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Globalization of Distinguished Supercuspidal Representations of GL(n)

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Abstract. An irreducible supercuspidal representation π of $G = \mathrm{GL}(n, F)$, where F is a nonarchimedean local field of characteristic zero, is said to be "distinguished" by a subgroup H of G and a quasicharacter χ of H if $\mathrm{Hom}_H(\pi, \chi) \neq 0$. There is a suitable global analogue of this notion for an irreducible, automorphic, cuspidal representation associated to $\mathrm{GL}(n)$. Under certain general hypotheses, it is shown in this paper that every distinguished, irreducible, supercuspidal representation may be realized as a local component of a distinguished, irreducible automorphic, cuspidal representation. Applications to the theory of distinguished supercuspidal representations are provided.

1 Introduction

This paper is devoted to providing evidence which supports the heuristic which, loosely stated, says that whatever is true for distinguished automorphic, cuspidal representations of $\mathrm{GL}(n)$ should also be true for distinguished supercuspidal representations of $\mathrm{GL}(n)$. Before defining "distinguishedness" and stating our results more precisely, let us provide a simple example which involves the pair $\big(\mathrm{GL}(2n), \mathrm{Sp}(2n)\big)$. Given a number field F, it is shown in [15] that there cannot exist any automorphic, cuspidal representations π of $\mathrm{GL}(2n, F_{\mathbb{A}})$ which are distinguished by $\mathrm{Sp}(2n, F_{\mathbb{A}})$ in the sense that the period integral

$$\int_{\mathrm{Sp}(2n,F)\backslash \mathrm{Sp}(2n,F_{\mathbb{A}})} \varphi(h)\, dh$$

is nonzero for some φ in the space of π. The corresponding result for supercuspidal representations, proved in [14], says that if F is a nonarchimedean local field of characteristic zero then there cannot exist any supercuspidal representations π of $\mathrm{GL}(2n, F)$ which are distinguished with respect to $\mathrm{Sp}(2n, F)$ in the sense that there exists a nonzero $\mathrm{Sp}(2n, F)$-invariant linear functional on the space of π. Our main theorem, Theorem 1, allows us to immediately deduce a local result, such as the one just cited, from the corresponding global result. In examples such as the case of $\big(\mathrm{GL}(n), U(n)\big)$ (considered in [12] and discussed below), this is useful since the

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global result is considerably simpler to prove than the local result.

Given a group G, a quasicharacter χ of a subgroup H and a representation π of G, one can consider the space $\mathrm{Hom}_H(\pi,\chi)$ of linear forms λ on the representation space W of π such that $\lambda\bigl(\pi(h)w\bigr) = \chi(h)\lambda(w)$, for all $h \in H$ and $w \in W$. For example, taking $\chi = 1$ and assuming that Frobenius reciprocity applies, then the nonvanishing of $\mathrm{Hom}_H(\pi,1)$ is equivalent to the existence of a suitable model for π as a space of functions on $H\backslash G$. It is therefore not surprising that the representations π with $\mathrm{Hom}_H(\pi,1) \neq 0$ turn out to be basic building blocks in the harmonic analysis on $H \backslash G$ in many examples. The central focus of this paper is the case in which $G = \mathrm{GL}(n,F)$, for some nonarchimedean local field F of characteristic zero, and π is an irreducible supercuspidal representation of G. In this setting, we will say that π is (H,χ)-*distinguished* if $\mathrm{Hom}_H(\pi,\chi) \neq 0$.

Now suppose that we are dealing with a nonarchimedean local field F_{v_0} which is a completion of some number field F at some finite place v_0. Consider the adele group $G_{\mathbb{A}} = \mathrm{GL}(n,F_{\mathbb{A}})$ and suppose that $H_{\mathbb{A}}$ is an adelic subgroup associated to a reductive F-subgroup of $\mathrm{GL}(n)$. Assume that $\chi = \otimes_v \chi_v$ is an automorphic character of $H_{\mathbb{A}}$, that is, a 1-dimensional automorphic representation of $H_{\mathbb{A}}$. An irreducible, automorphic, cuspidal representation π of $G_{\mathbb{A}}$ is said to be (H,χ)-*distinguished* if the restriction of χ to $F_{\mathbb{A}}^\times \cap H_{\mathbb{A}}$ agrees the corresponding restriction of the central character of π and if the space of π contains a function φ such that the period integral

$$P_\chi(\varphi) = \int_{(F_{\mathbb{A}}^\times \cap H_{\mathbb{A}})H\backslash H_{\mathbb{A}}} \varphi(h)\chi(h)^{-1}\, dh$$

is nonzero, where H denotes the group of F-rational points in $H_{\mathbb{A}}$. This definition is stated in a slightly broader context in the next section. Given an irreducible supercuspidal representation τ of $G_{v_0} = \mathrm{GL}(n,F_{v_0})$, we say that an irreducible, automorphic, cuspidal representation π of $G_{\mathbb{A}}$ is a *globalization* of τ if τ is equivalent to the local component of π at v_0. The existence of a globalization for τ is discussed in [1], [2], [3] and [6]. Our main result in the present paper states that every distinguished, irreducible, supercuspidal representation τ admits a distinguished globalization π.

2 Statement of the Main Result

Let F/F' be an extension of number fields of degree one or two. Our attention will be focused on a particular finite place v_0 of F' which is inert in F. Let w_0 be the place of F which lies above v_0. We consider the F'-group $\mathbf{G}$ which is obtained from the F-group GL_n by restriction of scalars. Let $G = \mathbf{G}(F') = \mathrm{GL}(n,F)$, $G_{\mathbb{A}} = \mathbf{G}(F'_{\mathbb{A}}) = \mathrm{GL}(n,F_{\mathbb{A}})$ and, when v is place of F', let $G_v = \mathbf{G}(F'_v)$. (Hereafter, for any F'-group, we use a similar pattern of notations.) Fix an automorphism ι of $\mathbf{G}$ of order two which is defined over F' and let $\mathbf{H}$ be the F'-subgroup of $\mathbf{G}$ consisting of the fixed points of ι. Let $\mathbf{Z}$ be the center of $\mathbf{G}$ and let $\mathbf{Z_H} = \mathbf{Z} \cap \mathbf{H}$.

Now fix a character $\omega = \otimes_v \omega_v$ of $Z_{\mathbb{A}}/Z$. We will consider irreducible, automorphic, cuspidal representations π of $G_{\mathbb{A}}$ with central character ω. As in the introduction, if χ is an automorphic character of $H_{\mathbb{A}}$ such that $\chi\omega^{-1}$ is trivial on $Z_{H,\mathbb{A}}$, we say that π is (H, χ)-*distinguished* if there exists φ in the space of π such that

$$P_\chi(\varphi) = \int_{Z_{H,\mathbb{A}} H \backslash H_\mathbb{A}} \varphi(h)\chi(h)^{-1}\, dh \neq 0.$$

We will prove:

Theorem 1 **(The Globalization Theorem)** *If τ is an (H_{v_0}, χ_{v_0})-distinguished, irreducible, supercuspidal representation of $G_{v_0} = \mathrm{GL}(n, F_{w_0})$ then there exists an (H, χ)-distinguished, irreducible, automorphic, cuspidal representation $\pi = \otimes_v \pi_v$ of $G_\mathbb{A} = \mathrm{GL}(n, F_\mathbb{A})$ such that $\pi_{v_0} \simeq \tau$.*

The proof will involve an analogue of Selberg's trace formula for the symmetric space $\mathbf{H} \backslash \mathbf{G}$. The traces which occur in Selberg's formula are defined by averaging a kernel function $k(x, y)$ over the diagonal (where $x = y$), whereas the basic objects in our formula are the averages of the values $k(x, y)$ with $x \in H_\mathbb{A}$ and $y = 1$. Our formula may be regarded as a very simple example of the "relative trace formulas" pioneered by Jacquet. The strategy of our proof is to give a relative trace analogue of an argument (on pp. 60–61 of [6]) used to demonstrate how to embed discrete series representations of $\mathrm{GL}(n)$ over nonarchimedean fields as local components of automorphic cuspidal representations of $\mathrm{GL}(n)$. The argument in [6] draws on [1], [2] and [3].

3 The Proof

Fix an (H_{v_0}, χ_{v_0})-distinguished, irreducible, supercuspidal representation τ of $G_{v_0} = \mathrm{GL}(n, F_{w_0})$ and a character $\omega = \otimes_v \omega_v$ of $Z_\mathbb{A}/Z_{H,\mathbb{A}} Z$ such that ω_{v_0} is the central character of τ. Given a test function $f = \otimes_v f_v \in C_c^\infty(G_\mathbb{A})$, we let

$$f'(g) = \int_{Z_\mathbb{A}} f(zg)\omega(z)\, dz,$$

for all $g \in G_\mathbb{A}$. The analogous local integrals define functions f'_v such that $f' = \otimes_v f'_v$. There is an associated automorphic kernel

$$K(x, y) = \sum_{\gamma \in Z \backslash G} f'(x^{-1}\gamma y),$$

where $x, y \in Z_\mathbb{A} G \backslash G_\mathbb{A}$. Let $L^2(G, \omega)$ be the space of L^2-classes of functions ϕ on $G \backslash G_\mathbb{A}$ which transform according to $\phi(zg) = \omega(z)\phi(g)$, where $z \in Z_\mathbb{A}$, $g \in G_\mathbb{A}$ and the L^2-inner product is given by:

$$(\phi_1, \phi_2) = \int_{Z_\mathbb{A} G \backslash G_\mathbb{A}} \phi_1(x)\overline{\phi_2(x)}\, dx.$$

Then f defines an operator $R(f)$ on $L^2(G, \omega)$ with kernel K:

$$R(f)\phi(x) = \int_{Z_\mathbb{A} G \backslash G_\mathbb{A}} K(x, y)\phi(y)\, dy.$$

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Equivalently, if $K_x(y) = K(x, y)$, then $R(f)\phi(x) = (\phi, \overline{K}_x)$.

If **P** is a parabolic subgroup of **G** then we let $\mathbf{N_P}$ denote the unipotent radical of **P**. A continuous function $\phi \in L^2(G, \omega)$ is cuspidal if, for every proper parabolic subgroup **P** of **G**, the integral $\int_{N_P \backslash N_{P\mathbb{A}}} \phi(nx)\, dn = 0$, for almost all $x \in G_{\mathbb{A}}$. The L^2-completion of the space of such functions is denoted $L^2_{\mathrm{cusp}}(G, \omega)$. The kernel K projects to a kernel K_{cusp} on $L^2_{\mathrm{cusp}}(G, \omega)$. The cuspidal kernel may be expressed as:

$$K_{\mathrm{cusp}}(x, y) = \sum_{\pi} K_{\pi}(x, y),$$

where we are summing over the irreducible, automorphic, cuspidal representations π of $G_{\mathbb{A}}$ with central character ω and K_π is given by

$$K_\pi(x, y) = \sum_{\phi \in \mathcal{B}_\pi} R(f)\phi(x)\overline{\phi(y)}$$

with $\mathcal{B}_\pi$ being an orthonormal basis of the space of π.

We will always assume that f'_{v_0} is a matrix coefficient of $\tilde{\tau}$, the contragredient of τ. Thus, if **P** is a proper parabolic subgroup of **G** then

$$\int_{N_{P,v_0}} f'_{v_0}(a_{v_0} n_{v_0} b_{v_0})\, dn_{v_0} = 0,$$

for all $a_{v_0}, b_{v_0} \in G_{v_0}$. Consequently, K_x is cuspidal for all x since

$$\int_{N_P \backslash N_{P,\mathbb{A}}} K_x(ny)\, dn = \sum_{\gamma \in G/ZN_P} \prod_{v} \int_{N_{P,v}} f'_v(x_v^{-1} \gamma n_v y_v)\, dn_v = 0.$$

Hence, $K = K_{\mathrm{cusp}}$.

We now define a distribution Λ_χ on $G_{\mathbb{A}}$ by:

$$\Lambda_\chi(f) = \int_{Z_{H,\mathbb{A}} H \backslash H_{\mathbb{A}}} K(h, 1)\chi(h)^{-1}\, dh.$$

This is the "relative trace" distribution referred to above.

Since $K = K_{\mathrm{cusp}}$, the spectral decomposition of $\Lambda_\chi(f)$ only has a cuspidal contribution. In other words, we have a decomposition:

$$(*) \qquad \Lambda_\chi(f) = \sum_{\pi} \sum_{\phi \in \mathcal{B}_\pi} P_\chi\big(R(f)\phi\big)\, \overline{\phi(1)},$$

where the latter sum is over the irreducible, automorphic cuspidal representations π of $G_{\mathbb{A}}$ with central character ω. In fact, due to the appearance of the factor $P_\chi\big(R(f)\phi\big)$, it is evident that only those π which are (H, χ)-distinguished can make a nonzero contribution to the sum. According to the Schur orthogonality relations, $\pi_{v_0}(f_{v_0}) = 0$ unless $\pi_{v_0} \simeq \tau$. Therefore the outer sum in $(*)$ may be regarded as a

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sum over the irreducible, automorphic, cuspidal representations π of $G_{\mathbb{A}}$ which are (H,χ)-distinguished, have central character ω and have τ as their local component at v_0. Thus if $\Lambda_\chi(f) \neq 0$, then there must exist at least one representation π with the properties just described. Therefore it suffices to show that f can be chosen so that $\Lambda_\chi(f) \neq 0$.

To obtain f so that $\Lambda_\chi(f) \neq 0$, we now develop the "geometric" side of our relative trace formula. We have:

$$\begin{aligned}\Lambda_\chi(f) &= \int_{Z_{H,\mathbb{A}}H\backslash H_{\mathbb{A}}} \sum_{\gamma\in Z\backslash G} f'(h^{-1}\gamma)\chi(h)^{-1}\,dh\\ &= \int_{H_{\mathbb{A}}/HZ_{H,\mathbb{A}}} \sum_{\gamma\in ZH\backslash G}\sum_{\beta\in Z_H\backslash H} f'(h\beta\gamma)\chi(h)\,dh\\ &= \sum_{\gamma\in ZH\backslash G}\prod_v \int_{H_v/Z_{H,v}} f'_v(h_v\gamma)\chi_v(h_v)\,dh_v.\end{aligned}$$

Letting

$$\Phi_v(f_v,\gamma) = \int_{H_v/Z_{H,v}} f'_v(h_v\gamma)\chi_v(h_v)\,dh_v,$$

we may summarize the above discussion as:

Theorem 2 **(The Relative Trace Formula for $\Lambda_\chi(f)$)** *Assume $f = \otimes f_v \in C_c^\infty(G_{\mathbb{A}})$ is such that f'_{v_0} is a matrix coefficient of $\tilde{\tau}$. Then*

$$\sum_\pi \sum_{\phi\in\mathcal{B}_\pi} P_\chi\big(R(f)\phi\big)\,\overline{\phi(1)} = \sum_{\gamma\in ZH\backslash G}\prod_v \Phi_v(f_v,\gamma),$$

where π ranges over the irreducible, (H,χ)-distinguished automorphic cuspidal representations of $G_{\mathbb{A}}$ with central character ω such that $\pi_{v_0} \simeq \tau$.

To prove the Globalization Theorem, it suffices merely to show that there exists some f such the right hand side of the above relative trace formula is nonzero. Indeed, if this is the case, then there must exist at least one π which makes a nonzero contribution to the left hand side and such a π must satisfy the requirements of the Globalization Theorem.

Let us refer to the integrals $\Phi_v(f_v,\gamma)$ as "local orbital integrals" and the product $\Phi(f,\gamma) = \prod_v \Phi_v(f_v,\gamma)$ as a "global orbital integral." The first step in our proof is to show that f may be chosen so that $\Phi(f,\gamma)$ is nonzero for some $\gamma \in G$. Once this is done, we show that, by altering one of the archimedean components of f, we can arrange things so that $\Phi(f,\gamma)$ is nonzero for exactly one $\gamma \in ZH \setminus G$.

Step 1 (Choosing f so that $\Phi(f,\gamma) \neq 0$, for some $\gamma \in G$.)

We must choose $\gamma \in G$ and $f = \otimes_v f_v \in C_c^\infty(G_{\mathbb{A}})$ so that $\Phi_v(f_v,\gamma) \neq 0$. The desired function f is constrained so that f'_{v_0} is a matrix coefficient of $\tilde{\tau}$ and, for almost all finite places v, the function f_v must be the characteristic function of the standard maximal compact subgroup K_v of G_v.

Appealing to the corollary to the Generalized Schur Orthogonality Relations in the Appendix below, we see that γ and f_{v_0} may be chosen so that f'_{v_0} is a matrix coefficient of $\tilde{\tau}$ and $\Phi_{v_0}(f_{v_0},\gamma) \neq 0$. In fact, we may take $\gamma = 1$, since left and right translates of matrix coefficients are again matrix coefficients.

Lemma 1 *For almost all finite places v, we have $Z_vK_v \cap H_v = (Z_v \cap H_v)(K_v \cap H_v)$.*

Proof Suppose $z \in Z_v$, $k \in K_v$ and $zk \in H_v$. Then $\iota(zk) = zk$ implies $z\iota(z)^{-1} = \iota(k)k^{-1} \in Z_v \cap K_v$. If one considers the various possibilities for $\iota|Z_v$, it is easy to check that there exists $w \in Z_v \cap K_v$ such that $z\iota(z)^{-1} = w\iota(w)^{-1}$. Since $zk = (zw^{-1})(wk)$, with $zw^{-1} \in Z_v \cap H_v$ and $wk \in K_v \cap H_v$, our claim follows. ∎

Now suppose v is a finite place other than v_0 which satisfies the condition of Lemma 1. Assume also that ω_v is unramified and χ_v is trivial on $K_v \cap H_v$. For such a place, we take f_v to be the characteristic function of K_v. We choose Haar measures on Z_v and $Z_{H,v} \setminus H_v$ normalized so that $Z_v \cap K_v$ and $Z_{H,v} \setminus (H_v \cap Z_vK_v)$ have measure one. Then, using the assumption that ω_v is unramified, we see that f'_v vanishes outside Z_vK_v and $f'_v(zk) = \omega_v(z)^{-1}$, whenever $z \in Z_v$ and $k \in K_v$. Consider now the orbital integral $\Phi_v(f_v, g)$. Clearly, this vanishes outside $Z_vH_vK_v$ and satisfies $\Phi_v(f_v, zhk) = \omega_v(z)^{-1}\chi_v(h)^{-1}\Phi_v(f_v, k)$, when $z \in Z_v$, $h \in H_v$ and $k \in K_v$. In fact, $\Phi_v(f_v, k) = 1$, under assumptions. (Indeed, $\Phi_v(f_v, k_v)$ is an integral whose integrand $f'_v(hk)\chi_v(h)$ vanishes unless $h \in Z_vK_v \cap H_v$. It follows from Lemma 1 and the assumption that $\chi_v|(K_v \cap H_v) = 1$ that the integrand is just the characteristic function of $Z_vK_v \cap H_v$.)

At this point, we have chosen f_v such that $\Phi_v(f_v, 1) \neq 0$ for almost all places v of F'. Consider now a place v for which f_v has not yet been chosen. It is elementary to describe the space of functions $\varphi_v(g) = \Phi_v(f_v, g)$ on G_v, as f_v varies over $C_c^\infty(G_v)$. Indeed, the functions φ_v are precisely the smooth functions on G_v whose support has compact image in $Z_vH_v \setminus G_v$ and which transform according to $\varphi_v(zhg) = \omega_v(z)^{-1}\chi_v(h)^{-1}\varphi_v(g)$, for all $z \in Z_v$, $h \in H_v$ and $g \in G_v$. Thus, we can choose f_v so that $\Phi_v(f_v, 1) \neq 0$.

Step 2 (Choosing f so that $\Phi(f,\gamma) \neq 0$, for exactly one $\gamma \in ZH \setminus G$.)

Fix an infinite place w_1 of F lying above a place v_1 of F'. Consider the set

$$\mathcal{S} = \{\iota(\gamma)^{-1}\gamma : \gamma \in G \text{ and } \Phi(f,\gamma) \neq 0\}.$$

We will show that this set has discrete image in $\mathrm{PGL}(n, F_{w_1})$ and from this deduce that we may shrink the support of f_{v_1} so that $\Phi(f,\gamma) \neq 0$ for exactly one $\gamma \in ZH \setminus G$. This is nearly identical to the strategy employed in [6] (pp. 60–61), however, there are some additional obstacles. These extra technicalities obscure the fundamental simplicity of the argument and we therefore advise the reader to consult [6] before reading our argument.

When trying to establish the discreteness of a subset of $\mathrm{PGL}(n, F_{w_1})$, it is perhaps advisable to keep in mind the following false argument that $\mathrm{PGL}(n,\mathbb{Z})$ is discrete in $\mathrm{PGL}(n,\mathbb{R})$: clearly, $\mathrm{GL}(n,\mathbb{Z})$ is discrete in $\mathrm{GL}(n,\mathbb{R})$ and thus, since modding out by the center preserves discreteness, $\mathrm{PGL}(n,\mathbb{Z})$ must be discrete in $\mathrm{PGL}(n,\mathbb{R})$. Unfortunately, the natural map $\mathrm{GL}(n,\mathbb{R}) \to \mathrm{PGL}(n,\mathbb{R})$ does not preserve discreteness. In fact, the set $\mathcal{S}$ of matrices in $\mathrm{GL}(n,\mathbb{R})$ with integer entries does not have discrete image in $\mathrm{PGL}(n,\mathbb{R})$ since $\mathbb{Q}^\times\mathcal{S} = \mathrm{GL}(n,\mathbb{Q})$ is dense in $\mathrm{GL}(n,\mathbb{R})$. The problem in the above argument disappears for SL_n, since SL_n has finite center. In other words, we obtain a valid proof that $\mathrm{PSL}(n,\mathbb{Z})$ is discrete in $\mathrm{PSL}(n,\mathbb{R})$. Then, using the fact that $\mathrm{PGL}(n,\mathbb{R})/\mathrm{PSL}(n,\mathbb{R})$ is finite, one can deduce that $\mathrm{PGL}(n,\mathbb{Z})$ is indeed discrete in $\mathrm{PGL}(n,\mathbb{R})$.

In light of the previous paragraph, we will work with the sets

$$\mathcal{S}_r = \{\beta \in \mathcal{S} : \det \beta = r\},$$

with r lying in the ring of integers $\mathfrak{O}_F$ of F. Suppose R is any (finite) set of representatives for $F^\times/(F^\times)^n$ such that $R \subset \mathfrak{O}_F$ and let R' be the image of R under $\kappa(\gamma) = \iota(\gamma)^{-1}\gamma$. Then the image of $\mathcal{S}$ in $\mathrm{PGL}(n,F_{w_1})$ is the same as the image of $\bigcup_{r\in R'} \mathcal{S}_r$. Since this is a finite union, it suffices to show that each $\mathcal{S}_r$ has discrete image in $\mathrm{PGL}(n,F_{w_1})$ or, equivalently, we must show that $\mathcal{S}_r$ is discrete in

$$\mathcal{G}_r = \{g \in \mathrm{GL}(n,F_{w_1}) : \det g = r\}.$$

To do this, it suffices to show for each r that:

(i) For almost all finite places w of F, the matrix entries of each $\beta \in \mathcal{S}_r$ lie in the ring of integers $\mathfrak{O}_w$ of F_w.

(ii) For all finite places w of F the matrix entries of each $\beta \in \mathcal{S}_r$ lie in a fixed compact subset of F_w.

Indeed, once this is done, we will have shown that the matrix entries of each $\beta \in \mathcal{S}_r$ have the form $\frac{a}{b}$, with $a, b \in \mathfrak{O}_F$ and b in some bounded set in F_{w_1}.

To prove (i), we first note that the function on G_v defined by $\phi_v(g) = \Phi_v(f_v, g)$ has support $Z_vH_vK_v$, for almost all finite places v of F'. Moreover, for almost all v, we have $\iota(K_v) = K_v$ and thus $\kappa(K_v) \subset K_v$, where κ is defined on G_v by the formula $\kappa(g) = \iota(g)^{-1}g$. Therefore, for almost all finite v, the image of the support of ϕ_v under κ is contained in Z_vK_v. If w is a place of F lying above such a place v of F', then the matrix entries of each $\beta \in Z_vK_v \cap \mathcal{G}_r$ lie in $\mathfrak{O}_w$. Condition (i) follows. On the other hand, if w is any finite place of F and w lies above the place v of F', then we have observed above that the support of ϕ_v has compact image in $Z_vH_v \setminus G_v$. Therefore, $\kappa\big(\mathrm{support}(\phi_v)\big) \cap \mathcal{G}_r$ is compact. This proves (ii).

We have now shown that the set $\mathcal{S}$ has discrete image in $\mathrm{PGL}(n,F_{w_1})$. Our function f_{v_1} may be taken to be a product of functions f_w on the groups $\mathrm{GL}(n,F_w)$ as w ranges over the places of F lying over v. We can shrink the support of f_{w_1} so that $\Phi_{v_1}(f_{v_1},1) \neq 0$, but $\Phi_{v_1}(f_{v_1},\gamma) = 0$, for all other $\gamma \in ZH \setminus G$. The simple relative trace formula reduces to:

$$\sum_{\pi} \sum_{\phi\in\mathcal{B}_\pi} P_\chi\big(R(f)\phi\big)\,\overline{\phi(1)} = \Phi(f,1),$$

and, since the right hand side is nonzero, the proof of the Globalization Theorem is complete. ■

4 Applications

All of the applications considered in this section involve the special case in which the character χ is trivial. In this situation, instead of saying a representation is "(H,χ)-distinguished," we simply say it is "H-distinguished" or, when the context is clear, "distinguished."

In the introduction, we have mentioned a simple application of the Globalization Theorem to representations associated to $\mathrm{Sp}(2n)\setminus\mathrm{GL}(2n)$. In that case, there do not exist any distinguished automorphic, cuspidal representations or, in the language of [5], the symmetric space $\mathrm{Sp}(2n)\setminus\mathrm{GL}(2n)$ is not "cuspidal." Another non-cuspidal symmetric space is given as follows. Suppose n_1 and n_2 are distinct positive integers and $n = n_1 + n_2$. Let ι be the automorphism of $\mathbf{G} = \mathrm{GL}_n$ defined by conjugating by the block matrix $\begin{pmatrix} 1_{n_1} & 0 \\ 0 & -1_{n_2} \end{pmatrix}$, where 1_k denotes the k-by-k identity matrix. Let $\mathbf{H}$ be the group of fixed points of ι. It is shown in [5] that there do not exist any H-distinguished automorphic, cuspidal representations π of $G_{\mathbb{A}}$. Applying the Globalization Theorem, we obtain:

Proposition 1 *Let $n = n_1 + n_2$, where n_1 and n_2 are distinct positive integers, and assume F is a local, nonarchimedean local field of characteristic zero. If π is a supercuspidal representation of $\mathrm{GL}(n,F)$ then there do not exist any nonzero linear forms on the space of π which are invariant under the group of block matrices $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, with $a \in \mathrm{GL}(n_1,F)$ and $b \in \mathrm{GL}(n_2,F)$.*

In the case in which $n_1 = n_2$, it is shown in [16] that the space of invariant linear forms has dimension at most one.

In order to describe another application, we recall a result which first appears in [13], but is also mentioned in [17] and [4]. We include an elementary proof conveyed to us by Hervé Jacquet. Assume F/F' is a quadratic extension of number fields whose nontrivial Galois automorphism is $x \mapsto \bar{x}$. Applying the nontrivial Galois automorphism to the entries of a matrix in $\mathbf{G}$, gives an automorphism of $\mathbf{G}$ which we also denote by $g \mapsto \bar{g}$. Fix a matrix $\eta \in G$ which is hermitian in the sense that ${}^t\bar{\eta} = \eta$ and take ι to be the automorphism of $\mathbf{G}$ given by $\iota(g) = \eta^t\bar{g}^{-1}\eta^{-1}$. Thus H is a unitary group in G. If π is an irreducible, automorphic, cuspidal representation of $G_{\mathbb{A}} = \mathrm{GL}(n,F_{\mathbb{A}})$, then we say that π is *Galois invariant* if π is equivalent to the representation $g \mapsto \pi(\bar{g})$ which acts on the space of π.

Proposition 2 *If π is an irreducible, automorphic, cuspidal representation of $G_{\mathbb{A}}$ which is H-distinguished then π must be Galois invariant.*

Proof At almost all places v of F' which are inert in F, the local representation is unramified and Galois invariant in the sense that π_v is equivalent to the representation $g \mapsto \pi_v(\bar{g})$ on the space of π_v. (Here, we have not used the fact that π is H-distinguished.) Suppose v is a place of F' which splits into two places w_1 and w_2 in F. Let $F_v = F'_v \oplus F'_v$, where $\mathrm{Gal}(F/F')$ acts on the direct sum by permuting coordinates. Fix an embedding of F in F'_v and use it to embed F in F_v via $x \mapsto (x,\bar{x})$.

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Then $G_v = \mathrm{GL}(n, F_v) = \mathrm{GL}(n, F'_v) \times \mathrm{GL}(n, F'_v)$. The unitary group H_v then consists of those $g = (g_1, g_2) \in G_v$ such that $g_2 = \bar{\eta}^t g_1^{-1} \bar{\eta}^{-1}$. The local component π_v is a product $\pi_1 \times \pi_2$ of representations (π_1, V_1) and (π_2, V_2) on $\mathrm{GL}(n, F'_v)$. Since π is H-distinguished, we have a nonzero linear form λ on the space of $V_1 \otimes V_2$ such that $\lambda\big(\pi_1(g)\xi_1 \otimes \pi_2(\bar{\eta}^t g^{-1} \bar{\eta}^{-1})\xi_2\big) = \lambda(\xi_1 \otimes \xi_2)$, for all $\xi_1 \in V_1$ and $\xi_2 \in V_2$. Therefore, the representation $g \mapsto \pi_2(\bar{\eta}^t g^{-1} \bar{\eta}^{-1})$ on V_2 must be equivalent to the contragredient of π_1. On the other hand, it follows from Theorem A in [7] that this representation is equivalent to the contragredient of π_2. Thus, $\pi_1 \simeq \pi_2$ and again π_v is Galois invariant. Since π is equivalent to $g \mapsto \pi(\bar{g})$ at almost all places, the Strong Multiplicity One Theorem for $\mathrm{GL}(n)$ now implies that π is Galois invariant. ∎

Corollary *Assume F/F' is a quadratic extension of nonarchimedean local fields of characteristic zero, and $\eta \in \mathrm{GL}(n, F)$ is hermitian with respect to F/F'. If π is an irreducible, supercuspidal representation of $\mathrm{GL}(n, F)$ which is distinguished with respect to the unitary group consisting of $g \in \mathrm{GL}(n, F)$ such that $g = \eta^t \bar{g}^{-1} \eta^{-1}$ then π must be Galois invariant.*

Proof Fix a hermitian matrix η in $\mathrm{GL}(n, F)$ and let $U(\eta)$ be the associated unitary subgroup of $\mathrm{GL}(n, F)$. There exists a quadratic extension k/k' of number fields such that $k_{w_0}/k'_{v_0} = F/F'$ for some place v_0 of k' which is inert in k and lifts to the place w_0 of k. If η happens to lie in $\mathrm{GL}(n, k)$ then our assertion follows immediately from the Globalization Theorem. Otherwise, we may choose $h \in \mathrm{GL}(n, F)$ and a hermitian matrix $\eta' \in \mathrm{GL}(n, k)$ such that $\eta = h\eta'{}^t\bar{h}$. Indeed, the orbit of η under the action of $\mathrm{GL}(n, F)$ by $g \cdot \eta = g\eta^t\bar{g}$ is determined by the class of $\det \eta$ modulo $N_{F/F'}(F^\times)$. Since $U(\eta) = hU(\eta')h^{-1}$ and since the corollary holds for the subgroup $U(\eta')$, it must also hold for the conjugate subgroup $U(\eta)$. ∎

The statement of the previous corollary may be framed more generally as follows. Let $\iota(g) = \eta^t \bar{g}^{-1} \eta^{-1}$ be the involution of $G = \mathrm{GL}(n, F)$ whose fixed point set is the unitary group $H = U(\eta)$. Then the corollary is equivalent to the statement that if π is H-distinguished then $\tilde{\pi} \circ \iota \simeq \pi$, since according to Gelfand/Kazhdan's Theorem A in [7], the contragredient $\tilde{\pi}$ of π is equivalent to the representation $g \mapsto \pi({}^t g^{-1})$.

One could consider the analogous statement when ι is an involution of $G = \mathrm{GL}(n, F)$ whose fixed point group is an orthogonal group. Though it is again true that distinguishedness implies $\tilde{\pi} \circ \iota \simeq \pi$, this statement is vacuous since the condition $\tilde{\pi} \circ \iota \simeq \pi$ reduces to $\pi \simeq \pi$ using the result of Gelfand-Kazhdan. For more details and references to the literature in this case, we refer the reader to [10].

One could also consider the case in which $G = \mathrm{GL}(n, F)$ and $H = \mathrm{GL}(n, F')$. In this case, the analogue of Proposition 2 is not known (to our knowledge), however, the analogue of the corollary of Proposition 2 may be obtained by local methods. Indeed, this is precisely Proposition 12 of [4]. The proof in [4] is a variant of the proof of Theorem 2.1 of [9]. Though the latter result is stated in the context of GL(2), the ideas in the proof apply generally. Both strategies of proof use the "relative character" distribution Θ_π attached to the distinguished supercuspidal representation π. Among the most basic properties of the relative character, are the facts that distinct representations always have distinct relative characters (Proposition 3 of [19]) and that relative character distributions are locally integrable [8]. Whereas the proof in [4] is self-contained, the alternate proof is perhaps conceptually simpler, at the expense of invoking the local integrability property.

5 Appendix: Generalized Schur Orthogonality

Fix a supercuspidal representation (τ, V) of a totally disconnected group G with center Z and let $(\tilde{\tau}, \tilde{V})$ be the contragredient. The Schur orthogonality relations state that if $\xi, \lambda \in V$ and $\tilde{\xi}, \tilde{\lambda} \in \tilde{V}$, then:

$$\int_{G/Z} \langle \xi, \tilde{\tau}(g)\tilde{\xi}\rangle \langle \tau(g)\lambda, \tilde{\lambda}\rangle \, dg = d(\tau)\langle \xi, \tilde{\lambda}\rangle\langle \lambda, \tilde{\xi}\rangle,$$

where dg is a suitably normalized Haar measure on G/Z and $d(\tau)$ is the formal degree of τ. We now prove the following generalization:

***Lemma 2* (Generalized Schur Orthogonality Relations)** *Suppose $\xi \in V$, $\tilde{\xi} \in \tilde{V}$, $\lambda \in \mathrm{Hom}_{\mathbb{C}}(\tilde{V}, \mathbb{C})$ and $\tilde{\lambda} \in \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$. Let f be the matrix coefficient of $\tilde{\tau}$ defined by $f(g) = \langle \xi, \tilde{\tau}(g)\tilde{\xi}\rangle$. Let $\tau(f)\lambda$ be the element of V defined by $\langle \tau(f)\lambda, \tilde{\mu}\rangle = \langle \lambda, \tilde{\tau}(\check{f})\tilde{\mu}\rangle$, for all $\tilde{\mu} \in \tilde{V}$, where $\check{f}(g) = f(g^{-1})$. Then $\langle \tau(f)\lambda, \tilde{\lambda}\rangle = d(\tau)\langle \xi, \tilde{\lambda}\rangle\langle \lambda, \tilde{\xi}\rangle$.*

Proof Let $K_1 \supset K_2 \supset \cdots$ be a basis of neighborhoods of the identity in G consisting of open, compact subgroups. Choose i large enough so that ξ and $\tilde{\xi}$ are K_i-fixed and thus f is bi-K_i-invariant. Choose $\lambda_i \in V$ and $\tilde{\lambda}_i \in \tilde{V}$ such that $\langle \lambda, \tilde{\mu}\rangle = \langle \lambda_i, \tilde{\mu}\rangle$, for all K_i-fixed vectors $\tilde{\mu} \in \tilde{V}$, and $\langle \mu, \tilde{\lambda}\rangle = \langle \mu, \tilde{\lambda}_i\rangle$, for all K_i-fixed vectors $\mu \in V$. Then we have:

$$\langle \tau(f)\lambda, \tilde{\lambda}\rangle = \langle \tau(f)\lambda, \tilde{\lambda}_i\rangle = \langle \lambda, \tilde{\tau}(\check{f})\tilde{\lambda}_i\rangle = \langle \lambda_j, \tilde{\tau}(\check{f})\tilde{\lambda}_i\rangle.$$

We apply the Schur orthogonality relations to obtain:

$$\langle \tau(f)\lambda, \tilde{\lambda}\rangle = d(\tau)\langle \xi, \tilde{\lambda}_i\rangle\langle \lambda_j, \tilde{\xi}\rangle = d(\tau)\langle \xi, \tilde{\lambda}\rangle\langle \lambda, \tilde{\xi}\rangle.$$

Hence, our claim has been proven. ∎

Assume now that H is a closed subgroup of G and χ is a character of H such that $\tau(z) = \chi(z)$, for all $z \in Z \cap H$. We apply the Generalized Schur Orthogonality Relations in the case in which $\lambda \in V \subset \mathrm{Hom}_{\mathbb{C}}(\tilde{V}, \mathbb{C})$ and $\tilde{\lambda}$ satisfies $\langle \tau(h)\mu, \tilde{\lambda}\rangle = \chi(h)\langle \mu, \tilde{\lambda}\rangle$, for all $h \in H$ and $\mu \in V$ or, in other words, $\tilde{\lambda} \in \mathrm{Hom}_H(\tau, \chi)$.

Corollary *Suppose $\xi \in V$, $\tilde{\xi} \in \tilde{V}$ are nonzero and f' is the matrix coefficient of $\tilde{\tau}$ defined by $f'(g) = \langle \xi, \tilde{\tau}(g)\tilde{\xi}\rangle$. Suppose $\tilde{\lambda} \in \mathrm{Hom}_H(\tau, \chi)$ is such that $\langle \xi, \tilde{\lambda}\rangle \neq 0$. Then $g \mapsto \int_{(Z\cap H)\backslash H} f'(hg)\chi(h)\, dh$ is a nonzero smooth function on G whose support has compact image in $ZH \setminus G$.*

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Proof Choose $\lambda \in V$ so that $\langle \lambda, \tilde{\xi} \rangle \neq 0$. Then we have:

$$d(\tau)\langle \xi, \tilde{\lambda} \rangle \langle \lambda, \tilde{\xi} \rangle = \langle \tau(f)\lambda, \tilde{\lambda} \rangle = \int_{Z\backslash G} \langle \xi, \tilde{\tau}(g)\tilde{\xi} \rangle \langle \tau(g)\lambda, \tilde{\lambda} \rangle \, dg$$

$$= \int_{ZH\backslash G} \Big(\int_{(Z\cap H)\backslash H} f'(hg)\chi(h)\, dh \Big) \langle \tau(g)\lambda, \tilde{\lambda} \rangle \, dg.$$

Since the left hand side is nonzero, our claim follows. ■

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