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STRONG MORITA EQUIVALENCE FOR THE DENJOY C^*-ALGEBRAS

BY

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ABSTRACT. The C^*-algebras associated with irrational rotations of the circle were classified up to strong Morita equivalence by M. A. Rieffel. As a corollary, he gave a complete classification of the C^*-algebras arising from irrational or Kronecker flows on the 2-torus up to $*$-isomorphism. Here, we extend the result to the so-called Denjoy homeomorphisms. Specifically, we give a necessary and sufficient condition for the strong Morita equivalence of two C^*-algebras arising from homeomorphisms of the circle without periodic points. As a corollary, we show that two C^*-algebras arising from flows on the 2-torus obtained from such homeomorphisms by the "flow under constant function" construction are $*$-isomorphic if and only if the flows themselves are topologically conjugate.

1. **Introduction.**

Let φ be a homeomorphism of the circle, S^1, without periodic points. We denote by $(\mathbf{Z}, S^1, \varphi)$ the free action of the group of integers which it generates, and by A_φ the associated crossed-product or transformation group C^*-algebra. Such transformation groups are classified (topologically) by a pair of invariants. The first is called the rotation number and denoted by $\rho(\varphi)$. It is an irrational number between 0 and 1. The second invariant, denoted $Q(\varphi)$, is a countable subset of the circle which is invariant under rotation by an angle of $2\pi\rho(\varphi)$. In fact, the set $Q(\varphi)$ is only defined up to a rigid rotation of the circle. For $\varphi = R_\alpha$, rotation through an angle of $2\pi\alpha$ (with α irrational), $\rho(\varphi) = \alpha$ and $Q(\varphi)$ is the empty set. The reader is referred to [6] or to Markley [3] for the appropriate definitions. Our set $Q(\varphi)$ is the complement of Markley's $T(\varphi)$.

The C^*-algebras A_φ were classified completely up to $*$-isomorphism in [6]. Here, in Section 2, we give a complete classification of them up to strong Morita equivalence (see Rieffel [8]). In Section 3, we use this result to give a complete classification of the C^*-algebras associated with Denjoy flows on the 2-torus.

These results extend those of Rieffel [7] who considered irrational rotations. Rieffel's results in [7] depend heavily on the work of M. Pimsner and D. Voiculescu ([4] and [5]) which completed the classification of the irrational

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rotation C^*-algebras up to $*$-isomorphism. Similarly, the results of [6] also rely on [5].

The notation used here is the same as in [6]. The reader is referred there for a description of the dynamics and the basic facts regarding the C^*-algebras which we are considering here. We let $\mathscr{K}$ denote the compact operators on a separable, infinite dimensional Hilbert space, S^1 denote the circle and [ ] and { } denote integer and fractional part, respectively. We use π to denote the usual covering map of $\mathbf{R}$ onto S^1 and T_r to denote translation by r on $\mathbf{R}$.

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2. **The main theorem.**

In this section, we will prove the following theorem.

THEOREM 1. *Let φ and Ψ be homeomorphisms of the circle without periodic points. The C^*-algebras A_φ and A_Ψ are strongly Morita equivalent if and only if there is* $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ *in* $GL(2, \mathbf{Z})$ *such that*

$$\text{(i)} \qquad \rho(\Psi) = \frac{a\rho(\varphi) + b}{c\rho(\varphi) + d}, \text{ and}$$

$$\text{(ii)} \qquad \pi^{-1}(Q(\Psi)) \approx \frac{1}{c\rho(\varphi) + d}\pi^{-1}(Q(\varphi)),$$

where $\approx$ denotes equality up to translation.

We suppose that φ and Ψ are homeomorphisms of the circle without periodic points. We further suppose that the C^*-algebras A_φ and A_Ψ are strongly Morita equivalent. Since we are dealing with separable C^*-algebras, this is equivalent to the existence of a $*$-isomorphism $\sigma: A_\varphi \otimes \mathscr{K} \to A_\Psi \otimes \mathscr{K}$ ([1]).

For any C^*-algebra, $\mathscr{A}$, there is an order-preserving bijection between the ideals of $\mathscr{A}$ and those of $\mathscr{A} \otimes \mathscr{K}$ ([8]). There is also a natural isomorphism between the groups $K_*(\mathscr{A})$ and $K_*(\mathscr{A} \otimes \mathscr{K})$ and in the case of K_0-groups it is actually an order isomorphism. We will suppress this in our notation. Therefore, $\mathscr{I}_\varphi \otimes \mathscr{K}$ and $\mathscr{I}_\Psi \otimes \mathscr{K}$ are the unique maximal ideals in $A_\varphi \otimes \mathscr{K}$ and $A_\Psi \otimes \mathscr{K}$, respectively, so that σ must carry the former to the latter isomorphically. Recall from [6] that $\mathscr{I}_\varphi$ decomposes into the direct sum of ideals $\mathscr{I}_\varphi^k$, $k = 1, 2, \ldots, n(\varphi)$, corresponding to the connected components of Prim($\mathscr{I}_\varphi$). The ideals $\mathscr{I}_\varphi^k$ are uniquely determined up to a choice of indexing and each is $*$-isomorphic to $C_0(\mathbf{R}) \otimes \mathscr{K}$. As Prim($\mathscr{I}_\varphi$) and Prim($\mathscr{I}_\varphi \otimes \mathscr{K}$) are homeomorphic, we conclude that (after a suitable choice of indexing) σ carries $\mathscr{I}_\varphi^k \otimes \mathscr{K}$ to $\mathscr{I}_\Psi^k \otimes \mathscr{K}$ for each $k = 1, 2, \ldots, n(\varphi) = n(\Psi)$.

We will suppress the natural $*$-isomorphism between $A_\varphi \otimes \mathscr{K}/\mathscr{I}_\varphi \otimes \mathscr{K}$ and $(A_\varphi/\mathscr{I}_\varphi) \otimes \mathscr{K} = D_\varphi \otimes \mathscr{K}$. As σ takes $\mathscr{I}_\varphi \otimes \mathscr{K}$ to $\mathscr{I}_\Psi \otimes \mathscr{K}$, we obtain a $*$-isomorphism between $D_\varphi \otimes \mathscr{K}$ and $D_\Psi \otimes \mathscr{K}$. This will also be denoted by σ. We then have the following commutative diagram.

(I)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{I}_\varphi \otimes \mathscr{K} & \longrightarrow & A_\varphi \otimes \mathscr{K} & \longrightarrow & D_\varphi \otimes \mathscr{K} & \longrightarrow & 0 \\
 & & \sigma \downarrow & & \sigma \downarrow & & \sigma \downarrow & & \\
0 & \longrightarrow & \mathscr{I}_\Psi \otimes \mathscr{K} & \longrightarrow & A_\Psi \otimes \mathscr{K} & \longrightarrow & D_\Psi \otimes \mathscr{K} & \longrightarrow & 0
\end{array}$$

From Propositions 2.2 and 2.3 of [7] and the uniqueness of the traces on D_φ and D_Ψ, we conclude that there is a positive real number r such that

$$\hat{Tr}_\Psi \circ \sigma_* = r \cdot \hat{Tr}_\varphi : K_0(D_\varphi) \to \mathbf{R}.$$

In [6], it was shown that $K_0(A_\varphi)$ is a subgroup of $K_0(D_\varphi)$ and, from the commutativity of the right square of diagram (I), it must be carried to $K_0(A_\Psi)$ in $K_0(D_\Psi)$ by σ_*. Again from [6],

$$\hat{Tr}_\varphi(K_0(A_\varphi)) = \mathbf{Z} + \rho(\varphi)\mathbf{Z}.$$

We conclude that

$$r(\mathbf{Z} + \rho(\varphi)\mathbf{Z}) = \mathbf{Z} + \rho(\Psi)\mathbf{Z}.$$

An elementary calculation then shows that

$$r = \frac{1}{c\rho(\varphi) + d} \text{ and } \rho(\Psi) = \frac{a\rho(\varphi) + b}{c\rho(\varphi) + d}$$

for some integers a, b, c and d with $ad - bc = \pm 1$. It remains to verify that condition (ii) is satisfied.

We recall the notation of [6]. Let $x_1, x_2, \ldots, x_{n(\varphi)}$ be generators for the groups $K_1(\mathscr{I}_\varphi^1), K_1(\mathscr{I}_\varphi^2), \ldots, K_1(\mathscr{I}_\varphi^{n(\varphi)})$, respectively, such that $i_*(x_1) = \ldots = i_*(x_{n(\varphi)})$ in $K_1(A_\varphi)$. Lemma 6.7 of [6] then asserts that

$$Q(\varphi) \approx \pi\left(\pm \hat{Tr}_\varphi\left(\bigcup_k \exp^{-1}(x_1 - x_k)\right)\right).$$

From the commutativity of diagram (I) above we obtain the following commutative diagram of K-groups.

(II)
$$\begin{array}{ccccccccc}
\longrightarrow & K_0(A_\varphi) & \longrightarrow & K_0(D_\varphi) & \xrightarrow{\exp} & K_1(\mathscr{I}_\varphi) & \xrightarrow{i_*} & K_1(A_\varphi) \\
 & \downarrow \sigma_* & & \downarrow \sigma_* & & \downarrow \sigma_* & & \downarrow \sigma_* \\
\longrightarrow & K_0(A_\Psi) & \longrightarrow & K_0(D_\Psi) & \xrightarrow{\exp} & K_1(\mathscr{I}_\Psi) & \xrightarrow{i_*} & K_1(A_\Psi)
\end{array}$$

For each k, $\sigma_*(x_k)$ is a generator of $K_1(\mathscr{I}_\Psi^k)$. Moreover, the commutativity of the right square in diagram (II) implies that $i_*(\sigma_*(x_1)) = \ldots = i_*(\sigma_*(x_{n(\varphi)}))$ in $K_1(A_\Psi)$. By Lemma 6.7 of [6], we have

$$Q(\Psi) \approx \pi\left(\pm \hat{Tr}_\Psi\left(\bigcup_k \exp^{-1}(\sigma_*(x_1) - \sigma_*(x_k))\right)\right).$$

Now we apply the commutativity of the centre square of diagram (II), so that

$$\exp^{-1}(\sigma_*(x_1) - \sigma_*(x_k)) = \sigma_*(\exp^{-1}(x_1 - x_k)),$$

for each k. This yields

$$\begin{aligned} Q(\Psi) &\approx \pi\left(\pm \hat{Tr}_\Psi\left(\bigcup_k \sigma_*(X_k)\right)\right) \\ &\approx \pi\left(\pm \hat{Tr}_\Psi \circ \sigma_*\left(\bigcup_k X_k\right)\right) \\ &\approx \pi\left(\pm r\hat{Tr}_\varphi\left(\bigcup_k X_k\right)\right). \end{aligned}$$

Since $\hat{Tr}_\varphi(X_k)$ is invariant under translation by 1 and by $\rho(\varphi)$, we may conclude that

$$\pi^{-1}(Q(\Psi)) \approx \pm r\hat{Tr}_\varphi\left(\bigcup_k X_k\right) \approx \frac{\pm 1}{c\rho(\varphi) + d}\pi^{-1}(Q(\varphi)),$$

as desired. Note that in order to remove the possibility of -1 in the numerator, we can simply replace a, b, c and d by their negatives. This completes the proof of the necessity of the condition.

In the proof of the sufficiency of the condition, the homeomorphisms φ and Ψ will be treated asymmetrically. Beginning with φ, we obtain a lifting $\tilde{\varphi}:\mathbf{R} \to \mathbf{R}$ such that $\tilde{\varphi}(0) \in (0, 1)$. This is equivalent to the property

$$\lim_{n\to\infty} \frac{\tilde{\varphi}^n(x)}{n} = \rho(\varphi), \quad x \in \mathbf{R},$$

without taking fractional part of the left hand side. (See [6].)

We then have an action of $\mathbf{Z}^2$ on $\mathbf{R}$ generated by the (commuting) homeomorphisms $\tilde{\varphi}$ and T_1. For i and j in $\mathbf{Z}$ and x in $\mathbf{R}$, define $(i, j) \cdot x = \tilde{\varphi}^i \circ T_1^j(x)$. We use $\langle (i, j) \rangle$ to denote the subgroup of $\mathbf{Z}^2$ generated by (i, j). Notice that $C^*(\langle (1, 0) \rangle, \mathbf{R}/\langle (0, 1) \rangle)$ is just A_φ.

In the general situation of a locally compact group G acting on a locally compact space X, we say that the action is wandering if, for every compact set $K \subset X$, the set $\{g \in G | g \cdot K \cap K \text{ is non-empty}\}$ is precompact in G. This condition will guarantee that the orbit space X/G is Hausdorff.

THEOREM 2. *Let $\mathbf{Z}^2$ act on a locally compact space X so that the action, restricted to any cyclic subgroup of $\mathbf{Z}^2$, is wandering. Let u and v be in $\mathbf{Z}^2$, be such that $\langle u\rangle + \langle v\rangle = \mathbf{Z}^2$. Let $\alpha = \left[\begin{smallmatrix} i & j \\ k & l\end{smallmatrix}\right]$ be in $GL(2, \mathbf{Z})$ and denote by $A(\alpha, u, v)$ the automorphism of $\mathbf{Z}^2$ defined by sending u to $iu + jv$ and v to $ku + 1v$. Then $C^*(U, X/V)$ and $C^*(A(\alpha, u, v)(U), X/(A(\alpha, u, v)(V)))$ are strongly Morita equivalent.*

PROOF. See the discussion on page 421 of Rieffel [7]. □

We will apply the result to the case of our action of $\mathbf{Z}^2$ on $\mathbf{R}$, with $u = (1, 0)$ and $v = (0, 1)$, and

$$\alpha = \begin{bmatrix} a & b \\ c & d\end{bmatrix},$$

the element of $GL(2, \mathbf{Z})$ given in our hypothesis relating φ and Ψ.

We must still verify that our action satisfies the wandering hypothesis of the theorem stated above and show that the transformation groups $(\mathbf{Z}, S^1, \Psi)$ and $(\langle (a, b)\rangle, \mathbf{R}/\langle (c, d)\rangle)$ are isomorphic. Denote the homeomorphisms $\tilde{\varphi}^a \circ T_1^b$ and $\tilde{\varphi}^c \circ T_1^d$ of $\mathbf{R}$ by ξ and η, respectively.

We begin with a simple lemma.

LEMMA 3. *Let i and j be any integers. Then*

$$\lim_{n\to\infty} \frac{(\tilde{\varphi}^i \circ T_1^j)^n(x)}{n} = i\rho(\varphi) + j,\ x \in \mathbf{R}.$$

PROOF.

$$\lim_{n\to\infty} \frac{(\tilde{\varphi}^i \circ T_1^j)^n(x)}{n} = \lim_{n\to\infty} \frac{\tilde{\varphi}^{in}(x) + jn}{n}$$

$$= i \lim_{n\to\infty} \frac{\tilde{\varphi}^{in}(x)}{in} + j$$

$$= i\rho(\varphi) + j.$$ □

Throughout the rest of the proof, we will assume that $c\rho(\varphi) + d$ is positive. The other case is similar.

It is a straightforward consequence of Lemma 3 that our action of $\mathbf{Z}^2$ on $\mathbf{R}$ satisfies the hypotheses of Theorem 2.

From Lemma 3, we see that we may write $\mathbf{R}$ as a disjoint union

$$\mathbf{R} = \bigcup_{k\in\mathbf{Z}} [\eta^k(0), \eta^{k+1}(0)).$$

Define $g:\mathbf{R} \to \mathbf{R}$ by

$$g(t) = k + \eta(0)^{-1}\eta^{-k}(t),\ t \in [\eta^k(0), \eta^{k+1}(0)).$$

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It is easy to verify that g is a homeomorphism of $\mathbf{R}$ and that $g \circ \eta \circ g^{-1} = T_1$.

Define a homeomorphism Ψ_0 of S^1 by

$$\Psi_0(\pi(x)) = \pi(g \circ \xi \circ g^{-1}(x)), \ x \in \mathbf{R}.$$

Since ξ and η commute, so must $g \circ \xi \circ g^{-1}$ and $g \circ \eta \circ g^{-1} = T_1$, from which we see that Ψ_0 is well defined. We also remark that we have an obvious choice for a lifting $\tilde{\Psi}_0$ of Ψ_0, namely, $g \circ \xi \circ g^{-1}$. It is clear that g implements an isomorphism between the transformation groups $(\langle (a, b) \rangle, \mathbf{R}/\langle (c, d) \rangle)$ and $(\mathbf{Z}, S^1, \Psi_0)$. It now remains for us to show that the latter is isomorphic to $(\mathbf{Z}, S^1, \Psi)$. To do this, we calculate $\rho(\Psi_0)$ and $Q(\Psi_0)$.

LEMMA 4.

$$\rho(\Psi_0) = \frac{a\rho(\varphi) + b}{c\rho(\varphi) + d}.$$

PROOF. For each positive integer k, there is an integer m_k such that

$$\eta^{m_k}(0) \leqq \xi^k(0) \leqq \eta^{m_k+1}(0).$$

Applying g, we obtain

$$T_1^{m_k}(0) \leqq \tilde{\Psi}_0^k(0) \leqq T_1^{m_k+1}(0), \text{ i.e. } m_k \leqq \tilde{\Psi}_0^k(0) \leqq m_k + 1,$$

and so,

$$\frac{m_k}{k} \leqq \frac{\tilde{\Psi}_0^k(0)}{k} \leqq \frac{m_k + 1}{k}.$$

Thus, it suffices to show that

$$\lim_{k\to\infty} \frac{m_k}{k} = \frac{a\rho(\varphi) + b}{c\rho(\varphi) + d}.$$

Recalling the first inequality above, we see that

$$\lim_{k\to\infty} \frac{\xi^k(0)}{\eta^{m_k}(0)}$$

exists and equals one. Hence, we obtain

$$\lim_{k\to\infty} \frac{m_k}{k} = \lim_{k\to\infty} \frac{m_k}{\eta^{m_k}(0)} \frac{\xi^k(0)}{k}$$

$$= (c\rho(\varphi) + d)^{-1}(a\rho(\varphi) + b),$$

by Lemma 3 applied to ξ and η. □

In order to find $Q(\Psi_0)$, we must find the semiconjugacy for Ψ_0. (See [6] or Markley [3].) Let h be the unique semiconjugacy for φ (i.e., $h \circ \varphi = R_{\rho(\varphi)} \circ h$)

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such that $h(0) = 0$. Let $\tilde{h}$ be the unique lifting of h to $\mathbf{R}$ such that $\tilde{h}(0) = 0$. It is immediate that $\tilde{h} \circ \tilde{\varphi} = T_{\rho(\varphi)} \circ \tilde{h}$ and $\tilde{h} \circ T_1 = T_1 \circ \tilde{h}$. So, for any integers i and j, $\tilde{h} \circ (\tilde{\varphi}^i \circ T_1^j) = T_{i\rho(\varphi)+j} \circ \tilde{h}$. Define $h_0 : S^1 \to S^1$ as follows:

$$h_0(\pi(x)) = \pi\left(\frac{1}{c\rho(\varphi) + d}\tilde{h} \circ g^{-1}(x)\right), \; x \in \mathbf{R}.$$

The fact that h_0 is well defined is seen from the following:

$$\tilde{h} \circ g^{-1} \circ T_k = \tilde{h} \circ \eta^k \circ g^{-1} = T^k_{c\rho(\varphi)+d} \circ \tilde{h} \circ g^{-1}.$$

It is routine to verify that h_0 is a semi-conjugacy for Ψ_0, i.e. $h_0 \circ \Psi_0 = R_{\rho(\Psi_0)} \circ h_0$.

LEMMA 5.

$$\pi^{-1}(Q(\Psi_0)) \approx \frac{1}{c\rho(\varphi) + d}\pi^{-1}(Q(\varphi)).$$

PROOF. We begin by identifying the unique maximal open Ψ_0-invariant proper subset of S^1. Let Y be the unique maximal open φ-invariant proper subset of S^1 and let $\tilde{Y} = \pi^{-1}(Y)$. It is clear that $\tilde{Y}$ is open and invariant under $\tilde{\varphi}$ and T_1 and that among proper subsets of $\mathbf{R}$, it is the unique maximal one with these properties. Then it follows that $g(\tilde{Y})$ is the unique maximal open $\tilde{\Psi}_0$- and T_1-invariant proper subset of $\mathbf{R}$. Thus, $\pi(g(\tilde{Y}))$ is the unique maximal open Ψ_0-invariant proper subset of S^1. By definition,

$$\begin{aligned} Q(\Psi_0) &\approx h_0(\pi(g(\tilde{Y}))) \\ &\approx \pi\left(\frac{1}{c\rho(\varphi) + d}\tilde{h} \circ g^{-1}(g(\tilde{Y}))\right) \\ &\approx \pi\left(\frac{1}{c\rho(\varphi) + d}\tilde{h}(\tilde{Y})\right). \end{aligned}$$

The set $(1/(c\rho(\varphi) + d))\tilde{h}(\tilde{Y})$ is invariant under translation by 1, so

$$\pi^{-1}(Q(\Psi_0)) \approx \frac{1}{c\rho(\varphi) + d}\tilde{h}(\tilde{Y}).$$

Since $\tilde{h}$ and $\tilde{Y}$ are liftings, $Q(\varphi)$, which is $h(Y)$, is the same as $\pi(\tilde{h}(\tilde{Y}))$. Again, the translation invariance of $\tilde{h}(\tilde{Y})$ then yields $\pi^{-1}(Q(\varphi)) \approx \tilde{h}(\tilde{Y})$. □

This completes the proof of Theorem 1.

3. **Denjoy flows on the 2-torus.**

We begin by setting out some notation and recalling the "flow under constant function" construction. Let φ be a homeomorphism of the circle without periodic points. We start with the topological space $[0, 1] \times S^1$ and identify the point $(1, s)$ with $(0, \varphi(s))$, for every s in S^1. This quotient space is homeomorphic to the usual 2-torus $S^1 \times S^1$. We obtain an action, $F\varphi$, of $\mathbf{R}$ on the quotient space by

$$F\varphi_t(r, s) = (\{r + t\}, \varphi^{[t+r]}(s)),$$

$t \in \mathbf{R}$, $r \in [0, 1]$, and $s \in S^1$.

For our purposes, it will be more useful to describe this action as follows. First, consider the following action of $\mathbf{Z}^2$ on $\mathbf{R}^2$:

$$\Phi_{(i,j)}(r, s) = (r - i, \tilde{\varphi}^i(s) + j), (i, j) \in \mathbf{Z}^2, (r, s) \in \mathbf{R}^2.$$

Define a flow F on $\mathbf{R}^2$ by $F_t(r, s) = (r + t, s)$. It is clear that these two actions commute and so we obtain a flow, $F\varphi$, on the quotient space $\mathbf{R}^2/\Phi$. Henceforth, we will denote this space by X_φ. It is homeomorphic to the quotient space defined above in an obvious way.

Following Rieffel, we use C_φ to denote the transformation group C^*-algebra $C(X_\varphi)x_{F\varphi}\ \mathbf{R}$.

THEOREM 6. *Let φ and Ψ be homeomorphisms of the circle having no periodic points. The following statements are equivalent.*

(i) *There is $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ in $GL(2, \mathbf{Z})$ such that*

$$\rho(\Psi) = \frac{a\rho(\varphi) + b}{c\rho(\varphi) + d},$$

$$\pi^{-1}(Q(\Psi)) \approx \frac{1}{c\rho(\varphi) + d}\pi^{-1}(Q(\varphi)).$$

(ii) *C_φ and C_Ψ are strongly Morita equivalent.*

(iii) *C_φ and C_Ψ are $*$-isomorphic.*

(iv) *There is a homeomorphism $\sigma: X_\varphi \to X_\Psi$ and a non-zero real number e such that*

$$\sigma \circ F\varphi_t \circ \sigma^{-1} = F\Psi_{et}, t \in \mathbf{R}.$$

That is, the flows $(X_\varphi, F\varphi)$ and $(X_\Psi, F\Psi)$ are topologically conjugate.

PROOF. The implications (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) are immediate. Theorem 1, along with the fact that C_φ is $*$-isomorphic to $A_\varphi \otimes \mathscr{K}$ ([2]), yields the implication (ii) $\Rightarrow$ (i).

For (i) $\Rightarrow$ (iv), we will produce a homeomorphism $\bar{\sigma}: \mathbf{R}^2 \to \mathbf{R}^2$ and a non-zero e in $\mathbf{R}$ such that

(a) $\bar{\sigma} \circ \Phi_{(a,b)} \circ \bar{\sigma}^{-1} = \Psi_{(1,0)}$,
(b) $\bar{\sigma} \circ \Phi_{(c,d)} \circ \bar{\sigma}^{-1} = \Psi_{(0,1)}$,
(c) $\bar{\sigma} \circ F_t \circ \bar{\sigma}^{-1} = F_{et}$, $t \in \mathbf{R}$.

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Since (a, b) and (c, d) together generate $\mathbf{Z}^2$, it will follow that $\bar{\sigma}$ carries Φ-orbits to Ψ-orbits and so $\bar{\sigma}$ drops to a homeomorphism from $X_\varphi = \mathbf{R}^2/\Phi$ to $X_\Psi = \mathbf{R}^2/\Psi$ which conjugates $F\varphi_t$ with $F\Psi_{et}$ by condition (c).

Let $\tilde{h}$ and g be as in Section 2. For convenience we will assume that Ψ and Ψ_0 are actually equal, rather than just conjugate. Set $e = c\rho(\varphi) + d$ and $f = -c$.

Define $\bar{\sigma}$ by $\bar{\sigma}(r, s) = (er - f\tilde{h}(s), g(s))$, $(r, s) \in \mathbf{R}^2$. A straightforward calculation shows that $\bar{\sigma}$ satisfies (a), (b), and (c). □

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