DENOMINATOR SEQUENCES OF CONTINUED FRACTIONS II

R. T. WORLEY

(Received 4 October 1971; revised 3 December 1971)

Communicated by E. S. Barnes

In part I, I considered the problem of discovering when, given an irrational α which has a simple continued fraction representation with convergents p_n/q_n, there exists α' for which the denominator sequence for convergents is a subsequence of (q_n). It was shown that such an α' exists if the continued fraction representation was "nearly periodic" with odd period. The following is a generalization of the results of part I to semi-regular continued fractions, where the problem seems to fit more naturally.

Let $a_0, \varepsilon_1, a_1, \varepsilon_2, \cdots, \varepsilon_n, a_n$ be integers satisfying (i) $\varepsilon_i = \pm 1$ for $1 \leqq i \leqq n$, and (ii) $a_i \geqq 1$ for $1 \leqq i \leqq n$. If the repeated fraction

$$a_0 + \varepsilon_1/(a_1 + \varepsilon_2/(\cdots/(a_{n-1} + \varepsilon_n/a_n)\cdots))$$

can be evaluated as a real number without dividing by zero we say that the symbol $\langle a_0, \varepsilon_1, a_1, \cdots, a_n \rangle$ is defined and is equal to that real number. It can be shown that division by zero never arises if $a_i + \varepsilon_{i+1} \geqq 1$ for $0 \leqq i \leqq n-1$.

By the infinite semi-regular continued fraction $\langle a_0, \varepsilon_1, a_1, \cdots \rangle$ is meant $\lim_{n\to\infty} \langle a_0, \varepsilon_1, a_1, \cdots, a_n \rangle$, where in writing $\langle a_0, \varepsilon_1, a_1, \cdots \rangle$ it is assumed that $\langle a_0, \varepsilon_1, a_1, \cdots, a_n \rangle$ is defined for all n and $a_i + \varepsilon_{i+1} \geqq 1$ for all i. It is well-known that the limit exists, and is irrational iff $a_i + \varepsilon_{i+1} > 1$ infinitely often.

It is also well-known that for all $n \geqq 1$, $\langle a_0, \varepsilon_1, \cdots, a_n \rangle = p_n/q_n$ where p_n and q_n are defined inductively by the properties

$$p_0 = a_0,\ q_0 = 1,\ p_1 = a_1 a_0 + \varepsilon_1,\ q_1 = a_1,$$
$$p_n = a_n p_{n-1} + \varepsilon_n p_{n-2} \quad (n \geqq 2),$$
$$q_n = a_n q_{n-1} + \varepsilon_n q_{n-2} \quad (n \geqq 2).$$

This notation will be preserved for the rest of this paper. In addition, for convenience, we shall take $a_0 \geqq 2$ in order to avoid difficulties with symbols $\langle \cdots, a_0 \rangle$, though plainly the denominators q_i $(i \geqq 0)$ are independent of a_0. We define $\langle a_0, \varepsilon_1, a_1, \cdots \rangle$ to be nearly periodic with period (p, r) if p, r are non-negative integers with $p > 1$ such that for each $n \geqq 1$ at least one of the following two equations holds, where the expressions are considered as $(2p-3)$-tuples

(i) $(a_{np+r}, \varepsilon_{np+r}, a_{np+r-1}, \cdots, \varepsilon_{(n-1)p+r+3}, a_{(n-1)p+r+2})$
$$= (a_{np+r+2}, \varepsilon_{np+r+3}, a_{np+r+3}, \cdots, a_{(n+1)p+r})$$

(ii) $(a_{(n-1)p+r+2}, \varepsilon_{(n-1)p+r+3}, \cdots, a_{np+r-1}, \varepsilon_{np+r}, a_{np+r})$
$$= (a_{np+r-2}, \varepsilon_{np+r+3}, a_{np+r+3}, \cdots, a_{(n+1)p+r}).$$

Thus "nearly periodic" is independent of the initial terms as there is no restriction on the size of r. The condition allows the numbers $\varepsilon_{np+r+1}, a_{np+r+1}, \varepsilon_{np+r+2}$ to be arbitrary within the restrictions on a semi-regular continued fraction, while the blocks between these groups of three elements must recur in the same or reverse order.

Because we shall be reversing the order of elements of continued fractions we need the following lemma.

LEMMA 1. *Let $\langle a_0,\varepsilon_1,a_1,\cdots\rangle$ be a semi-regular continued fraction and let, for non-negative integers r,n with $n \geqq r$,*

$$(b_0,\rho_1,b_1,\rho_2,\cdots,b_r) = (a_n,\varepsilon_n,a_{n-1},\varepsilon_{n-1},\cdots,\varepsilon_{n-r+1},a_{n-r})$$

as $(2r+1)$-tuples. Then $\langle b_0,\rho_1,b_1,\cdots,b_r\rangle$ is defined and is positive. Furthermore it is not greater than 1 if and only if $r = 0$ and $b_0 = 1$ or $r > 0$, $b_0 = 1$ and $\rho_1 = -1$.

PROOF. The result is trivial for $r = 0$. If $r = 1$ then $\langle b_0,\rho_1,b_1\rangle = a_n + \varepsilon_n/a_{n-1}$ is defined. If $\varepsilon_n = 1$ then plainly $\langle b_0,\rho_1,b_1\rangle > a_n \geqq 1$. If $\varepsilon_n = -1$ and $a_n \geqq 2$ then since $a_{n-1} \geqq 1-\varepsilon_n = 2$ we have $\langle b_0,\rho_1,b_1\rangle \geqq 3/2 > 1$. If $\varepsilon_n = -1$ and $a_n = 1$ then $a_{n-1} \geqq 2$ again and

$$0 < \tfrac{1}{2} \leqq 1 - a_n^{-1} < 1.$$

This proves the lemma for $r = 1$. The proof is now completed by induction Assume truth for $r = k$. Then

$$\langle b_0,\rho_1,b_1,\cdots,b_{k+1}\rangle = b_0 + \rho_1/\langle b_1,\rho_2,\cdots,b_{k+1}\rangle$$

and is defined since $\langle b_1,\cdots,b_{k+1}\rangle > 0$ by assumption. In addition, $\langle b_1,\cdots,b_{k+1}\rangle \leqq 1$ only if $b_1 = 1$, i.e. $a_{n-1} = 1$. Then $\varepsilon_n \geqq 1-a_{n-1} = 0$ and so $\varepsilon_n = 1$, i.e. $\rho_1 = 1$. In this case $\langle b_0,\rho_1,b_1,\cdots,b_{k+1}\rangle \geqq 2$. Otherwise $\langle b_1,\cdots,b_{k+1}) > 1$ and so (i) if $\rho_1 = 1$ we have $\langle b_0,\cdots,b_{k+1}\rangle > 1$,

(ii) if $\rho_1 = -1$ we have $b_0 - 1 < \langle b_0,\cdots,b_{k+1}\rangle \leqq b_0$

and so $\langle b_0,\cdots,b_{k+1}\rangle \leqq 1$ if and only if $b_0 = 1$ and $\rho_1 = -1$. This completes the induction step.

Another lemma that is easily proved by induction is

LEMMA 2. *Let $\langle a_0,\varepsilon_1,a_1,\cdots\rangle$ be an infinite semiregular continued fraction* *and let r, s be non-negative integers with $s > r$. Then*

$$\langle a_r, \varepsilon_{r+1}, a_{r+1}, \cdots, a_s \rangle \geqq 1$$

with equality if and only if a_r, $a_{r+1}, \cdots, a_{s-1}$ are all 2, $\varepsilon_{r+1}, \cdots, \varepsilon_s$ are all -1 and $a_s = 1$.

The following lemmas are the natural generalization of the results of part I and are proved in the same manner.

LEMMA 3. *Let k, l, m be positive integers with $m > l$. Define P_m, Q_m, R_m, S_m by*

$$P_m/Q_m = \langle a_{m+1}, \varepsilon_{m+2}, a_{m+2}, \cdots, a_{m+k} \rangle$$

$$R_m/S_m = \langle 0, 1, a_m, \varepsilon_m, a_{m-1}, \cdots, a_{m-l+2} \rangle$$

where the second is interpreted as 0/1 if $l = 1$ and is defined by lemma 1. Then

$$q_{m+k} = P_m q_m + \varepsilon_{m+1} Q_m q_{m-1},$$

$$\varepsilon_m \varepsilon_{m-1} \cdots \varepsilon_{m-l+2} q_{m-l} = (-1)^{l-1}(S_m q_{m-1} - R_m q_m),$$

and similarly with the q_i replaced by p_i.

COROLLARY. *If $\langle a_0, \varepsilon_1, a_1, \cdots \rangle$ is nearly periodic with period (p, r) then for all $n \geqq 1$,*

(i) $$q_{(n+1)p+r} = c_{n+1} q_{np+r} + \eta_{n+1} q_{(n-1)p+r}$$

(ii) $$p_{(n+1)p+r} = c_{n+1} p_{np+r} + \eta_{n+1} p_{(n-1)p+r}$$

where

$$c_{n+1} = P_{np+r} + \varepsilon_{np+r+1} R_{np+r}$$

$$\eta_{n+1} = (-1)^{p-1} \varepsilon_{np+r+1} \varepsilon_{np+r} \cdots \varepsilon_{(n-1)p+r+2}.$$

It should be noted that the first part of the conclusion of Lemma 1 is basically Perron [1] p. 15 eq. (24) while the result, for $\langle a_0, \varepsilon_1, a_1, \cdots \rangle$ nearly periodic with period (d,p,r), that $Q_{np+r} = S_{np+r}$, is basically Perron p. 12 equation (18). The following lemma can be proved easily by induction.

LEMMA 4. *Let $d_2, \cdots, d_s$ be positive integers, let $\rho_2, \cdots, \rho_s$ satisfy $\rho_i = \pm 1$ $(2 \leqq i \leqq s)$, let X_0, X_1, Y_0, Y_1 be non-negative integers and let $X_2, \cdots, X_s$ $Y_2, \cdots, Y_s$ be defined inductively by*

$$X_m = d_m X_{m-1} + \rho_m X_{m-2} \qquad (2 \leqq m \leqq s).$$

$$Y_m = d_m Y_{m-1} + \rho_m Y_{m-2}$$

Then

$$X_s/Y_s = (X_1 \beta_s + \rho_2 X_0)/(Y_1 \beta_s + \rho_2 Y_0)$$

where $\beta_s = \langle d_2, \rho_3, d_3, \cdots, \rho_s, d_s \rangle$.

The following theorem and its corollary are the main results.

THEOREM. *Let* $\alpha = \langle a_0, \varepsilon_1, a_1, \cdots \rangle$ *be such that for positive integers* $c_2, c_3, c_4, \cdots$

(i) $q_{i_{n+1}} = c_{n+1} q_{i_n} + \eta_{n+1} q_{i_{n-1}}$ $(n \geqq 1)$, and

(ii) $p_{i_{n+1}} = c_{n+1} p_{i_n} + \eta_{n+1} p_{i_{n-1}}$ $(n \geqq 1)$

for a subsequence (q_{i_n}) *of* (q_i), *where* $\eta_i = \pm 1$ *and* $c_i + \eta_{i+1} \geqq 1$ *for* $i \geqq 2$ *and* $\eta_2 = 1$ *if* $q_{i_1} \leqq q_{i_0}$.

If q_{i_0} *and* q_{i_1} *are relatively prime then there exists* α' *of the form* $a\alpha + b$ *with* a *and* b *rational such that* α' *has a semi-regular continued fraction expansion for which the convergents are* A_n/B_n *with* $B_{n+u} = q_{i_n}$ *for* $n \geqq 0$, *where* $u \geqq 0$ *is integral and* $u = 0$ *if* $q_{i_0} = 1$.

PROOF. If $q_{i_0} = 1$ then plainly the denominators of convergents to $\alpha' = \langle 0, 1, q_{i_1}, \eta_2, c_2, \eta_3, c_3 \cdots \rangle$ are $q_{i_0}, q_{i_1} \cdots$. However if $q_{i_0} > 1$ define $q_{i_{-1}}, q_{i_{-2}}, \cdots$ and $c_1, c_0, c_{-1}, \cdots$ inductively by

$$q_{i_{j+2}} = c_{j+2} q_{i_{j+1}} + \eta_{j+2} q_{i_j} \ (j \leqq -1; 0 \leqq q_{i_j} \leqq q_{i_{j+1}} - 1)$$

where if $q_{i_1} > 2q_{i_0}$ then η_{j+2} is always chosen as $+1$ and if $q_{i_1} < 2q_{i_0}$ then η_{j+2} is always chosen as -1. The process is considered to stop when $q_{i_t} = 0$ is first reached. It is easy to verify that the convergents to

$$\alpha' = \langle 0, 1, c_{t+2}, \eta_{t+2}, c_{t+3}, \cdots \rangle$$

have denominators $q_{i_{t+1}}, q_{i_{t+2}}, \cdots$.

To find the relation between α and α' we observe that by lemma 4

$$\alpha = \lim_{n \to \infty} p_{i_n}/q_{i_n} = (p_{i_1}\beta + \eta_2 p_{i_0})/(q_{i_1}\beta + \eta_2 q_{i_0})$$

where $\beta = \langle c_2, \eta_3, c_3, \cdots \rangle$. But we have, by Perron p. 7 eq. (11),

$$\alpha' = (y\beta + \eta_2 x)/(q_{i_1}\beta + q_{i_0}\eta_2)$$

where $y/q_{i_1} = \langle 0, 1, c_{t+2}, \varepsilon_{t+3}, \cdots, c_1 \rangle$ and $x/q_{i_0} = \langle 0, 1, c_{t+2}, \cdots, c_0 \rangle$, these expressions being interpreted as $1/q_{i_1}$ and $0/1$ if $q_{i_0} = 1$. Comparing the above formulae gives

$$\alpha' = (p_{i_1} q_{i_0} - p_{i_0} q_{i_1})^{-1} (\alpha(y q_{i_0} - x q_{i_1}) + x p_{i_1} - y p_{i_0})$$

which is clearly of the desired form. It remains to set $u = -(t+1)$.

It will be noticed that if $(q_{i_1}, q_{i_0}) = d \neq 1$ then carrying out the above procedure on $d^{-1} q_{i_0}$ and $d^{-1} q_{i_1}$ gives α' as above, the denominators of the convergents being $d^{-1} q_{i_{t+1}}, d^{-1} q_{i_{t+2}}, \cdots$, and we have

$$\begin{aligned} \alpha' &= (y\beta + \eta_2 x)/(d^{-1} q_{i_1}\beta + d^{-1} q_{i_0}\eta_2) \\ &= a\alpha + b \end{aligned}$$

for rational a and b, as before.

COROLLARY. *If* α *is irrational with continued fraction expansion* $\alpha = \langle a_0, \varepsilon_1, a_1, \cdots \rangle$ *nearly periodic with period* (p, r), *then*

(a) *If* $r = 0$ *or* $r = a_1 = 1$ *then* $q_r = 1$ *and the theorem with* $i_n = np + r$ *yields the existence of* α' *the denominators of convergents to which form a subsequence* $(B_n) = (q_{i_n+kp})$, *for a suitable integer* k, *of the denominator sequence of convergents to* α.

(b) *If* $(q_r, q_{p+r}) = 1$ *then the theorem with* $i_n = np + r + kp$ *for a suitable integer* k *yields* α' *the denominators of convergents to an expansion of which, apart from an initial few, form a subsequence* $(B_{n+u}) = (q_{i_n})$ *of the denominator sequence of convergents to* α.

(c) *If* $(q_r, q_{p+r}) = d > 1$ *then by the remarks above a similar conclusion holds except that the denominators of convergents to* α' *need to be multiplied by* d.

PROOF. It is only necessary to show the conditions of the theorem are satisfied. The corollary to Lemma 3 is used. Plainly $\eta_i = \pm 1$. We must show that $c_i + \eta_{i+1} \geqq 1$ for $i \geqq 2$. Since $\eta_{n+2} \geqq -1$ and $c_{n+1} \geqq P_{np+r} - R_{np+r}$ for $n \geqq 1$ this condition is satisfied if $P_{np+r}/Q_{np+r} > 1$ and $R_{np+r}/S_{np+r} < 1$ for then $Q_{np+r} = S_{np+r}$ implies that $P_{np+r} \geqq R_{np+r} + 2$. Now suppose that $P_{np+r}/Q_{np+r} \leqq 1$ By lemma 2 this implies that

$$\langle a_{np+r+1}, \varepsilon_{np+r+2}, a_{np+r+3}, \cdots, a_{(n+1)p+r} \rangle = \langle 2, -1, 2, \cdots, 2, -1, 1 \rangle.$$

But then

$$\begin{aligned} \eta_{n+2} &= (-1)^{p-1} \varepsilon_{np+r+2} \varepsilon_{np+r+3} \cdots \varepsilon_{(n+1)p+r} \varepsilon_{(n+1)p+r+1} \\ &= (-1)^{p-1} (-1)^{p-1} \varepsilon_{(n+1)p+r+1} \\ &= \varepsilon_{(n+1)p+r+1} \\ &\geqq 1 - a_{(n+1)p+r} = 0. \end{aligned}$$

Thus $\eta_{n+2} = 1$ and so $c_{n+1} + \eta_{n+2} \geqq 1$. Similarly if we suppose $R_{np+r}/S_{np+r} \geqq 1$ then using lemma 1 we have

$$\langle a_{np+r}, \varepsilon_{np+r}, \cdots, a_{(n-1)p+r+2} \rangle = \langle 1, -1, \cdots \rangle \text{ or } \langle 1 \rangle.$$

But $a_{np+r} = 1$ implies $\varepsilon_{np+r+1} \geqq 1 - 1 = 0$, so $\varepsilon_{np+r+1} = 1$ and hence $c_{n+1} = P_{np+r} + \varepsilon_{np+r+1} R_{np+r} \geqq 1 + 1 = 2$, since neither P_{np+r}/Q_{np+r} nor R_{np+r}/S_{np+r} can be zero. It remains to satisfy the condition $\eta_2 = 1$ if $q_{i_1} \leqq q_{i_0}$. We do this by choosing $i_n = (n+k)p + r$ such that $q_{i_1} > q_{i_0}$. This is possible since α is assumed irrational and $\lim_{k \to \infty} |\alpha - p_{kp+r}/q_{kp+r}| = 0$.

It will be noted that if a nearly periodic simple continued fraction expansion of α is chosen then the continued fraction expansion of α' that is obtained is simple if the period is odd and the "integer above" expansion if the period is even. For this expansion we have, eventually, $c_{n+1} = P_{np+r} + R_{np+r}$ as all ε are 1.

https://doi.org/10.1017/S1446788700028822 Published online by Cambridge University Press

By lemmas 1 and 2 we have $P_{np+r}/Q_{np+r} > 1$ and $R_{np+r}/S_{np+r} \neq 0$. Thus $P_{np+r} \geqq 2$ and $R_{np+r} \geqq 1$ and so $c_{n+1} \geqq 3$. The convergents h/k to α' corresponding to these c therefore satisfy $|\alpha' - h/k| < 1/2k^2$ and so by a well-known result are convergents of the simple continued fraction for α'. Thus α and α' have simple continued fraction expansions with a common subsequence of denominators.

References

[1] O. Perron, *Die Lehre von den Kettenbrüchen* (Chelsea).
[2] R. T. Worley, 'Denominator Sequences of Continued Fractions I,' *Aust. Math. Soc.* 15 (1973), 112–116.

Monash University
Clayton, Victoria, 3168
Australia