A NOTE ON GAMMA FUNCTIONS

by G. N. WATSON

Various improvements in the formula

$$\frac{2}{\pi}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdots\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdots,$$

which was discovered by Wallis in 1669, were studied by D. K. Kazarinoff in No. 40 of these *Notes* (December 1956). He began by quoting from text-books the formula

$$\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}=\frac{1}{\sqrt{\{\pi(n+\theta)\}}},\quad 0<\theta<\tfrac{1}{2}\,;\ n=1,\,2,\,\ldots$$

(I do not remember having seen this formula stated explicitly; but it, like the original formula, is an immediate consequence of taking $z=\frac{1}{2}\pi$ in the canonical product for $\sin z$.) He then sharpened this result by proving that the inequality satisfied by θ could be replaced by $\frac{1}{4}<\theta<\frac{1}{2}$. He deduced this inequality as a special case of the corresponding inequality for Gamma functions, establishing the latter by applying some rather elaborate analysis to an integral formula due to Legendre.

I have noticed that the results which he obtained (and more) are almost immediate consequences of a special case of the Gaussian formula for the hypergeometric function with final element equal to unity, as I now proceed to show.

We write

$$\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}\equiv f(x)\equiv\frac{1}{\sqrt{\{x+\theta(x)\}}},$$

taking $x+\frac{1}{2}$ positive (or zero) throughout the following work.

Then, by the formula just mentioned, we have

$$\theta(x)=-x+x\,\frac{\Gamma(x)\Gamma(x+1)}{\Gamma^2(x+\frac{1}{2})}=-x+xF(-\tfrac{1}{2},\,-\tfrac{1}{2}\,;\ x\,;\ 1)$$

$$=\sum_{m=1}^{\infty}\frac{(-\frac{1}{2})_m(-\frac{1}{2})_m}{m\,!\,(x+1)_{m-1}},$$

with the usual notation

$$(z)_0=1,\quad (z)_m=z(z+1)\ldots(z+m-1),\quad (m=1,\,2,\,3,\,\ldots)\,;$$

the condition $x+\frac{1}{2}\geqslant 0$ amply secures the convergence of the series.

Now each term of the series

$$\sum_{m=2}^{\infty}\frac{(-\frac{1}{2})_m(-\frac{1}{2})_m}{m\,!\,(x+1)_{m-1}}$$

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is positive and decreases as x is increased. It follows immediately that $\theta(x)$ is a monotonic decreasing function of x; we also have

$$\theta(-\tfrac{1}{2})=\tfrac{1}{2},$$

while

$$\theta(0)=\pi^{-1}=0{\cdot}318\ldots,\quad \theta(1)=4\pi^{-1}-1=0{\cdot}273\ldots$$

We now consider what happens when $x\to\infty$. For $x\geqslant 1$ we have

$$\begin{aligned}0<\theta(x)-\tfrac{1}{4}&=\frac{1}{x+1}\sum_{m=2}^{\infty}\frac{(-\frac{1}{2})_m(-\frac{1}{2})_m}{m!\,(x+2)_{m-2}}\\ &\leqslant\frac{1}{x+1}\sum_{m=0}^{\infty}\frac{(-\frac{1}{2})_m(-\frac{1}{2})_m}{m!\,(3)_{m-2}}\\ &\to 0 \text{ as } x\to\infty.\end{aligned}$$

This establishes the sharpened theorem, namely $\theta(x)>\frac{1}{4}$ in place of $\theta(x)>0$ for a continuous variable $x(x\geqslant-\frac{1}{2})$, which was discovered and proved by Kazarinoff for x positive.

I have not tried to ascertain whether the monotonic property of $\theta(x)$ (which is evident by my method) can be obtained by Kazarinoff's method.

The inequalities

$$\tfrac{1}{4}\leqslant\theta(x)\leqslant\tfrac{1}{2},\quad(x\geqslant-\tfrac{1}{2});\quad \tfrac{1}{4}<\theta(x)\leqslant\pi^{-1},\quad(x\geqslant 0)$$

are evident from the foregoing results.

We leave to the reader the task of deducing various inequalities from the relation

$$x+\theta(x)=\frac{x^2}{x-\frac{1}{2}+\theta(x-\frac{1}{2})}$$

for appropriate ranges of values of x.

Next consider positive integral values (n) of x only, zero included.

For $n=0$ we have $\theta(0)=\pi^{-1}$, as already stated.

For $n=1, 2, 3, \ldots$ we have

$$\frac{1\,.\,3\,.\,5\ldots(2n-1)}{2\,.\,4\,.\,6\ldots(2n)}\sqrt{\pi}=\frac{1}{\sqrt{\{n+\theta(n)\}}},$$

$$\theta(1)=0{\cdot}273\ldots\geqslant\theta(n)>\tfrac{1}{4},$$

so that the change in the value of $\theta(n)$ as n runs through these positive integral values is not particularly large.

I might mention that this is not my first encounter with the function here denoted by $f(x)$. It is proved by E. W. Hobson, *Spherical and Ellipsoidal Harmonics* (1931), §192 that the function satisfies the rather weak inequality

$$f(x)<1/\sqrt{(x-\tfrac{1}{2})},\quad(x>\tfrac{1}{2}).$$

When I saw the paged proofs of the book, I remarked to Hobson that the inequality could be easily obtained from the modified form of the First Eulerian Integral

$$f(x)=\frac{2}{\sqrt{\pi}}\int_0^{\frac{1}{2}\pi}\cos^{2x}\theta\,d\theta=\frac{2}{\sqrt{\pi}}\int_0^{\infty}\exp(-xt^2)\,\frac{t\exp(-\frac{1}{2}t^2)dt}{\sqrt{\{1-\exp(-t^2)\}}},$$

and he agreed with me ; but it did not seem worth while going to the trouble and expense of replacing his work by mine.

By using the fairly obvious inequalities

$$\sqrt{\{1-\exp(-t^2)\}} \leqslant t, \quad \frac{t\exp(-\frac{1}{4}t^2)}{\sqrt{\{1-\exp(-t^2)\}}} = \frac{t}{\sqrt{(2\sinh\frac{1}{2}t^2)}} \leqslant 1,$$

we have, for $x > -\frac{1}{4}$,

$$\frac{2}{\sqrt{\pi}}\int_0^\infty \exp\{-(x+\tfrac{1}{2})t^2\}dt < f(x) < \frac{2}{\sqrt{\pi}}\int_0^\infty \exp\{-(x+\tfrac{1}{4})t^2\}dt,$$

that is to say

$$1/\sqrt{(x+\tfrac{1}{2})} < f(x) < 1/\sqrt{(x+\tfrac{1}{4})}.$$

These are the results given by Kazarinoff ; and Hobson's inequality is a weakened version of the right-hand inequality.

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