A(X) = \{\alpha \in I(X) : g(\alpha) = d(\alpha)\}.$$

In particular, for uncountable X, they described all maximal subsemigroups of $A(X)$, some of which involve a maximal subsemigroup of $G(X)$, the symmetric group on X. However, little seems to be known about such semigroups of permutations.

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In [4], Hotzel describes many different types of maximal subsemigroups of $BL(p,q)$, the *Baer–Levi semigroup* defined on X, when $|X| = p \geq q \geq \aleph_0$ (for the definition, see [2, Section 8.1]). In addition, in [4, page 157], he remarks that $\{\pi \in G(X) : |A\pi \setminus A| < |A|\}$ is a maximal subsemigroup of $G(X)$ if and only if $|A| = 1$ or $\aleph_0 \leq |A| \leq |X \setminus A|$ (see [2, Section 3]). However, Hotzel does not prove this assertion in [4], albeit a *tour-de-force* in many other ways, even though he often uses the assumption that $\aleph_0 \leq |A| \leq |X \setminus A|$ in [4, Section 3] (see Corollary 3.10 and elsewhere). Furthermore, we cannot find any statement like Hotzel's assertion anywhere else in the literature.

In [4, page 154], Hotzel gives a brief summary of what was known about maximal subgroups of $G(X)$ in 1995. In fact, maximal subgroups of $G(X)$ have been extensively studied, particularly when X is infinite. For later developments, see [1, 3] and the references therein. In [4, page 153], Hotzel remarks that some maximal subgroups of $G(X)$ are also maximal as subsemigroups of $G(X)$. In [3, Section 10], the authors provide several examples of such maximal subsemigroups of $G(X)$. However, here we focus on *nongroup* maximal subsemigroups of $G(X)$, since that is what is needed to support the main result in [11, Section 4].

In Section 2, we prove Hotzel's assertion (as quoted above) and, in Section 4, we prove a linear version of it (for interest, and to support a linear version of [11, Section 4] which naturally arises from [8]). In Section 3, we observe that, in many cases, $G(X)$ is not isomorphic to $G(V)$, the general linear group on an infinite-dimensional vector space, even though their algebraic properties are similar, for example, the description herein of some of their maximal subsemigroups. This is akin to work in [10, 11] (see Section 3 for more details).

2. Infinite symmetric group

In what follows, $Y = A \,\dot\cup\, B$ means that Y is a *disjoint* union of A and B, and we let id_X denote the identity of $G(X)$. Also, following standard practice in transformation semigroup theory, we compose mappings from left to right.

We adapt the convention introduced in [2, Volume 2, page 241]: namely, if $\alpha \in G(X)$, then we write

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix}$$

and take as understood that the subscript i belongs to some (unmentioned) index set I, that the abbreviation $\{a_i\}$ denotes $\{a_i : i \in I\}$, and that $X = \{a_i\} = \{x_i\}$ and $x_i\alpha^{-1} = a_i$ for each i. In addition, if $X = A \,\dot\cup\, B = C \,\dot\cup\, D$, where $|A| = |C|$ and $|B| = |D|$, we often write

$$\alpha = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

to indicate that $\alpha \in G(X)$ consists of some (unspecified) bijection from A to C, together with a bijection from B to D.

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For all other notation and terminology in semigroup theory, we refer the reader to [2, 5].

Clearly, $G(X)$ contains subsemigroups that are not groups. For example, if $X = \mathbb{Z}$ and $n\alpha = n + 1$ for all $n \in \mathbb{Z}$, then the cyclic semigroup $\langle\alpha\rangle$ is a subsemigroup of $G(X)$ but $\alpha^{-1} \notin \langle\alpha\rangle$, and this idea can be extended to any infinite X. The next result provides infinitely many examples of nongroup maximal subsemigroups of $G(X)$.

Proposition 2.1. *Let Y be a subset of X such that $\aleph_0 \leq |Y| \leq |X \setminus Y|$. Then the set*

$$H(Y) = \{\pi \in G(X) : |Y\pi \setminus Y| < |Y|\}$$

is a maximal subsemigroup of $G(X)$ that is not a group.

Proof. Let $|Y| = m \leq n = |X \setminus Y|$ and write $X = Y \dot\cup Z$. Clearly,

$$H(Y) = \{\pi \in G(X) : |Y\pi \cap Z| < m\}.$$

Suppose that $\alpha, \beta \in H(Y)$. Since $Y\alpha = (Y\alpha \cap Y) \cup (Y\alpha \cap Z)$, we have

$$Y\alpha\beta \cap Z \subseteq (Y\beta \cap Z) \cup ((Y\alpha \cap Z)\beta \cap Z).$$

So, $|Y\alpha\beta \cap Z| \leq |Y\beta \cap Z| + |Y\alpha \cap Z| < m + m = m$ and $\alpha\beta \in H(Y)$.

Write $Y = \{a_k\} \dot\cup \{a_j\} = \{b_j\} \dot\cup \{c_j\}$ and $Z = \{x_j\} \dot\cup \{x_i\} = \{y_k\} \dot\cup \{y_i\}$, where $|K| < m = |J|$ and $|I| = n$. Define $\alpha \in G(X)$ by

$$\alpha = \begin{pmatrix} a_k & a_j & x_j & x_i \\ y_k & b_j & c_j & y_i \end{pmatrix}. \tag{2.1}$$

Clearly, $Y\alpha \cap Y = \{b_j\}$ and $Y\alpha \cap Z = \{y_k\}$, so $\alpha \in H(Y)$. But $Y\alpha^{-1} \cap Z = \{x_j\}$, so $\alpha^{-1} \notin H(Y)$. That is, $H(Y)$ is a subsemigroup of $G(X)$ that is not a group.

To show that $H(Y)$ is maximal, we let $h \in G(X) \setminus H(Y)$ and show that every $g \in G(X) \setminus H(Y)$ belongs to $\langle H(Y), h\rangle$, the semigroup generated by $H(Y) \cup \{h\}$: in other words, $G(X) = \langle H(Y), h\rangle$. To do this, we consider four cases.

First, note that if $h, g \notin H(Y)$, then $|Yh \cap Z| \geq m$ and $|Yg \cap Z| \geq m$. But $|Yh| = |Yg| = |Y| = m$, and so $|Yh \cap Z| = m = |Yg \cap Z|$. In addition, h can be written as

$$h = \begin{pmatrix} Y \cap Zh^{-1} & Y \cap Yh^{-1} & Z \cap Zh^{-1} & Z \cap Yh^{-1} \\ Yh \cap Z & Yh \cap Y & Zh \cap Z & Zh \cap Y \end{pmatrix}.$$

By this, we mean that, for each set in the first row, there is a bijection (determined by the permutation h) between it and the set below it. Note that, in general, one or more of the intersections may be empty. However, if $Zh \cap Y = \emptyset$, then $Zh = Zh \cap Z$, and so $Z = Z \cap Zh^{-1}$ and $Y = Y \cap Yh$, and likewise for other possibilities.

Case 1. Suppose that $|Z \cap Yg^{-1}| < m$ and $|Z \cap Yh^{-1}| < m$. Since $|Z| = n \geq m$, we have $|Z \cap Zg^{-1}| = n = |Z \cap Zh^{-1}|$. Also, $|Yg^{-1}| = |Yh^{-1}| = |Y| = m$ implies that $|Y \cap Yg^{-1}| = m = |Y \cap Yh^{-1}|$. Write

$$Y \cap Yg^{-1} = B_1 \dot\cup B_2, \quad |B_1| = m,\ |B_2| = |Z \cap Yh^{-1}|,$$
$$Y \cap Yh^{-1} = C_1 \dot\cup C_2, \quad |C_1| = m,\ |C_2| = |Z \cap Yg^{-1}|,$$

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and consider $\pi_1, \pi_2 \in G(X)$, defined as follows:

$$\pi_1 = \begin{pmatrix} Y \cap Zg^{-1} & B_1 & Z \cap Yg^{-1} & Z \cap Zg^{-1} & B_2 \\ Y \cap Zh^{-1} & C_1 & C_2 & Z \cap Zh^{-1} & Z \cap Yh^{-1} \end{pmatrix},$$
$$h = \begin{pmatrix} Y \cap Zh^{-1} & C_1 & C_2 & Z \cap Zh^{-1} & Z \cap Yh^{-1} \\ Yh \cap Z & C_1h & C_2h & Zh \cap Z & Zh \cap Y \end{pmatrix},$$
$$\pi_2 = \begin{pmatrix} Yh \cap Z & C_1h & C_2h & Zh \cap Z & Zh \cap Y \\ Yg \cap Z & B_1g & Zg \cap Y & Zg \cap Z & B_2g \end{pmatrix}.$$

Clearly, π_1 and π_2 are well-defined permutations of X (this depends, in part, on our choice of sets and their cardinals). Now $Y\pi_1 \cap Z = Z \cap Yh^{-1}$, which has cardinal less than m by supposition, so $\pi_1 \in H(Y)$. Also, $Y\pi_2 \cap Z = \emptyset$, so $\pi_2 \in H(Y)$. Moreover, $g = \pi_1 h \pi_2$ provided we define π_2 as follows. If, for example, $x \in B_1$, then π_2 maps C_1h to B_1g via $x\pi_1 h \mapsto xg$, and likewise for each set in the first row of π_2.

Case 2. Suppose that $|Z \cap Yg^{-1}| < m$ and $|Z \cap Yh^{-1}| = m$. Then $|Y \cap Yg^{-1}| = m$ and $|Z \cap Zg^{-1}| = n$, but now $|Y \cap Yh^{-1}|$ and $|Z \cap Zh^{-1}|$ may be unequal and less than m and n, respectively. In fact, this case is more complicated than the others. Clearly, if $m < n$, then $|Z \cap Zh^{-1}| = n$. On the other hand, if $m = n$, then $|Z \cap Zg^{-1}| = m \geq |Z \cap Zh^{-1}|$. Write

$$\begin{aligned} Y \cap Yg^{-1} &= B_1 \dot\cup B_2, & |B_1| &= m,\ |B_2| = |Y \cap Yh^{-1}|, \\ Z \cap Zg^{-1} &= C_1 \dot\cup C_2, & |C_1| &= m,\ |C_2| = |Z \cap Zh^{-1}|, \\ Z \cap Yh^{-1} &= E_1 \dot\cup E_3, & |E_1| &= m,\ |E_3| = |Z \cap Yg^{-1}|, \\ Y \cap Zh^{-1} &= D_1 \dot\cup D_2, & |D_1| &= |D_2| = m, \\ &= F_1 \dot\cup F_2 \dot\cup F_3, & |F_1| &= |F_2| = m,\ |F_3| = |E_3|. \end{aligned}$$

Now define $\pi_1, \pi_2, \pi_3 \in G(X)$ as follows:

$$\pi_1 = \begin{pmatrix} Y \cap Zg^{-1} & B_1 & B_2 & C_2 & C_1 & Z \cap Yg^{-1} \\ D_2 & D_1 & Y \cap Yh^{-1} & Z \cap Zh^{-1} & E_1 & E_3 \end{pmatrix},$$
$$h = \begin{pmatrix} D_2 & D_1 & Y \cap Yh^{-1} & Z \cap Zh^{-1} & E_1 & E_3 \\ D_2h & D_1h & Yh \cap Y & Zh \cap Z & E_1h & E_3h \end{pmatrix},$$
$$\pi_2 = \begin{pmatrix} D_2h & D_1h & Yh \cap Y & Zh \cap Z & E_1h & E_3h \\ F_2 & Z \cap Yh^{-1} & Y \cap Yh^{-1} & Z \cap Zh^{-1} & F_1 & F_3 \end{pmatrix},$$
$$h = \begin{pmatrix} F_2 & Z \cap Yh^{-1} & Y \cap Yh^{-1} & Z \cap Zh^{-1} & F_1 & F_3 \\ F_2h & Zh \cap Y & Yh \cap Y & Zh \cap Z & F_1h & F_3h \end{pmatrix},$$
$$\pi_3 = \begin{pmatrix} F_2h & Zh \cap Y & Yh \cap Y & Zh \cap Z & F_1h & F_3h \\ Yg \cap Z & B_1g & B_2g & C_2g & C_1g & Zg \cap Y \end{pmatrix}.$$

Clearly, π_1, π_2 and π_3 are well-defined permutations of X. Note that $Y\pi_1 \cap Z = \emptyset$, so $\pi_1 \in H(Y)$. Also, $E_1h \subseteq Y$ and $F_1 \subseteq Y$ (and likewise for E_3h and F_3), so $Y\pi_2 \cap Z = \emptyset$ and $\pi_2 \in H(Y)$. In addition, $F_ih \subseteq Z$ and $B_jg \subseteq Y$ for $i = 1, 2, 3$ and $j = 1, 2$. Therefore,

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$Y\pi_3 \cap Z = \emptyset$ and $\pi_3 \in H(Y)$. Furthermore, by defining π_3 in a suitable manner, we have $g = \pi_1 h \pi_2 h \pi_3$.

Case 3. Suppose that $|Z \cap Yg^{-1}| = m$ and $|Z \cap Yh^{-1}| < m$. In this event, $|Y \cap Yh^{-1}| = m$ and $|Z \cap Zh^{-1}| = n$, whereas $|Z \cap Zg^{-1}|$ and $|Y \cap Yg^{-1}|$ are unknown. As before, if $m < n$, then $|Z \cap Zg^{-1}| = n$; if $m = n$, then $|Z \cap Zh^{-1}| = m \geq |Z \cap Zg^{-1}|$. Write

$$\begin{aligned} Z \cap Yg^{-1} &= B_1 \dot\cup B_2 \dot\cup B_3, & |B_1| = |B_2| = m,\ |B_3| = |Z \cap Yh^{-1}|,\\ Y \cap Zh^{-1} &= C_1 \dot\cup C_2, & |C_1| = m,\ |C_2| = |Y \cap Yg^{-1}|,\\ Z \cap Zh^{-1} &= D_2 \dot\cup D_1, & |D_2| = m,\ |D_1| = |Z \cap Zg^{-1}|. \end{aligned}$$

Now define $\pi_1, \pi_2 \in G(X)$ as follows:

$$\begin{aligned} \pi_1 &= \begin{pmatrix} Y \cap Zg^{-1} & Y \cap Yg^{-1} & B_1 & B_2 & Z \cap Zg^{-1} & B_3 \\ C_1 & C_2 & Y \cap Yh^{-1} & D_2 & D_1 & Z \cap Yh^{-1} \end{pmatrix},\\ h &= \begin{pmatrix} C_1 & C_2 & Y \cap Yh^{-1} & D_2 & D_1 & Z \cap Yh^{-1} \\ C_1 h & C_2 h & Yh \cap Y & D_2 h & D_1 h & Zh \cap Y \end{pmatrix},\\ \pi_2 &= \begin{pmatrix} C_1 h & C_2 h & Yh \cap Y & D_2 h & D_1 h & Zh \cap Y \\ Yg \cap Z & Yg \cap Y & B_1 g & B_2 g & Zg \cap Z & B_3 g \end{pmatrix}. \end{aligned}$$

Clearly, π_1 and π_2 are permutations of X. Also, $Y\pi_1 \cap Z = \emptyset = Y\pi_2 \cap Z$, so $\pi_1, \pi_2 \in H(Y)$. As before, $g = \pi_1 h \pi_2$ if π_2 is defined suitably.

Case 4. Suppose that $|Z \cap Yg^{-1}| = m$ and $|Z \cap Yh^{-1}| = m$. In this case, each of the cardinals $|Y \cap Yg^{-1}|, |Z \cap Zg^{-1}|$ and $|Y \cap Yh^{-1}|, |Z \cap Zh^{-1}|$ is unknown. If $m = n$, write

$$\begin{aligned} Z \cap Yg^{-1} &= B_1 \dot\cup B_2 \dot\cup B_3, & |B_1| = |Y \cap Yh^{-1}|,\ |B_2| = |Z \cap Zh^{-1}|,\ |B_3| = m,\\ Y \cap Zh^{-1} &= C_1 \dot\cup C_2 \dot\cup C_3, & |C_1| = m,\ |C_2| = |Y \cap Yg^{-1}|,\ |C_3| = |Z \cap Zg^{-1}|, \end{aligned}$$

and define $\pi_1, \pi_2 \in G(X)$ as follows:

$$\begin{aligned} \pi_1 &= \begin{pmatrix} Y \cap Zg^{-1} & Y \cap Yg^{-1} & Z \cap Zg^{-1} & B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 & Y \cap Yh^{-1} & Z \cap Zh^{-1} & Z \cap Yh^{-1} \end{pmatrix},\\ h &= \begin{pmatrix} C_1 & C_2 & C_3 & Y \cap Yh^{-1} & Z \cap Zh^{-1} & Z \cap Yh^{-1} \\ C_1 h & C_2 h & C_3 h & Yh \cap Y & Zh \cap Z & Zh \cap Y \end{pmatrix},\\ \pi_2 &= \begin{pmatrix} C_1 h & C_2 h & C_3 h & Yh \cap Y & Zh \cap Z & Zh \cap Y \\ Yg \cap Z & Yg \cap Y & Zg \cap Z & B_1 g & B_2 g & B_3 g \end{pmatrix}. \end{aligned}$$

On the other hand, if $m < n$, then $|Z \cap Zg^{-1}| = |Z \cap Zh^{-1}| = n$. In this case, write

$$\begin{aligned} Z \cap Yg^{-1} &= B_1 \dot\cup B_2, & |B_1| = m,\ |B_2| = |Y \cap Yh^{-1}|,\\ Y \cap Zh^{-1} &= C_1 \dot\cup C_2, & |C_1| = m,\ |C_2| = |Y \cap Yg^{-1}|, \end{aligned}$$

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and define $\pi_1, \pi_2 \in G(X)$ as follows:

$$\pi_1 = \begin{pmatrix} Y \cap Zg^{-1} & Y \cap Yg^{-1} & Z \cap Zg^{-1} & B_1 & B_2 \\ C_1 & C_2 & Z \cap Zh^{-1} & Z \cap Yh^{-1} & Y \cap Yh^{-1} \end{pmatrix},$$
$$h = \begin{pmatrix} C_1 & C_2 & Z \cap Zh^{-1} & Z \cap Yh^{-1} & Y \cap Yh^{-1} \\ C_1h & C_2h & Zh \cap Z & Zh \cap Y & Yh \cap Y \end{pmatrix},$$
$$\pi_2 = \begin{pmatrix} C_1h & C_2h & Zh \cap Z & Zh \cap Y & Yh \cap Y \\ Yg \cap Z & Yg \cap Y & Zg \cap Z & B_1g & B_2g \end{pmatrix}.$$

In both cases, $\pi_1, \pi_2 \in G(X)$ and $Y\pi_1 \cap Z = \emptyset = Y\pi_2 \cap Z$. Hence, $\pi_1, \pi_2 \in H(Y)$ and, as before, $g = \pi_1 h \pi_2$ for a suitably defined π_2. □

We note that the α defined in (2.1) belongs to $H(Y)$ but

$$\alpha \notin H(Z) = \{\beta \in G(X) : |Z\beta \cap Y| < n\},$$

even when $n = m$. Also, there are 2^n subsets of X with cardinal n, and hence there are 2^n partitions of X into two subsets, each with cardinal n. Consequently, there are at least 2^n distinct nongroup subsemigroups of $G(X)$ that are maximal.

Hotzel's claim follows easily from Proposition 2.1.

THEOREM 2.2. *Let A be a subset of X. Then the set*

$$H(A) = \{\pi \in G(X) : |A\pi \setminus A| < |A|\}$$

is a maximal subsemigroup of $G(X)$ if and only if $|A| = 1$ or $\aleph_0 \le |A| \le |X \setminus A|$.

PROOF. First, suppose that $|A| = m$ for some $m \in \mathbb{N} \setminus \{1\}$, and write $A = \{a_1, a_2, \ldots, a_m\}$. Let $X \setminus A = \{x_1, x_2, \ldots, x_m\} \dot\cup \{y_i\}$, with $|I| = n \ge \aleph_0 > m$, and define $\alpha, \beta \in G(X)$ by

$$\alpha = \begin{pmatrix} a_1 & a_2 & \ldots & a_m & x_1 & x_2 & \ldots & x_m & y_i \\ a_1 & x_2 & \ldots & x_m & x_1 & a_2 & \ldots & a_m & y_i \end{pmatrix},$$

$$\beta = \begin{pmatrix} a_1 & a_2 & \ldots & a_m & x_1 & x_2 & \ldots & x_m & y_i \\ x_1 & a_2 & \ldots & a_m & a_1 & x_2 & \ldots & x_m & y_i \end{pmatrix}.$$

It is easy to verify that $|A\alpha \setminus A| = m - 1$ and $|A\beta \setminus A| = 1$, and hence $\alpha, \beta \in H(A)$. But $|A(\alpha\beta) \setminus A| = m$, and so $\alpha\beta \notin H(A)$. Therefore, $H(A)$ is not a semigroup when $2 \le |A| < \aleph_0$.

On the other hand, if A is infinite but $|A| > |X \setminus A|$, then $|A| = |X|$. Given that $\pi \in G(X)$, $A\pi \setminus A \subseteq X \setminus A$. Therefore, $|A\pi \setminus A| \le |X \setminus A| < |A|$, and so $H(A) = G(X)$. Thus, we have just proved that $|A| = 1$ or $\aleph_0 \le |A| \le |X \setminus A|$ when $H(A)$ is a maximal subsemigroup of $G(X)$.

Conversely, assume that $|A| = 1$ and write $A = \{a\}$. Clearly,

$$H(A) = \{\pi \in G(X) : A\pi \setminus A = \emptyset\} = \{\pi \in G(X) : a\pi = a\}.$$

It is not difficult to see that $H(A)$ is a subsemigroup of $G(X)$. In fact, it is a subgroup of $G(X)$. Next, we prove that $H(A)$ is a maximal subsemigroup. To do this, we let $h \in G(X) \setminus H(A)$ and we show that $g \in \langle H(A), h\rangle$ for every $g \in G(X) \setminus H(A)$. Since $h, g \notin H(A)$, it follows that $ah = b$ and $ag = c$ for some $b, c \in X \setminus \{a\}$. Also, there exist $d, e \in X \setminus \{a\}$ such that $dh = a$ and $eg = a$. Write $X = \{a, e\} \dot\cup Y = \{a, d\} \dot\cup Z$ and define π_1, π_2 in $G(X)$ by

$$\pi_1 = \begin{pmatrix} a & e & Y \\ a & d & Z \end{pmatrix}, \quad \pi_2 = \begin{pmatrix} a & b & Zh \\ a & c & Yg \end{pmatrix}.$$

Clearly, $\pi_1, \pi_2 \in H(A)$ and $g = \pi_1 h \pi_2$ if π_2 is suitably defined. Thus, $H(A)$ is a maximal subsemigroup of $G(X)$ if $|A| = 1$. By Proposition 2.1, this is also true if $\aleph_0 \le |A| \le |X \setminus A|$. □

3. An isomorphism problem

In [9], the authors proved that the Baer–Levi semigroup $BL(p, q)$ of type (p, q) defined on an infinite set is never isomorphic to its linear counterpart $GS(m, n)$ defined on an infinite-dimensional vector space (for the definitions, see [9]). This is surprising since $BL(p, q)$ and $GS(m, n)$ have many algebraic properties in common. Likewise, in [10], the same authors showed that the symmetric inverse semigroup $I(X)$ defined on an arbitrary set is almost never isomorphic to the analogous semigroup $I(V)$ defined on an arbitrary vector space V over a field F. In Section 2, we provided a family of nongroup maximal subsemigroups of $G(X)$. Before we do the same for $G(V)$ in Section 4, we observe that, although $G(X)$ and $G(V)$ are similar in character, these groups are almost never isomorphic. In fact, it is well known that the centre of $G(X)$ is $\{\mathrm{id}_X\}$ and the centre of $G(V)$ is $\{k\,\mathrm{id}_V : k \in F \setminus \{0\}\}$. Thus, $G(X)$ and $G(V)$ are never isomorphic if $|F| > 2$.

THEOREM 3.1. *Any semigroup S can be embedded in $T(V)$, the semigroup of all linear transformations of some vector space V with dimension $|S|$, if S contains an identity (or $|S| + 1$ if S does not contain an identity).*

PROOF. Write $S^1 = \{a_i\}$. Let F be any field and let F_i be a copy of F for each $i \in I$. As in [6, page 182, Remark (c)], we let V be the vector space $\sum F_i$ over F whose basis can be identified in a natural way with $\{a_i\}$: that is, $\sum F_i$ is the set of all $(r_i)_{i\in I}$, where $r_i \in F_i$ and at most finitely many r_i are nonzero.

For each $x \in S$, let $\rho_x : S^1 \to S^1, a_i \mapsto a_i x$, be a mapping of the basis $\{a_i\}$ into itself (note that ρ_x may not be injective). Hence, ρ_x can be extended by linearity to an element of $T(V)$ that we also denote by ρ_x. Clearly, for each $x, y \in S$, ρ_{xy} and $\rho_x \circ \rho_y$ agree on the basis $\{a_i\}$, and hence they agree on all of V. That is, the mapping $\rho : S \to T(V)$, $x \to \rho_x$, is a homomorphism. Moreover, since $1 \in \{a_i\}$, $\rho_x = \rho_y$ implies that $x = y$, so ρ is injective and the result follows. □

It is well known that, if $|X| = m \ge \aleph_0$, then $|G(X)| = 2^m$ (compare [9, page 479]). To determine the cardinal of $G(V)$ when $\dim V = n \ge \aleph_0$, we first recall [7, Volume II,

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page 245]: if V is a vector space over a field F and $\dim V = n \geq \aleph_0$, then $|V| = n \times |F|$. From this, we deduce (as in [9, page 480]) that $|T(V)| = |V|^n$ and thus

$$|T(V)| = \begin{cases} 2^n & \text{if } |F| \leq n, \\ |F|^n & \text{if } |F| > n. \end{cases}$$

LEMMA 3.2. *If* $\dim V = n \geq \aleph_0$, *then the cardinal of* $G(V)$ *is* $|V|^n$.

PROOF. Suppose that $\{a_i\}$ is a basis for V and $|F| \geq 3$. In this case, if $k_i \in F \setminus \{0, 1\}$ for each i, then $\{k_i a_i\}$ is a basis for V that differs from $\{a_i\}$. Now each bijection $k_i a_i \mapsto a_i$ extends by linearity to some $\pi_k \in G(V)$, and $\pi_k \neq \pi_{k'}$ for $k \neq k'$. Note that there are $|F|^n$ bases for V of the form $\{k_i a_i\}$. Since $|G(A)| = 2^n$ for each basis A of V,

$$|G(V)| \geq 2^n.|F|^n = (n.|F|)^n = |V|^n.$$

But $|V|^n = |T(V)| \geq |G(V)|$, and equality follows. Now suppose that $|F| = 2$. For each fixed $i_0 \in I$, write $I' = I \setminus \{i_0\}$ and choose $j_0 \in I'$. Then $\{a_i : i \in I'\} \dot\cup \{a_{i_0} + a_{j_0}\}$ is a basis for V and so the number of bases for V is at least n. Hence,

$$|G(V)| \geq n.2^n = (n.2)^n = |V|^n,$$

and thus we also have equality in the case $|F| = 2$. □

In passing, we observe that, if $|F| = 2$ and $2^m \neq 2^n$, where $m, n \geq \aleph_0$ as above, then $G(X)$ is not isomorphic to $G(V)$.

4. General linear group

In Section 2, we provided a family of nongroup maximal subsemigroups of the symmetric group $G(X)$ on an infinite set X. Here, we do the same for $G(V)$, the general linear group on an infinite-dimensional vector space V. When $\alpha \in G(V)$, we take the notation displayed at the start of Section 2 to mean that α is the extension by linearity to the whole of V of a bijection between bases $\{a_i\}$ and $\{x_i\}$ for V. The subspace U of V generated by a linearly independent subset $\{u_i\}$ of V is denoted by $\langle u_i \rangle$, and we write $\dim U = |I|$. Observe that, given that $\alpha \in G(V)$ and $U \leq V$,

$$U = \langle u_i \rangle \oplus \langle u_j \rangle \quad \text{if and only if } U\alpha = \langle u_i \alpha \rangle \oplus \langle u_j \alpha \rangle.$$

Our next result is the linear analogue of Proposition 2.1. In the set case, the complement of Y in X is unique and this makes the definition of appropriate mappings in $G(X)$ and the proof of maximality simpler. But the problem here is that, given a subspace W of V, we may not fix a complementary subspace U of W in V and define all linear mappings in $G(V)$ necessary for the proof of our result by considering the images of W and of the fixed U. The concept of quotient space plays an important role, since it simplifies the task, given all possible choices of complementary subspaces. In fact, we are mainly concerned with codimensions of subspaces in a vector space, and we know that $\operatorname{codim} W = \dim V/W$.

Proposition 4.1. *Let W be a subspace of V with $\aleph_0 \leq \dim W = m \leq \operatorname{codim} W$. If*

$$H(W) = \{\alpha \in G(V) : \dim W\alpha/(W\alpha \cap W) < m\},$$

then $H(W)$ is a maximal subsemigroup of $G(V)$ that is not a group.

Proof. Suppose that $\alpha, \beta \in H(W)$ and let $W\alpha \cap W = \langle a_i \rangle$. Write $W\alpha = \langle a_i \rangle \oplus \langle b_j \rangle$, where $|J| = \dim W\alpha/(W\alpha \cap W) < m$. Since $\alpha \in G(V)$, there exist unique w_i, w_j in W such that $a_i = w_i\alpha$ and $b_j = w_j\alpha$ for every i and every j. It is not difficult to see that $W = \langle w_i \rangle \oplus \langle w_j \rangle$. On the other hand, $\langle a_i \rangle \subseteq W$ and we may write $W = \langle a_i \rangle \oplus \langle a_\ell \rangle$. Moreover, $\{a_i\} \dot\cup \{a_\ell\} \dot\cup \{b_j\}$ is a linearly independent subset of V and

$$V = \langle a_i \rangle \oplus \langle a_\ell \rangle \oplus \langle b_j \rangle \oplus \langle b_t \rangle,$$

where T may be empty. Also, if $v_\ell, v_t \in V$ are such that $v_\ell\alpha = a_\ell$ and $v_t\alpha = b_t$, for each ℓ and each t, then $V = \langle w_i \rangle \oplus \langle w_j \rangle \oplus \langle v_\ell \rangle \oplus \langle v_t \rangle$. Note that

$$\alpha = \begin{pmatrix} w_i & w_j & v_\ell & v_t \\ a_i & b_j & a_\ell & b_t \end{pmatrix}.$$

Write $\langle a_i\beta \rangle \cap W = \langle c_r \rangle$ and $\langle a_i\beta \rangle = \langle c_r \rangle \oplus \langle c_s \rangle$. Similarly, let $\langle b_j\beta \rangle \cap W = \langle d_x \rangle$ and $\langle b_j\beta \rangle = \langle d_x \rangle \oplus \langle d_y \rangle$. Then

$$W\beta = \langle a_i\beta \rangle \oplus \langle a_\ell\beta \rangle = \langle c_r \rangle \oplus \langle c_s \rangle \oplus \langle a_\ell\beta \rangle$$

and

$$W\alpha\beta = \langle a_i\beta \rangle \oplus \langle b_j\beta \rangle = \langle c_r \rangle \oplus \langle c_s \rangle \oplus \langle d_x \rangle \oplus \langle d_y \rangle.$$

Since $\langle c_r \rangle \oplus \langle d_x \rangle \subseteq W$, we have $\dim W\alpha\beta/(W\alpha\beta \cap W) \leq |S| + |Y|$. But we also have $|J| = \dim W\alpha/(W\alpha \cap W) < m$ and $|Y| \leq |J|$. Also,

$$|S| = \dim \langle a_i\beta \rangle/(\langle a_i\beta \rangle \cap W) \leq \dim W\beta/(W\beta \cap W) < m.$$

Thus, $|S| + |Y| < m + m = m$, and $\alpha\beta \in H(W)$.

Now write $W = \langle w_k \rangle \oplus \langle w_j \rangle = \langle v_j \rangle \oplus \langle u_j \rangle$ and $V = W \oplus \langle a_j \rangle \oplus \langle a_i \rangle = W \oplus \langle b_k \rangle \oplus \langle b_i \rangle$, where $|K| < |J| = m \leq n = |I| = \operatorname{codim} W$. Define $\alpha \in G(V)$ by

$$\alpha = \begin{pmatrix} w_k & w_j & a_j & a_i \\ b_k & v_j & u_j & b_i \end{pmatrix}.$$

Clearly, $W\alpha = \langle b_k \rangle \oplus \langle v_j \rangle$ and so $\dim W\alpha/(W\alpha \cap W) = |K| < m$. Thus, $\alpha \in H(W)$. But $W\alpha^{-1} = \langle w_j \rangle \oplus \langle a_j \rangle$, and hence $\dim W\alpha^{-1}/(W\alpha^{-1} \cap W) = |J| = m$ and $\alpha^{-1} \notin H(W)$. In other words, we have just shown that $H(W)$ is a subsemigroup of $G(V)$ that is not a group.

To show that $H(W)$ is maximal, we let $h \in G(V) \setminus H(W)$ and we show that, for every $g \in G(V) \setminus H(W)$, we have $g \in \langle H(W), h \rangle$. Given that $h, g \in G(V) \setminus H(W)$, we have $\dim Wh/(Wh \cap W) \geq m$ and $\dim Wg/(Wg \cap W) \geq m$. But $\dim Wh = \dim Wg = \dim W = m$, since $h, g \in G(V)$. Therefore,

$$\dim Wh/(Wh \cap W) = \dim Wg/(Wg \cap W) = m.$$

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Write $Wg^{-1} \cap W = \langle a_i g^{-1} \rangle$ and $Wg^{-1} = \langle a_i g^{-1} \rangle \oplus \langle b_\ell g^{-1} \rangle$. Then $W = \langle a_i \rangle \oplus \langle b_\ell \rangle$. Also, we may write $W = \langle a_i g^{-1} \rangle \oplus \langle u_j g^{-1} \rangle$ and $V = \langle a_i g^{-1} \rangle \oplus \langle u_j g^{-1} \rangle \oplus \langle b_\ell g^{-1} \rangle \oplus \langle u_k g^{-1} \rangle$. Clearly, $|L| + |K| = \operatorname{codim} W$ and $|I| + |J| = |I| + |L| = m$. Since $W = \langle a_i \rangle \oplus \langle b_\ell \rangle$ and $Wg = \langle (a_i g^{-1})g \rangle \oplus \langle (u_j g^{-1})g \rangle = \langle a_i \rangle \oplus \langle u_j \rangle$,

$$m = \dim Wg/(Wg \cap W) \le |J| \le \dim Wg = m,$$

and hence $|J| = m$.

Proceeding similarly, write $Wh^{-1} \cap W = \langle c_p h^{-1} \rangle$, $Wh^{-1} = \langle c_p h^{-1} \rangle \oplus \langle d_q h^{-1} \rangle$ and $W = \langle c_p h^{-1} \rangle \oplus \langle v_r h^{-1} \rangle$. Then $W = \langle c_p \rangle \oplus \langle d_q \rangle$ and we may write

$$V = \langle c_p h^{-1} \rangle \oplus \langle v_r h^{-1} \rangle \oplus \langle d_q h^{-1} \rangle \oplus \langle v_s h^{-1} \rangle.$$

As before, we may conclude that $|P| + |Q| = |P| + |R| = m$, $|Q| + |S| = \operatorname{codim} W$ and $|R| = m$.

Case 1. Suppose that $\dim Wh^{-1}/(Wh^{-1} \cap W) < m$ and $\dim Wg^{-1}/(Wg^{-1} \cap W) < m$. Then $|L| = \dim Wg^{-1}/(Wg^{-1} \cap W) < m \le \operatorname{codim} W$, so $|K| = \operatorname{codim} W$ and $|I| = m$. Since $|J| = |I|$, we may write $\{u_j g^{-1}\}$ as $\{u_i g^{-1}\}$.

Analogously, $|Q| < m = |P| = |R|$ and $|S| = \operatorname{codim} W$. Therefore, we may write $\{c_p h^{-1}\}$, $\{v_r h^{-1}\}$ and $\{v_s h^{-1}\}$ as $\{c_i h^{-1}\}$, $\{v_i h^{-1}\}$ and $\{v_k h^{-1}\}$, respectively. Since $|Q| < |I|$, $|L| < |I|$ and $|I| = m \ge \aleph_0$, we may write

$$\langle a_i g^{-1} \rangle = \langle w_i g^{-1} \rangle \oplus \langle w_q g^{-1} \rangle, \quad \langle c_i h^{-1} \rangle = \langle y_i h^{-1} \rangle \oplus \langle y_\ell h^{-1} \rangle.$$

Now define π_1, π_2 in $G(V)$ by

$$\pi_1 = \begin{pmatrix} u_i g^{-1} & w_i g^{-1} & b_\ell g^{-1} & u_k g^{-1} & w_q g^{-1} \\ v_i h^{-1} & y_i h^{-1} & y_\ell h^{-1} & v_k h^{-1} & d_q h^{-1} \end{pmatrix}, \quad \pi_2 = \begin{pmatrix} v_i & y_i & y_\ell & v_k & d_q \\ u_i & w_i & b_\ell & u_k & w_q \end{pmatrix}.$$

Clearly, $g = \pi_1 h \pi_2$. It is not difficult to see that $W\pi_1 = \langle v_i h^{-1} \rangle \oplus \langle y_i h^{-1} \rangle \oplus \langle d_q h^{-1} \rangle$. But $\langle v_i h^{-1} \rangle \oplus \langle y_i h^{-1} \rangle \subseteq W$, and so $\dim W\pi_1/(W\pi_1 \cap W) \le |Q| < m$. On the other hand, $W\pi_2 = \langle w_i \rangle \oplus \langle b_\ell \rangle \oplus \langle w_q \rangle \subseteq W$, and so $\dim W\pi_2/(W\pi_2 \cap W) = 0 < m$. Thus, $\pi_1, \pi_2 \in H(W)$.

Case 2. Suppose that $\dim Wh^{-1}/(Wh^{-1} \cap W) = m$ and $\dim Wg^{-1}/(Wg^{-1} \cap W) < m$. Then $|Q| = m$, $|L| < m$, and this inequality implies that $|I| = m \le \operatorname{codim} W = |K|$, but $|P|$ is unknown (at most m). Also, $|Q| + |S| = |I| + |S| = \operatorname{codim} W$ and we may write $\{u_j g^{-1}\}$, $\{v_r h^{-1}\}$ and $\{d_q h^{-1}\}$ as $\{u_i g^{-1}\}$, $\{v_i h^{-1}\}$ and $\{d_i h^{-1}\}$, respectively. Also, let

$$\begin{aligned} \langle a_i g^{-1} \rangle &= \langle w_i g^{-1} \rangle \oplus \langle w_p g^{-1} \rangle, \\ \langle u_k g^{-1} \rangle &= \langle y_i g^{-1} \rangle \oplus \langle y_s g^{-1} \rangle, \\ \langle d_i h^{-1} \rangle &= \langle x_i h^{-1} \rangle \oplus \langle x_\ell h^{-1} \rangle, \\ \langle v_i h^{-1} \rangle &= \langle z_i h^{-1} \rangle \oplus \langle z_i^* h^{-1} \rangle = \langle z_i' h^{-1} \rangle \oplus \langle z_i'' h^{-1} \rangle \oplus \langle z_\ell''' h^{-1} \rangle, \end{aligned}$$

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and define π_1, π_2, π_3 in $G(V)$ by

$$\pi_1 = \begin{pmatrix} u_ig^{-1} & w_ig^{-1} & w_pg^{-1} & y_sg^{-1} & y_ig^{-1} & b_\ell g^{-1} \\ z_i^*h^{-1} & z_ih^{-1} & c_ph^{-1} & v_sh^{-1} & x_ih^{-1} & x_\ell h^{-1} \end{pmatrix},$$

$$\pi_2 = \begin{pmatrix} z_i^* & z_i & c_p & v_s & x_i & x_\ell \\ z_i''h^{-1} & d_ih^{-1} & c_ph^{-1} & v_sh^{-1} & z_i'h^{-1} & z_\ell'''h^{-1} \end{pmatrix},$$

$$\pi_3 = \begin{pmatrix} z_i'' & d_i & c_p & v_s & z_i' & z_\ell''' \\ u_i & w_i & w_p & y_s & y_i & b_\ell \end{pmatrix}.$$

Clearly, $g = \pi_1 h \pi_2 h \pi_3$. Also,

$$W\pi_1 = \langle z_i^*h^{-1}\rangle \oplus \langle z_ih^{-1}\rangle \oplus \langle c_ph^{-1}\rangle,$$
$$W\pi_2 = \langle c_ph^{-1}\rangle \oplus \langle z_i'h^{-1}\rangle \oplus \langle z_\ell'''h^{-1}\rangle,$$
$$W\pi_3 = \langle w_i\rangle \oplus \langle w_p\rangle,$$

and so, for $i = 1, 2, 3$, $W\pi_i \subseteq W$ and $\dim W\pi_i/(W\pi_i \cap W) = 0 < m$, that is, $\pi_i \in H(W)$.

Case 3. Suppose that $\dim Wh^{-1}/(Wh^{-1} \cap W) < m$ and $\dim Wg^{-1}/(Wg^{-1} \cap W) = m$. In this event, $|I| \le m = |J| = |L| \le \operatorname{codim} W = |L| + |K|$. On the other hand, $|Q| < m = |P| = |R| \le \operatorname{codim} W = |S|$. Thus, we may write $\{b_\ell g^{-1}\}$, $\{c_ph^{-1}\}$ and $\{v_rh^{-1}\}$ as $\{b_jg^{-1}\}$, $\{c_jh^{-1}\}$ and $\{v_jh^{-1}\}$, respectively. Also, write

$$\langle b_jg^{-1}\rangle = \langle w_jg^{-1}\rangle \oplus \langle z_jg^{-1}\rangle \oplus \langle w_qg^{-1}\rangle,$$
$$\langle v_jh^{-1}\rangle = \langle x_jh^{-1}\rangle \oplus \langle x_ih^{-1}\rangle,$$
$$\langle v_sh^{-1}\rangle = \langle y_jh^{-1}\rangle \oplus \langle y_kh^{-1}\rangle,$$

and define π_1, π_2 in $G(V)$ by

$$\pi_1 = \begin{pmatrix} u_jg^{-1} & a_ig^{-1} & w_jg^{-1} & z_jg^{-1} & u_kg^{-1} & w_qg^{-1} \\ x_jh^{-1} & x_ih^{-1} & c_jh^{-1} & y_jh^{-1} & y_kh^{-1} & d_qh^{-1} \end{pmatrix},$$

$$\pi_2 = \begin{pmatrix} x_j & x_i & c_j & y_j & y_k & d_q \\ u_j & a_i & w_j & z_j & u_k & w_q \end{pmatrix}.$$

Clearly, $g = \pi_1 h \pi_2$. Also,

$$W\pi_1 = \langle x_jh^{-1}\rangle \oplus \langle x_ih^{-1}\rangle,$$
$$W\pi_2 = \langle w_j\rangle \oplus \langle w_q\rangle,$$

and hence $\dim W\pi_1/(W\pi_1 \cap W) = \dim W\pi_2/(W\pi_2 \cap W) = 0 < m$. In other words, $\pi_1, \pi_2 \in H(W)$.

Case 4. Suppose that $\dim Wh^{-1}/(Wh^{-1} \cap W) = m$ and $\dim Wg^{-1}/(Wg^{-1} \cap W) = m$. Then $|I| \le m = |L| = |J| \le |L| + |K| = \operatorname{codim} W$ and $|P| \le m = |Q| = |R| \le |Q| + |S| = \operatorname{codim} W$. Thus, we may write $\{b_\ell g^{-1}\}$, $\{d_qh^{-1}\}$ and $\{v_rh^{-1}\}$ as $\{b_jg^{-1}\}$, $\{d_jh^{-1}\}$ and $\{v_jh^{-1}\}$, respectively.

If $m < \operatorname{codim} W$, then $|K| = \operatorname{codim} W = |S|$. Write $\{v_sh^{-1}\}$ as $\{v_kh^{-1}\}$ and

$$\langle b_jg^{-1}\rangle = \langle w_jg^{-1}\rangle \oplus \langle w_pg^{-1}\rangle, \quad \langle v_jh^{-1}\rangle = \langle x_jh^{-1}\rangle \oplus \langle x_ih^{-1}\rangle.$$

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Now define π_1, π_2 in $G(V)$ by

$$\pi_1 = \begin{pmatrix} u_j g^{-1} & a_i g^{-1} & u_k g^{-1} & w_p g^{-1} & w_j g^{-1} \\ x_j h^{-1} & x_i h^{-1} & v_k h^{-1} & c_p h^{-1} & d_j h^{-1} \end{pmatrix},$$

$$\pi_2 = \begin{pmatrix} x_j & x_i & v_k & c_p & d_j \\ u_j & a_i & u_k & w_p & w_j \end{pmatrix}.$$

If $m = \operatorname{codim} W$, then $|K| \leq m$ and $|S| \leq m$. Write

$$\langle b_j g^{-1} \rangle = \langle w_p g^{-1} \rangle \oplus \langle w_s g^{-1} \rangle \oplus \langle w_j g^{-1} \rangle,$$
$$\langle v_j h^{-1} \rangle = \langle x_j h^{-1} \rangle \oplus \langle x_i h^{-1} \rangle \oplus \langle x_k h^{-1} \rangle,$$

and define π_1, π_2 in $G(V)$ by

$$\pi_1 = \begin{pmatrix} u_j g^{-1} & a_i g^{-1} & u_k g^{-1} & w_p g^{-1} & w_s g^{-1} & w_j g^{-1} \\ x_j h^{-1} & x_i h^{-1} & x_k h^{-1} & c_p h^{-1} & v_s h^{-1} & d_j h^{-1} \end{pmatrix},$$

$$\pi_2 = \begin{pmatrix} x_j & x_i & x_k & c_p & v_s & d_j \\ u_j & a_i & u_k & w_p & w_s & w_j \end{pmatrix}.$$

It is easy to see that, in both cases, $W\pi_1, W\pi_2 \subseteq W$, and so $\pi_1, \pi_2 \in H(W)$. Moreover, $g = \pi_1 h \pi_2$. □

As for the set case, it is not difficult to see that $H(W)$ is a nongroup maximal subsemigroup of $G(V)$ if and only if $\aleph_0 \leq \dim W \leq \operatorname{codim} W$. In fact, as we prove in our next result, the linear version of Hotzel's claim holds.

THEOREM 4.2. *Let W be a subspace of V. Then the set*

$$H(W) = \{\pi \in G(V) : \dim W\pi/(W\pi \cap W) < \dim W\}$$

is a maximal subsemigroup of $G(V)$ if and only if $\dim W = 1$ *or* $\aleph_0 \leq \dim W \leq \operatorname{codim} W$.

PROOF. By Proposition 4.1, if $\aleph_0 \leq \dim W \leq \operatorname{codim} W$, then $H(W)$ is a maximal subsemigroup of $G(V)$. Now assume that $\dim W = 1$ and let $W = \langle w \rangle$, with $w \neq 0$. Since $\dim W\pi = 1$ for each $\pi \in G(V)$, it follows that $\dim W\pi/(W\pi \cap W) < 1$ if and only if $W\pi = W$, and hence

$$H(W) = \{\pi \in G(V) : w\pi = k\pi \text{ for some } k \in F \setminus \{0\}\}.$$

It is easy to see that $H(W)$ is a subsemigroup of $G(V)$ that is a group. Given $h, g \in G(V) \setminus H(W)$,

$$Wh \cap W = \{0\} = Wg \cap W.$$

Therefore, there exist nonzero $u, v, a, b \notin \langle w \rangle$ such that $wh = u$, $wg = v$, $ah = w$ and $bg = w$. Write $V = \langle w \rangle \oplus \langle a \rangle \oplus \langle a_i \rangle = \langle w \rangle \oplus \langle b \rangle \oplus \langle b_i \rangle$, and define $\pi_1, \pi_2 \in G(V)$ by

$$\pi_1 = \begin{pmatrix} w & b & b_i \\ w & a & a_i \end{pmatrix}, \quad \pi_2 = \begin{pmatrix} w & u & a_i h \\ w & v & b_i g \end{pmatrix}.$$

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Clearly, $\pi_1, \pi_2 \in H(W)$, since $w\pi_i = w$ for $i = 1, 2$. Also,

$$\pi_1 h \pi_2 = \begin{pmatrix} w & b & b_i \\ v & w & b_i g \end{pmatrix} = g.$$

Thus, if $\dim W = 1$, then $H(W)$ is a maximal subsemigroup of $G(V)$.

Conversely, suppose that $\dim W = m$ with $m \in \mathbb{N} \setminus \{1\}$, and write $W = \langle w_1, \ldots, w_m \rangle$. Since V is infinite-dimensional, we may write $V = \langle w_1, \ldots, w_m \rangle \oplus \langle u_1, \ldots, u_m \rangle \oplus \langle v_i \rangle$, where $|I| = \dim V$. Now define $\alpha, \beta \in G(V)$ by

$$\alpha = \begin{pmatrix} w_1 & w_2 & \ldots & w_m & u_1 & u_2 & \ldots & u_m & v_i \\ w_1 & u_2 & \ldots & u_m & u_1 & w_2 & \ldots & w_m & v_i \end{pmatrix},$$

$$\beta = \begin{pmatrix} w_1 & w_2 & \ldots & w_m & u_1 & u_2 & \ldots & u_m & v_i \\ u_1 & w_2 & \ldots & w_m & w_1 & u_2 & \ldots & u_m & v_i \end{pmatrix}.$$

Clearly, $\dim W\alpha/(W\alpha \cap W) = \dim \langle u_2, \ldots, u_m \rangle = m - 1$ and $\dim W\beta/(W\beta \cap W) = \dim \langle u_1 \rangle = 1$, so $\alpha, \beta \in H(W)$. But

$$\alpha\beta = \begin{pmatrix} w_1 & w_2 & \ldots & w_m & u_1 & u_2 & \ldots & u_m & v_i \\ u_1 & u_2 & \ldots & u_m & w_1 & w_2 & \ldots & w_m & v_i \end{pmatrix},$$

and hence $\dim W\alpha\beta/(W\alpha\beta \cap W) = \dim \langle u_1, u_2, \ldots, u_m \rangle = m$. Therefore, $\alpha\beta \notin H(W)$ and $H(W)$ is not a semigroup. Next, assume that W is infinite-dimensional and $\dim W > \operatorname{codim} W$. Given that $\pi \in G(V)$, $\dim W\pi/(W\pi \cap W) \leq \operatorname{codim} W < \dim W$. Thus, $H(W) = G(V)$. In other words, we have just proved that $H(W)$ is not a maximal subsemigroup of $G(V)$ when $\dim W \in \mathbb{N} \setminus \{1\}$ or W is infinite-dimensional but $\dim W > \operatorname{codim} W$. □

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SUZANA MENDES-GONÇALVES, Centro de Matemática,
Universidade do Minho, 4710 Braga, Portugal
e-mail: d233@math.uminho.pt

R. P. SULLIVAN, School of Mathematics and Statistics,
University of Western Australia, Nedlands, Western Australia 6009, Australia
e-mail: bob@maths.uwa.edu.au