A GENERALIZATION OF THE BANG-BANG PRINCIPLE OF LINEAR CONTROL THEORY*

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In a paper by LaSalle [1] on linear time optimal control the following lemma is proved:

LEMMA. Let Ω be the set of all r-dimensional vector functions $u(\tau)$ measurable on $[0, t]$ with $|u_i(\tau)| \leq 1$. Let Ω^o be the subset of functions $u^o(\tau)$ with $|u_i^o(\tau)| = 1$. Let $Y(\tau)$ be any $(n \times r)$ matrix function in $L^1([0, t])$. Define

$$A(t) = \{ \int_0^t Y(\tau)u(\tau)d\tau ;\ u \in \Omega \}$$

and

$$A^o(t) = \{ \int_0^t Y(\tau)u^o(\tau)d\tau ;\ u^o \in \Omega^o \} .$$

Then $A^o(t)$ is closed and $A(t) = A^o(t)$.

In theorem 1 we generalize the above lemma to the case where the functions u take their values on an arbitrary convex set in R^m which may vary with time. Theorem 2 gives a characterization of the boundary points of a set which corresponds to the set $A(t)$ of the lemma.

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It should be pointed out that the above lemma has been generalized in another direction by L. W. Neustadt [3]. However, theorem 1 of this note and Neustadt's result have an intersection which is larger than the above lemma. For example the result of the remark at the end of this note can also be found in [3].

Let K_o be a compact convex set in R^n and assume:

1. For each t in some measurable set $I \subset R$ containing the point $t = 0$, with Lebesgue measure $\mu(I) < \infty$, that there corresponds a mapping $F_t: K_o \to K_t \subset R^n$, where K_t is a compact convex set.

2. There exist a ball in R^n, $S(0, p)$, of radius $p < \infty$ such that $K_t \subset S(0, p)$ for all $t \in I$.

3. If x_o is on the boundary, $\partial(K_o)$, of K_o then the mapping $F_t(x_o) = x_t \in \partial(K_t)$ for $t \in I$, and is the value of a measurable mapping $x: I \to R^n$ evaluated at the point t.

4. If $\{x_i(0)\}$ is countably dense in the set of extremal points of K_o then for each $t \in I$ $\{x_i(t)\}$ is countably dense in the set of extremal points of K_t.

THEOREM 1. _If_ $f: I \to R^n$ _is a measurable mapping such that_ $f(t) \in K_t$ _for each_ $t \in I$ _then there exists a measurable mapping_ $\bar{f}: I \to R^n$ _such that_ $\bar{f}(t) \in \partial(K_t)$ _for each_ $t \in I$ _and_ $\int_I f(t)dt = \int_I \bar{f}(t)dt$.

Proof. Consider the family U of measurable maps $z: I \to R^n$ with the property that $z(t) \in \partial(K_t)$ for each $t \in I$. Let

$$A = \{\int_I z(t)dt \mid z \in U\}.$$

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The theorem will be proved if we can show that $\int_I f(t)dt \in A$.

To do this we show the following:

(i) A is convex.

(ii) A is closed and hence since $K_t \subset S(0, p)$, for each $t \in I, A$ is compact. Then using (i) and (ii) we show $\int_I f(t)dt \in A$.

Proof of (i). Suppose z_i, $i = 1, 2$, are in U. Let $r_i = \int_I z_i(t)dt$, $i = 1, 2$. Then by Lyapunov's Theorem on the Range of a Vector Measure [2], given any $\alpha \in (0, 1)$ there exists a measurable set $D \subset I$ such that $\int_D z_i(t)dt = \alpha r_i$, $i = 1, 2$, and $\mu(D) = \alpha\mu(I)$. Let $z = C_D z_1 + (1 - C_D)z_2$ where C_D is the characteristic function of D. Obviously $z(t) \in \partial(K_t)$ for each $t \in I$, and $\int_I z(t)dt = \int_D z_i(t)dt + \int_{I-D} z_2(t)dt = \alpha r_1 + (1 - \alpha)r_2$ which shows the convexity of A.

(ii) Proof that A is closed. Since A is convex, to show A is closed it is only necessary to prove that the extremal points of $\bar{A}$ (which is also convex) belong to A. Suppose r is an extremal point of $\bar{A}$. Then there exists a sequence $\{r_n\} \subset A$ converging to r. Each r_n has the representation $r_n = \int_I z_n(t)dt$ with $z_n \in U$. Using theorem 4 of Blackwell's paper [2] we can find a subsequence in $\{z_n\}$, which for convenience is assumed to be the original sequence, such that $\lim_{n\to\infty} z_n(t) = z(t)$ [a.e.] on I. The Lebesgue bounded convergence theorem shows that $\lim_{n\to\infty} \int_I z_n(t)dt = \int_I z(t)dt$ i.e., $\lim_{n\to\infty} r_n = r$.

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Let $T_1 = \{t: \lim_{n\to\infty} z_n(t) = z(t)\}$, $T_2 = I - T_1$. Thus the measure of T_2 is zero and $z(t) \in \partial(K_t)$ for $t \in T_1$. Define $\bar{z}(t) = \begin{cases} z(t) & \text{on } T_1 \\ z_1(t) & \text{on } T_2 \end{cases}$. Then obviously $\int_I \bar{z}(t)dt = r$ which shows that $A = \bar{A}$.

Finally we show that $r \in A$. Suppose $r \notin A$. Since A is a compact convex set in R^n there exists a point $y \in R^n$ such that $y \cdot r > y \cdot p + \alpha$ for all $p \in A$ and some $\alpha > 0$. Let $\epsilon = \frac{\alpha}{2\mu(I)}$ and consider the sets $E_j(\alpha) = \{t: y \cdot f(t) < y \cdot x_j(t) + \frac{\alpha}{2\mu(I)}\}$ where $\{x_j(t)\}$ are the mappings postulated in assumption 4. It is easy to verify that $\bigcup_{j=1}^{\infty} E_j(\alpha) = I$. Let $E_1 = E_1(\alpha), \ldots, E_k = E_k(\alpha) - \bigcup_{j=1}^{k-1} E_j(\alpha)$. Then the $\{E_k\}$ form a measurable partition of I. Let C_{E_j} be the characteristic function of E_j, $j = 1, 2, \ldots$. Define $z = \sum_{j=1}^{\infty} C_{E_j} x_j$.

Then $p = \int_I z(t)dt \in A$ and $y \cdot \int_I f(t)dt < y \cdot \int_I z(t)dt + \frac{\alpha}{2}$.

i.e. $y \cdot r < y \cdot p + \frac{\alpha}{2}$. But this implies

$y \cdot p + \frac{\alpha}{2} > y \cdot r > y \cdot p + \alpha$ and hence $\frac{\alpha}{2} > \alpha$ which is a contradiction since $\alpha > 0$. Hence $r \in A$ and the theorem is proved.

THEOREM 2. *Let* $f(t) \in K_t$ *for each* $t \in I$. *Then the point* $p = \int_I f(t)dt$ *is in* $\partial(A)$ *if and only if there exists a* $y \neq 0$ in R^n *such that* $y \cdot f(t) = \sup y \cdot x_i(t)$ a.e. *on* I.

Proof. If f has the above property then since $\int_I \sup_i y \cdot x_i(t)dt = \int_I y \cdot f(t)\, dt = y \cdot p$ it follows that p must be in $\partial(A)$.

Next we suppose that $f(t) \in K_t$ for each $t \in I$ and that $p = \int_I f(t)dt \in \partial(A)$. Since A is convex it follows that there exists a $y \neq 0$ in R^n such that $y \cdot p \geqq y \cdot a$ for all $a \in A$.

Assume that f does not possess the property stated in the theorem. This implies that the set $E = \{t: y \cdot f(t) < \sup_i y \cdot x_i(t)\}$ is a measurable set with Lebesgue measure $\mu(E) > 0$. We consider the sequence $\{\frac{1}{2^k}\}$, $k = 1, 2, \ldots,$ and define a corresponding sequence of measurable sets $E_k = \{t: y \cdot f(t) + \frac{1}{2^k} < \sup_i y \cdot x_i(t)\} \cap E$. It is evident that $\bigcup_{k=1}^{\infty} E_k = E$. Hence since E has a positive measure there exists a j and an $\alpha > 0$ such that $\mu(E_j) = \alpha$.

Let $E_{ji} = \{t: y \cdot f(t) + \frac{1}{2^{j+1}} < y \cdot x_i(t)\} \cap E_j$. Obviously $\bigcup_i E_{ji} = E_j$. Since $\mu(E_j) = \alpha > 0$ there is an i such that $\mu(E_{ji}) > 0$.

Define

$$\bar{f}(t) = \begin{cases} f(t) & \text{on } I - E_{ji} \\ x_i(t) & \text{on } E_{ji} \end{cases}$$

Then $y \cdot \int_I \bar{f}(t)dt = y \cdot \int_{E_{ji}} x_i(t)dt + y \cdot \int_{I-E_{ji}} f(t)dt$

$$> y \cdot \int_{E_{ji}} f(t)dt + y \cdot \int_{I-E_{ji}} f(t)dt = y \cdot p$$ which contradicts the assumption that $p \in \partial(A)$.

Remark. As an application of theorem 1 we consider the system

$$\dot{x} = A(t)x + B(t)u(t) , \tag{1}$$

where $A(t)$ is a continuous $n \times n$ matrix, $B(t)$ is a continuous $n \times m$ matrix and u is in the family U of measurable mappings from $[0, \infty) \to K$ a compact convex set in R^m.

For any $u \in U$ the solution of sytem (1) with initial condition x_o at time t_o is given by

$$x_u(t) = X(t) [x_o + \int_{t_o}^{t} X^{-1}(s) B(s) u(s) ds] , \tag{2}$$

where $X(t)$ is a fundamental matrix of the system $\dot{X}(t) = A(t)X(t)$ with $X(t_o) = I$ the identity matrix.

Since linear mappings have the property that they map convex sets into convex sets and the extremal points of such sets into the extremal points of the image sets we see that the mappings $K_s : K \to R^n$ given by $X^{-1}(s) B(s) K$ satisfy assumptions 1-3. Moreover if $\{u_i\}$ is dense in the set of extremal points on K then $\{K_s(u_i)\} = \{X^{-1}(s) B(s) u_i\}$ will also be dense on the set of extremal points of $X^{-1}(s) B(s) K$. Thus assumption 4 is also satisfied.

Suppose we are given a $u \in U$ and $T < +\infty$. Let $x_u(T) = x_1$. Applying theorem 1 it follows that there exists a $\bar{u} \in U$ with the property that $x_{\bar{u}}(T) = x_1$ and for each $t \in [t_o, T]$ $\bar{u}(t) \in \partial(K_t)$.

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